GENERAL RELATIVITY TUTORIAL

Problem set 2, 16.09.2016.

■ PROBLEM 5 Constant acceleration, part I.

Consider the equation of motion in SR for a point particle

$$\frac{d\vec{p}}{dt} = \vec{F} \; , \qquad \vec{p} = m\gamma\vec{v} \; , \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \; , \qquad \beta = \frac{v}{c} \; . \tag{5.1}$$

Consider the situation where $\vec{F}$ points along the x-axis and has the constant value $F = mg$. Assume that the velocity is zero at the time $t = 0$.

(a) Show that the motion of the particle is hyperbolic,

$$\left(x + \frac{c^2}{g}\right)^2 - (x^0)^2 = \left(\frac{c^2}{g}\right)^2 \; , \tag{5.2}$$

where $x^0 = ct$ and $y = z = 0$.

(b) Let **I** denote the intertial system where the particle is at rest at $t = 0$. Show that the proper time (times c) of a clock following the accelerated particle is given by

$$\tau = \frac{c^2}{g} \operatorname{arcsinh}\left(\frac{gx^0}{c^2}\right) \; , \tag{5.3}$$

i.e.

$$x^0 = \frac{c^2}{g} \sinh\left(\frac{g\tau}{c^2}\right) \; . \tag{5.4}$$

(c) Show that the transformation from an inertial system **I'**, where the accelerated particle is at rest at proper time τ at $x' = 0 = x^{0\prime}$ (and $y' = z' = 0$), to **I** is given by

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c^2}\right) - 1\right] + x' \cosh\left(\frac{g\tau}{c^2}\right) + x^{0\prime} \sinh\left(\frac{g\tau}{c^2}\right) \; , \tag{5.5}$$

$$x^0 = \frac{c^2}{g} \sinh\left(\frac{g\tau}{c^2}\right) + x' \sinh\left(\frac{g\tau}{c^2}\right) + x^{0\prime} \cosh\left(\frac{g\tau}{c^2}\right) \; . \tag{5.6}$$

■ PROBLEM 6 Rotating coordinate system.

In order to find the line element of a uniformly rotating reference frame, we can start from flat space in cylindrical coordinates,

$$ds'^2 = g'_{\mu\nu} dx'^\mu dx'^\nu = -c^2 dt'^2 + d\rho'^2 + dz'^2 + \rho'^2 d\varphi'^2 \; , \tag{6.1}$$

and perform the following spatial transformation for axis of rotation z,

$$\rho = \rho' \ , \quad z = z' \ , \quad \varphi = \varphi' + \omega t \ , \tag{6.2}$$

where ω is the constant angular velocity of rotation.

(a) What is the line element ds^2 of the rotating coordinate system?

(b) What would be the circumference of a circle in terms of the coordiante ρ as measured by an observer in a rotating frame? Discuss the physical meaning of the case when $\rho \geq \frac{c}{\omega}$.

Hint: An observer on a given space-time measures the time interval between two events as $d\tau^2 = g_{00} dt^2$, and the space interval as $dl^2 = \left(g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}} \right) dx^i dx^j$.