GENERAL RELATIVITY

Tutorial problem set 16, 13.01.2017.

■ **PROBLEM 38** Motion of a particle in Kerr space-time.

The line element of the Kerr space-time in Boyer-Linquist coordinates is

$$ds^2 = -\left[1 - \frac{2MG_N r}{\rho^2}\right] dt^2 - \frac{4MG_N ar\sin^2\theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left[r^2 + a^2 + \frac{2MG_N ra^2\sin^2\theta}{\rho^2}\right]\sin^2\theta\, d\phi^2 \ , \tag{38.1}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta \ , \qquad \Delta = r^2 - 2MG_N r + a^2 \ . \tag{38.2}$$

In this problem you will examine the orbits of massive particles and light particles that lie in the equatorial plane $\theta = \frac{\pi}{2}$.

(a) Write the metric (38.1) in the equatorial plane $\theta = \frac{\pi}{2}$.

(b) Since the metric (38.1) does not depend on coordinates t and ϕ we can immediately conclude that there are two Killing vectors in this space-time,

$$K^\mu = (1, 0, 0, 0) \ , \qquad R^\mu = (0, 0, 0, 1) \ , \tag{38.3}$$

associated to time translation symmetry and axial symmetry, respectively. When there are Killing vectors, there are conserved quantities along geodesics associated to them. In this case those are energy and the z-component of the angular momentum,

$$E = -K_\mu \frac{dx^\mu}{d\lambda} \ , \qquad \ell = R_\mu \frac{dx^\mu}{d\lambda} \ . \tag{38.4}$$

Express $dt/d\lambda$ and $d\phi/d\lambda$ in terms of E and ℓ and the radial coordinate.

(c) Using results from (a) and (b) find the effective potential for the motion of a massless particle in the equatorial plane,

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + W_{\text{eff}}(E, \ell; r) = \frac{E^2}{2} \ . \tag{38.5}$$

(d) Find the effective potential for massive particles (take $\lambda = \tau$),

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(E, \ell; r) = \frac{E^2 - 1}{2} \ . \tag{38.6}$$

(e) Consider a massive particle initially infalling radially ($\ell = 0$) towards the Kerr black hole, and with negligible initial kinetic energy ($E = 1$ – why this value?). Kerr space-time is stationary, but it is not static (black hole is rotating), and as an effect it will dragg the initially radially infalling particle with its rotation, i.e. $d\phi/dr$ will not be constant. Show that

$$\frac{d\phi}{dr} = -\frac{2MG_N a}{r\Delta}\left[\frac{2MG_N}{r}\left(1+\frac{a^2}{r^2}\right)\right]^{-1/2} . \tag{38.7}$$

The angle $\Delta\phi(r_0)$ the trajectory of the particle has swept from $r = \infty$ to some $r = r_0$ can in principle be calculated by integrating this expression from ∞ to r_0. Will this particle eventually fall into the black hole?

(f) In the limit of extremal Kerr black hole where $a = MG_N$, find the two possible circular light-like orbits. Are they stable or unstable? What are their angular velocities?

There are two circular photon orbits in the non-extremal case as well,

$$r_1 = 2MG_N\left\{1+\cos\left[\frac{2}{3}\arccos\left(-\frac{|a|}{MG_N}\right)\right]\right\} , \tag{38.8}$$

$$r_2 = 2MG_N\left\{1+\cos\left[\frac{2}{3}\arccos\left(\frac{|a|}{MG_N}\right)\right]\right\} . \tag{38.9}$$

Convince yourself that they reduce to the right limit when $a \to 0$.

■ PROBLEM 39 The ergosphere and ergoregion of the Kerr black hole.

(a) Determine the boundary of the region in Kerr space-time inside of which it is impossible for an observer to remain at fixed coordinates (r, θ, ϕ). Express this boundary in a form $r(\theta)$. Note that this region extends outside of the event horizon at $r_+ = MG_N + \sqrt{M^2G_N^2 - a^2}$. The region of spacetime outside of the horizon where there are no stationary observers possible is called the ergoregion and its boundary is called the ergosphere. It is an effect of the rotation of the black hole. Sketch the ergoregion, ergosphere and the horizon.

(b) Although static observers are not possible inside the ergosphere, observers with fixed just r and θ coordinates that rotate with constant angular velocity as seen from infinity ($d\phi/dt = \Omega = \text{const.}$) along with the black hole are allowed. Show that it is so.

(c) Show that the minimal angular velocity of the massless particle at the horizon r_+ is

$$\Omega_H = \left(\frac{d\phi}{dt}\right)_{\min}(r_+) = \frac{a}{r_+^2 + a^2} . \tag{39.1}$$

(d) Show that the Killing vector $\chi^\mu = K^\mu + \Omega_H R^\mu$ (where K^μ and R^μ are defined in problem 1) has a Killing horizon, and determine where it is.