GENERAL RELATIVITY TUTORIAL

Problem set 3, 23.09.2016.

■ PROBLEM 7 Surfaces and manifolds.

Consider the following surfaces imbeded in 3-dimensional Euclidean space: (i) a sphere, (ii) a torus, (iii) surface of a cone, shown in Figure 2.

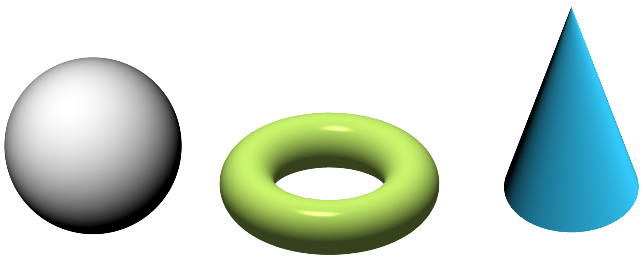

Figure 1: From left to right: a sphere, a torus, a cone.

(a) Construct a set of charts (atlas) for each of these surfaces, i.e. construct coordinate systems on these surfaces by expressing them in terms of 3-dimensional Cartesian coordinates. What can you say about the tip of the cone (in the context of differentiable manifolds)?

(b) Find what an infinitesimal line element is on these surfaces (in one particular chart). *Hint:* Remember that for 3-dimensional Euclidean space the line element in Cartesian coordinates is just $dl^2 = dx^2 + dy^2 + dz^2$.

■ PROBLEM 8 Coordinate transformations.

(a) How would the product of a covariant and a contravariant vector, $A^\mu B_\mu$, transform under general coordinate transformations?

(b) What about the quantity $g_{\mu\nu} g^{\nu\rho} = \delta_\mu{}^\rho$?

(c) Can you show that the derivative of a covariant vector, $\partial_\mu A_\nu$, does not transform as a tensor? Show that for linear coordinate transformations (*i.e.* Lorentz transformations) $\partial_\mu A_\nu$ is indeed a tensor.

(d) Check that the transformation rules (4.10) and (4.13) in the lecture notes (http://www.staff.science.uu.nl/~hooft101/lectures/genrel_2010.pdf) form a group, i.e. the transformation $x \to u$ yields the same tensor as the sequence $x \to v \to u$. Make use of the fact that partial differentiation obeys

$$\frac{\partial x^\mu}{\partial u^\nu} = \frac{\partial x^\mu}{\partial v^\alpha} \frac{\partial v^\alpha}{\partial u^\nu} \, . \tag{8.1}$$

■ **PROBLEM 9** (Anti)-symmetrizing tensors.

For any 2-tensor $T_{\mu\nu}$ (in four dimensions) we define the tensors

$$T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \ , \qquad T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \ . \tag{9.1}$$

These are, respectively, called symmetric and antisymmetric part of the tensor $T_{\mu\nu}$.

(a) Is it true that for any tensor $A_{\mu\nu} = A_{(\mu\nu)} + A_{[\mu\nu]}$? How many components do $A_{(\mu\nu)}$ and $A_{[\mu\nu]}$ have?

(b) If $B_{\mu\nu} = B_{\nu\mu}$ and $C_{\mu\nu} = -C_{\nu\mu}$, prove that (A is a generic tensor)

$$A_{\mu\nu}B^{\mu\nu} = A_{(\mu\nu)}B^{\mu\nu} \ , \qquad A_{\mu\nu}C^{\mu\nu} = A_{[\mu\nu]}C^{\mu\nu} \ . \tag{9.2}$$

For higher rank tensors, this concept can be generalized to obtain totally symmetric and anti-symmetric tensors by definitions

$$T_{(\mu_1\mu_2\ldots\mu_k)} = \frac{1}{k!}\sum_{\sigma\in S^k} T_{\sigma(\mu_1)\sigma(\mu_2)\ldots\sigma(\mu_k)} \ , \tag{9.3}$$

$$T_{[\mu_1\mu_2\ldots\mu_k]} = \frac{1}{k!}\sum_{\sigma\in S^k} (-1)^{\pi(\sigma)} T_{\sigma(\mu_1)\sigma(\mu_2)\ldots\sigma(\mu_k)} \ , \tag{9.4}$$

respectively, where T is an arbitrary covariant tensor of rank k. In (9.3) and (9.4), S^k denotes the group of k objects, and π the so-called parity of the permutation, with $\pi(\sigma) = 0$ if σ is an even permutation, and $\pi(\sigma) = 1$ if it is an odd permutation.

(c) Consider the case of an arbitrary 3-tensor $T_{\mu\nu\rho}$. How many components does it have? Write explicitly the form of $T_{(\mu\nu\rho)}$ and $T_{[\mu\nu\rho]}$.

(d) For an arbitrary 3-tensor $T_{\mu\nu\rho}$ is it true that $T_{\mu\nu\rho} = T_{(\mu\nu\rho)} + T_{[\mu\nu\rho]}$? Prove your result.