GENERAL RELATIVITY

Tutorial problem set 17, 20.01.2017.

■ **PROBLEM 40** Gauge invariant perturbations.

Consider a slightly perturbed Minkowski space,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \ , \tag{40.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the canonical Minkowski metric, and $h_{\mu\nu}$ is the small perturbation. The perturbation can be decomposed into parts that transform differently under *spatial rotations*, which is given in the usual way as

$$t \to \widetilde{t} = t \ , \qquad x^i \to \widetilde{x}^i = R^i{}_j \, x^j \ , \qquad \boldsymbol{R}\boldsymbol{R}^\top = \mathbf{1}, \quad \det(\boldsymbol{R}) = 1 \ . \tag{40.2}$$

(a) Show that under spatial rotations: (i) h_{00} transforms as a scalar, (ii) h_{0i} transforms as a vector, (iii) h_{ij} transforms as a tensor.

According to the Helmholtz's theorem on vector calculus, h_{0i} can be decomposed into two distinct parts, a transverse (divergence-free) part, and a longitudinal (curl-free) part,

$$h_{0i} = B_i + \partial_i S \ , \tag{40.3}$$

where $\partial^i B_i = 0$ and $\nabla \times (\nabla S) = 0$, and where we have used the result that the curl-free part can be written as a gradient of some scalar function S. In a similar way, a 2-tensor under spatial rotations can be decomposed as

$$h_{ij} = 2\delta_{ij}\psi + 2\partial_i\partial_j E + \partial_i F_j + \partial_j F_i + h_{ij}^{TT} \ . \tag{40.4}$$

Here ψ and E are scalar functions, F_j is a transverse (divergence-free) vector, and h_{ij}^{T} is a *transverse traceless tensor*,

$$\partial^i h_{ij}^{TT} = 0 \ , \qquad (h^{TT})^i{}_i = 0 \ . \tag{40.5}$$

It is also customary to relabel the scalar under spatial rotations,

$$h_{00} = 2\phi \ . \tag{40.6}$$

(b) Convince yourself that this decomposition preserves the number of degrees of freedom originally contained in $h_{\mu\nu}$.

(c) Under the infinitesimal diffeomorphism transformation (which is an active coordinate transformation, as opposed to a passive one), the metric perturbation transforms as

$$h_{\mu\nu}(x) \to \widetilde{h}_{\mu\nu}(x) = h_{\mu\nu}(x) + \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) \ , \tag{40.7}$$

where $h_{\mu\nu}$ are ξ_μ are of the same order.

Utilizing the fact that ξ_μ can be decomposed with respect to spatial rotations into a scalar ξ_0, and a vector $\xi_i = \xi_i^T + \partial_i \xi^S$, derive the following transformation rules for $h_{00}, B_i, S, \psi, E, F_i$ and h_{ij}^T under gauge (diffeomorphism) transformation (40.7)

$$\phi(x) \to \widetilde{\phi}(x) = \phi(x) + \partial_0 \xi_0(x) \ , \tag{40.8}$$

$$S(x) \to \widetilde{S}(x) = S(x) + \xi_0(x) + \partial_0 \xi^S(x) \ , \tag{40.9}$$

$$\psi(x) \to \widetilde{\psi}(x) = \psi(x) \ , \tag{40.10}$$

$$E(x) \to \widetilde{E}(x) = E(x) + \xi^S(x) \ , \tag{40.11}$$

$$B_i(x) \to \widetilde{B}_i(x) = B_i(x) + \partial_0 \xi_i^T(x) \ , \tag{40.12}$$

$$F_i(x) \to \widetilde{F}_i(x) = F_i(x) + \xi_i^T(x) \ , \tag{40.13}$$

$$h_{ij}^{TT}(x) \to \widetilde{h}_{ij}^{TT}(x) = h_{ij}^{TT}(x) \ . \tag{40.14}$$

(d) Show that, apart from h_{ij}^{TT} and ψ which are gauge invariant, there are two other combinations of quantities in (40.8)-(40.13) that are gauge invariant. Show that all these make six gauge invariant degrees of freedom in total (this makes sense, since four can be eliminated by fixing the gauge).

