GENERAL RELATIVITY

Tutorial problem set 15, 23.12.2016.

■ **PROBLEM 37** Tolman-Oppenheimer-Volkoff equation.

In this problem you will examine more closely the general relativistic models of spherical massive objects (such as stars). Namely, you will study spherically symmetric static solutions of the inhomogeneous Einstein equation.

If a star has a radius larger than its Schwarzschild radius, the space-time outside of it will be described by Schwarzschild metric (Birkhoff's theorem). In order to say something about the space-time inside the star we have to solve the Einstein equation for the particular source,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu} \ , \tag{37.1}$$

meaning we have to know someting about the composition of the star.

Since we are looking for a spherically symmetric and static solution, we assume the following line element

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \ . \tag{37.2}$$

(The relevant geometric quantities for this metric are given at the end of this problem, just specialize to time-independent α and β.)

(a) Calculate the Ricci scalar, and the non-vanishing components of the Einstein tensor.

The star will be modeled by a perfect fluid, whose energy-momentum tensor is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \ , \tag{37.3}$$

where we assume to be in the rest frame of the fluid, $u_\mu = (-e^\alpha, 0, 0, 0)$, and because of the assumed spherical symmetry, $\rho = \rho(r)$ and $p = p(r)$.

(b) Calculate all the non-vanishing components of the fluid's energy-momentum tensor.

(c) Write down all the independent components of the Einstein equation (there are three of them).

(d) Notice that the (tt) equation involves only β and ρ. Show that it can be rewritten as

$$\partial_r\left(r - re^{-2\beta}\right) = 8\pi G_N r^2 \rho \ . \tag{37.4}$$

Introduce a new function

$$m(r) = \frac{1}{2G_N}\left(r - re^{-2\beta}\right) \ , \tag{37.5}$$

and show that the line element can then be written as

$$ds^2 = -e^{2\alpha}dt^2 + \left(1 - \frac{2G_N m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \ . \tag{37.6}$$

We have substituted function $\beta(r)$ for function $m(r)$ in a way so g_{rr} would look like a Schwarzschild one, but with a 'radially dependent mass'. Because of Birkhoff's theorem, we know that outside of the star the solution must be a Schwarzschild one, and therefore $m(R_*) = M$ must correspond to the total mass of the star (as seen by the distant observer), where R_* is the radius of the star. Find the expression for $m(r)$ in terms of $\rho(r)$, assuming $m(0) = 0$.

(e) Show that the (rr) equation can be written as

$$\frac{d\alpha}{dr} = \frac{G_N m(r) + 4\pi G_N r^3 p}{r(r - 2G_N m(r))} \ . \tag{37.7}$$

(f) Combine the equation from (e) together with the $\vartheta\vartheta$ equation to get

$$\frac{dp}{dr} = -\frac{(\rho + p)[G_N m(r) + 4\pi G r^3 p]}{r[r - 2G_N m(r)]} \ . \tag{37.8}$$

This is the Tolman-Oppenheimer-Volkoff equation for hydrostatic equilibrium. (An easier way of obtaining this equation is to consider the conservation equation for the energy-momentum tensor instead of the $\vartheta\vartheta$ equation).

(g) In order to get a closed system of equations, we need one more apart from the Tolman-Oppenheimer-Volkoff equation, and that is the equation of state for the fluid, $p = p(\rho)$ (or some equivalent one). Then, in principle (37.8) represents an equation for $p(r)$.

One can get a quite simple and semi-realistic model of a star by assuming it is made out of an incompressible ideal fluid. It means that its energy density $\rho(r)$ is constant throughout,

$$\rho(r) = \begin{cases} \rho_* & , \quad r < R_* \\ 0 & , \quad r > R_* \end{cases} \tag{37.9}$$

where R_* is the radius of the star. This assumption serves instead of the equation of state. Solve the Tolman-Oppenheimer-Volkoff equation to find the pressure inside the star. What is the boudary condition for pressure you are going to assume? Express the final answer in the form $p(r) = \rho_* P(r)$, where P should be expressed in terms of G_N, M, R_* and r.

(h) By considering the pressure at the center of the star show that the maximum mass a star of the type considered here of a given radius R_* can be

$$M_{max} = \frac{4R_*}{9G_N} \ , \tag{37.10}$$

i.e. that there are no static solutions for a mass larger than this. This is a special case of the Buchdahl's theorem (but the result holds more generally). Can you give a physical interpretation of this result?

The non-vanishing components of the Ricci tensor for metric (37.2) are

$$R_{tt} = e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] , \tag{37.11}$$

$$R_{rr} = -\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta\right] , \tag{37.12}$$

$$R_{\vartheta\vartheta} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1 , \qquad R_{\varphi\varphi} = R_{\vartheta\vartheta}\sin^2\vartheta . \tag{37.13}$$