GENERAL RELATIVITY

Tutorial problem set 14, 16.12.2016.

■ **PROBLEM 35** Kruskal-Szekeres coordinates.

In this problem you will show that the singularity at Schwarzschild radius, $R_S = 2MG_N$, of the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \ , \tag{35.1}$$

can be removed by making use of different coordinates.

We start by defining the *Kruskal-Szekeres* coordinates,
for $r > R_S$:

$$R = \left(\frac{r}{R_S} - 1\right)^{1/2} \exp\left(\frac{r}{2R_S}\right) \cosh\left(\frac{t}{2R_S}\right) \ , \tag{35.2}$$

$$T = \left(\frac{r}{R_S} - 1\right)^{1/2} \exp\left(\frac{r}{2R_S}\right) \sinh\left(\frac{t}{2R_S}\right) \ , \tag{35.3}$$

and for $r < R_S$:

$$R = \left(1 - \frac{r}{R_S}\right)^{1/2} \exp\left(\frac{r}{2R_S}\right) \sinh\left(\frac{t}{2R_S}\right) \ , \tag{35.4}$$

$$T = \left(1 - \frac{r}{R_S}\right)^{1/2} \exp\left(\frac{r}{2R_S}\right) \cosh\left(\frac{t}{2R_S}\right) \ . \tag{35.5}$$

(a) Show that the metric in Kruskal-Szekeres takes the following form

$$ds^2 = -\frac{4R_S^3}{r} e^{-\frac{r}{R_S}} (dT^2 - dR^2) + r^2 d\Omega^2 \ . \tag{35.6}$$

(b) How is the radial null line parametrized, in terms of R and T?

In order to get a better understanding of this space-time, we can study the diagram in Fig. 1. Every point in this diagram is supposed to represent a sphere (coordinates ϑ and φ are suppressed). The results from part (b) of this problem tell us that the light cones in coordinates T and R are as open as in SR.

(c) What is the shape of the lines of constant r, for $r > R_S$? And for $r < R_S$? Can a massive object stay at constant r inside the horizon?

(d) Draw the lines of constant t. What happens for $t \to \infty$?

(e) Draw the path of an infalling observer.

(f) Which regions are connected to region I by light signals?

Read section 5.7 from Carroll's book for a discussion of the properties of Schwarzschild black hole in Kruskal-Szekeres coordinates.

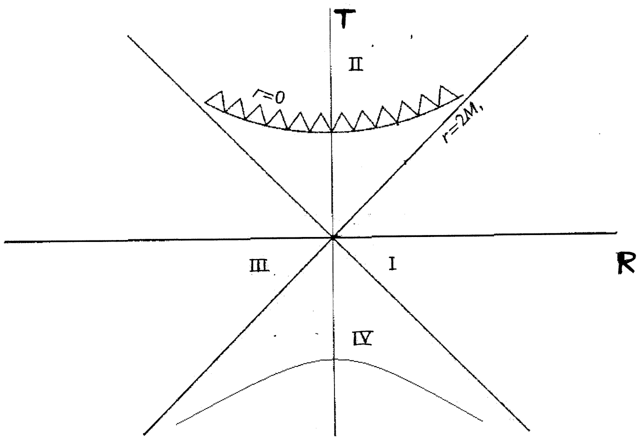

Figure 1: Kruskal diagram.

■ **PROBLEM 36** Dimming of objects falling into a black hole.

Consider an observer outside of a Schwarzschild black hole sitting at constant Schwarzschild coordinates $(r_0, \vartheta = \frac{\pi}{2}, \varphi = 0)$. At some point $t = 0$ he releases a beacon that starts falling towards the black hole, while constantly emitting a monochromatic light signal (of frequency ν_{em}) back to the observer (in the radial direction). In this problem you will determine what is the signal received by the observer.

(a) One generally needs to be careful what kind of velocities one defines in GR. Here we can define at least three ones for the beacon of certain use:

(i) The coordinate velocity, $v = \frac{dr}{dt}$, that measures what is the coordinate distance dr that the beacon travels during coordinate time dt,

(ii) The r-component of the 4-velocity, $u_r = \frac{dr}{d\tau}$, that measures the coordinate distance dr the beacon travels during its own proper time interval $d\tau$,

(iii) The 'velocity of a stationary observer', $w_R = \frac{dr}{d\tau_R}$, which is the velocity of the beacon that would be measured by an observer at the fixed coordinate R, at the moment the beacon is passing him by, where τ_R is the proper time of that observer.

Calculate these three velocities as functions of r. Rather than solving the geodesic equations, make use of the Schwarzschild line element, and the conserved quantity E derived in problem 32 of the tutorial problem set 12. Fix E by imposing initial conditions.

(b) Find the relation between the coordinate time and the proper time of a static observer at R, and between coordinate time and the proper time of the beacon at r.

In order to find the frequency of the signal received by the observer at r_0, we will imploy a similar strategy as in problem 34 of the tutorial problem set 13. We will relate the proper time (of the beacon) between the emission of the two successive light crests, and the proper time of the observer at r_0 between the absorption of these two crests.

(c) The beacon emits the first crest at (t_{em1}, r_{em1}), and the second one (t_{em2}, r_{em2}). The coordinate time between the emission of two successive crests is $\Delta t_{em} = t_{em2} - t_{em1}$, and during that time the beacon has moved by $\Delta r_{em} = r_{em2} - r_{em1}$. Calculate the coordinate time Δt_{re} between the reception of the two crests by the observer at r_0. In order to do that, calculate the coorinate time it takes each crest to reach the observer.

(d) We can safely assume that the time interval between the emission of the two crests is much smaller than any other relevant scale. Expand the expression obtained in (c) to linear order in Δt_{em}, and express the ratio $\Delta t_{re}/\Delta t_{em}$ in terms of r_0, $R_S = 2MG_N$ and $r_{em1} \equiv r$.

(e) Using the expressions found in (b) and (d) show that the frequency received by the observer is

$$\nu_{re} = \nu_L \sqrt{\frac{A(r)}{A(r_0)}} \times \frac{\sqrt{A(r)}}{\sqrt{A(r_0)} + \sqrt{A(r_0) - A(r)}} \,, \tag{36.1}$$

where $A(r) = 1 - \frac{R_s}{r}$. Compare this result to the one obtained in problem 34 of the tutorial problem set 13. Interpret where does the extra term here come from.