(e) You were asked in the last homework problem set to show that for metric (40.1) the Riemann tensor is

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\Big(\partial_\rho \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\mu h_{\nu\rho} - \partial_\rho \partial_\mu h_{\nu\sigma} - \partial_\sigma \partial_\nu h_{\mu\rho}\Big) \ . \tag{40.15}$$

Show that the Ricci tensor is

$$R_{\mu\nu} = \frac{1}{2}\Big(\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h\Big) \ , \tag{40.16}$$

where indices are defined to be raised by the Minkowski metric $\partial^\alpha = \eta^{\alpha\beta}\partial_\beta$, d'Alembertian is an ordinary flat space one $\Box = \partial^2 = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$, and the trace is $h = \eta^{\alpha\beta}h_{\alpha\beta}$.

(f) Using the fact that the Einstein equation in vacuum reduces to $R_{\mu\nu} = 0$, write out all the equations in terms of $\phi, S, \psi, E, B_i, F_i$ and h_{ij}^{TT} and show that only the gauge invariant combinations of these quantities appear in the equations of motion.

(g) Now we show that the only dynamical quantity in these equations is h_{ij}^{TT}.

By taking appropriate linear combinations of the components of the Einstein equation show that

$$\nabla^2 \psi = 0 \ . \tag{40.17}$$

If we assume that the perturbation vanishes at infinity, this equation has only one solution (remember electrostatics). From here the equations and solutions for the other non-dynamical gauge invariant perturbations follow. Show that the only equation that remains is

$$\Box h_{ij}^{TT} = 0 \ . \tag{40.18}$$

Therefore, h_{ij}^{TT} represents the two physical degrees of freedom of gravity in the absence of matter.

■ **PROBLEM 41** Response of matter to gravitational waves.

The equation of motion for the physical degrees of freedom of gravity propagating in Minkowski space-time is

$$\Box h_{ij}^{TT} = 0 \ , \tag{41.1}$$

where h_{ij}^{TT} is a transverse traceless tensor,

$$\partial^i h_{ij}^{TT} = 0 \ , \qquad (h^{TT})^i{}_i = 0 \ . \tag{41.2}$$

Equation of motion (41.2) obviously has propagating waves as solutions.

(a) Assume a plane wave propagating in an arbitrary direction,

$$h_{ij}^{TT} = A_\sigma \epsilon_{ij}^\sigma \cos\left(\omega t - \vec{k}\cdot\vec{x}\right) \ . \tag{41.3}$$

Determine the dispersion relation.

Now assume a wave propagating in the positive z-direction ($\vec{k} = k\hat{z}$) and show that there are two independent polarizations of a gravitational wave, a *plus* one and a *cross* one, corresponding to polarization tensors

$$\epsilon^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \qquad \epsilon^\times = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{41.4}$$

Next we want to determine what is the effect this wave has on a group of test particles initially distributed in a circle perpendicular to the propagation direction of the wave. In order to do that we will solve the geodesic deviation equation. Consider the separation vector between two nearby particle trajectories,

$$\frac{D^2}{d\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} S^\sigma \ . \tag{41.5}$$

(b) Assume the test particles to move slowly, $dx^\alpha/d\tau \approx (1,0,0,0)$. Show that by expanding the left- and the right-hand side of equation (41.5) to lowest non-vanishing order the geodesic deviation equation becomes

$$\frac{\partial^2}{\partial t^2} S_i = \frac{1}{2} S^j \frac{\partial^2}{\partial t^2} h_{ij}^{TT} \ . \tag{41.6}$$

(c) Consider a gravitational wave as in (41.3) with $A_\times = 0$. Show that (41.6) has the following perturbative solution (A_+ serves as a perturbation parameter in equation (41.6))

$$S^1 = \left(1 + \frac{A_+}{2} \cos(\omega t - kz)\right) S^1(0) \ , \tag{41.7}$$

$$S^2 = \left(1 - \frac{A_+}{2} \cos(\omega t - kz)\right) S^2(0) \ . \tag{41.8}$$

Sketch how the particles move assuming they were distributed in a circle at the initial moment.

(d) Repeat the analysis in (c) for $A_+ = 0$.