GENERAL RELATIVITY

Tutorial problem set 11, 25.11.2016.

■ PROBLEM 28 Energy conditions.

(a) The action for the scalar field is given by

$$S_\phi = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) \ , \tag{28.1}$$

where $V(\phi)$ is the potential. Under the assumption that $V(\phi) > 0$ show that the scalar field energy-momentum tensor satisfies the dominant energy condition.

(b) The action for the electromagnetic field is given by

$$S_{EM} = \int d^4x\sqrt{-g}\left(-\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\right) \ . \tag{28.2}$$

Show that the energy momentum tensor of the electromagnetic field satisfies the dominant energy condition.

■ PROBLEM 29 Birkhoff's theorem.

The most general spherically symmetric line element is of the form

$$ds^2 = -A(t,r)dt^2 + 2B(t,r)dtdr + C(t,r)dr^2 + D(t,r)r^2\Big(d\vartheta^2 + \sin^2\vartheta d\varphi^2\Big) \ . \tag{29.1}$$

In problems (a) and (b) you will show that this line element can be reduced to a simpler looking one via coordinate transformation. Therefore, the simpler form of the line element will be just as general as (29.1).

(a) Show that the coordinate transformation of the form

$$r \to \bar{r} = \bar{r}(r,t) \ , \tag{29.2}$$

can be chosen such that (after suitable redefinitions) line element reduces to

$$ds^2 = -a(t,r)dt^2 + 2b(t,r)dtdr + c(t,r)dr^2 + r^2\Big(d\vartheta^2 + \sin^2\vartheta d\varphi^2\Big) \ . \tag{29.3}$$

(b) Show that the coordinate transformation of the form

$$t \to \bar{t} = \bar{t}(t,r) \ , \tag{29.4}$$

which eliminates the mixed term $dtdr$, always exists and thus the most general spherically symmetric line element (after suitable redefinition of functions) can be written in the form

$$ds^2 = -e^{\nu(t,r)}dt^2 + e^{\mu(t,r)}dr^2 + r^2\Big(d\vartheta^2 + \sin^2\vartheta d\varphi^2\Big) \ . \tag{29.5}$$

Hint: The non-vanishing components of the Riemann tensor for the line element (29.5) are given to you to be,

$$R^0{}_{101} = \frac{1}{4}e^{\mu-\nu}\left[2\partial_t^2\mu + (\partial_t\mu)^2 - \partial_t\nu\partial_t\mu\right] + \frac{1}{4}\left[\partial_r\nu\partial_r\mu - 2\partial_r^2\nu - (\partial_r\nu)^2\right] \ , \tag{29.6}$$

$$R^0{}_{202} = -\frac{1}{2}re^{-\mu}\partial_r\nu \ , \qquad R^0{}_{303} = -\frac{1}{2}re^{-\mu}\sin^2\vartheta\,\partial_r\nu \ , \tag{29.7}$$

$$R^0{}_{212} = -\frac{1}{2}re^{-\nu}\partial_t\mu \ , \qquad R^0{}_{313} = -\frac{1}{2}re^{-\nu}\sin^2\vartheta\,\partial_t\mu \ , \tag{29.8}$$

$$R^1{}_{212} = \frac{1}{2}re^{-\mu}\partial_r\mu \ , \qquad R^1{}_{313} = \frac{1}{2}re^{-\mu}\sin^2\vartheta\,\partial_r\mu \ , \qquad R^2{}_{323} = (1-e^{-\mu})\sin^2\vartheta \ , \tag{29.9}$$

and the non-vanishing components of the Ricci tensor are R_{00}, R_{11}, R_{22}, R_{33}, and $R_{01} = R_{10}$. You do not need to calculate the Ricci scalar, why?

After casting the line element in a simpler form (but equally general), in the remaining problems you will determine what functions ν and μ are from the Einstein equation.

(c) Show that the tr component of the Einstein equation implies that μ is independent of time, $\mu(t,r) = \mu(r)$.

(d) Use (a linear combination of) the remaining components of the Einstein equation to show that

$$\nu(t,r) = -\mu(r) + f(t) \ . \tag{29.10}$$

(e) Use these results to conclude that the most general spherically symmetric solution to the Einstein equation in the vacuum (after a redefinition of the time coordinate) can be written as

$$ds^2 = -e^{-\mu(r)}dt^2 + e^{\mu(r)}dr^2 + r^2\Big(d\vartheta^2 + \sin^2\vartheta d\varphi^2\Big) \ . \tag{29.11}$$

(f) Show that the other components of the Einstein equation imply

$$e^{-\mu(r)} = 1 - \frac{R_S}{r} \ , \tag{29.12}$$

where R_S is a real constant, so that we obtain the Schwarzschild solution.

Schwarzschild geometry is the most general asymptotically flat spherically symmetric solution to the Einstein equation. This is the Birkhoff-Jebsen theorem.