GENERAL RELATIVITY

Tutorial problem set 4, 30.09.2016.

■ PROBLEM 10 Volume element.

The volume element can be written in terms of an n-form as

$$d^n x \equiv \frac{1}{n!}\widetilde{\epsilon}_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n} \ , \tag{10.1}$$

where $\widetilde{\epsilon}_{\mu_1 \ldots \mu_n}$ is the totally antisymmetric Levi-Civita symbol.

(a) Show that $\sqrt{-g}d^n x$ is the invariant volume element under coordinate transformations, where $g \equiv \det(g_{\mu\nu})$. Conclude from there that $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\,\widetilde{\epsilon}_{\mu\nu\rho\sigma}$ is a tensor.

(b) Can you discuss what happens to the volume element in $n = 3$ space when you introduce a left-handed instead of right-handed coordinate system? Hint: Consider first a left-handed and right-handed coordinate system on an $n = 2$ surface embeded in $n = 3$ space, and a surface element on it, and think of orientability.

■ PROBLEM 11 The Eötvös experiment.

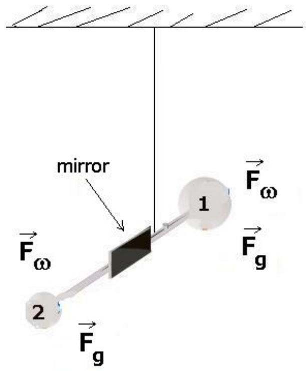

Figure 1: Torsion balance modelling the one used by Loránd Eötvös. Two weights on the ends of the rod are made of different materials. The mirror reflects the light beam such that one can detect the rotation due to the torque.

An experimental set-up in Figure 1 shows an idealized picture of how Eötvös tested the Equivalence principle (i.e. $M_{grav.} = M_{inert.}$). The rod in the laboratory frame (the one tied to the surface of the Earth) is aligned to be perpendicular to the meridian. Since this is not an inertial frame, the two weights experience a centrifugal force, apart from gravitational one.

The gravitational force on the surface of the Earth is

$$\vec{F}_g = -G_N M_\oplus M_{grav.} \frac{\vec{r}}{r^3} \ , \tag{11.1}$$

while the centrifugal force is

$$\vec{F}_\omega = M_{inert.}\omega^2 \left(\vec{r} - \frac{(\vec{\omega} \cdot \vec{r})}{\omega^2}\vec{\omega} \right) \ , \tag{11.2}$$

where ω is Earth's angular velocity, and r is the Earth's radius.

(a) Convince yourself of the validity of these two formulas. Draw the two forces acting on objects in the torsion balance of Figure 1.

(b) the main idea of the experiment was to check whether $\frac{M^{(1)}_{inert.}}{M^{(1)}_{grav.}} = \frac{M^{(2)}_{inert.}}{M^{(2)}_{grav.}}$ for different materials. If this were true, then the Equivalence principle holds. (The actual value of the ratio is somewhat ambiguous and depends on the definition. The value is fixed to be 1 by choosing the appropriate Newton's constant.) Show that the measurable misalignment between the forces $\vec{F}^{(i)} = \vec{F}^{(i)}_g + \vec{F}^{(i)}_\omega$ acting on two objects is

$$\alpha = \frac{|\vec{F}^{(1)} \times \vec{F}^{(2)}|}{|\vec{F}^{(1)}||\vec{F}^{(2)}|} = \left| \frac{M^{(1)}_{inert.}}{M^{(1)}_{grav.}} - \frac{M^{(2)}_{inert.}}{M^{(2)}_{grav.}} \right| \frac{(\vec{\omega} \cdot \vec{r})|\vec{\omega} \times \vec{r}|r}{G_N M_\oplus} \,. \tag{11.3}$$

Assume that the gravitational force is much stronger than the centrifugal one.

(c) Using the previous result of the previous point, what would happen if somebody carried out the Eötvös experiment at the equator? What is the most favourable latitude to carry out the experiment (namely, at which latitude the possible misalignment would be largest)?

■ PROBLEM 12 Relativistic particle in an electromagnetic field.

The properties of the electromagnetic field are governed by the vector potential A^μ. It is given by

$$A^\mu(x^\nu) = \left(\varphi(x^\nu), \vec{A}(x^\nu)\right).^1 \tag{12.1}$$

The action of a charged particle in the presence of an electromagnetic field is given by

$$S = -mc \int_a^b \mathrm{d}\tau - \frac{e}{c} \int A_\mu(x^\nu) \mathrm{d}x^\mu , \tag{12.2}$$

where $d\tau = \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}/c$ is the proper time, m denotes particle's mass and the charge e of the particle governs its interaction strength with the electromagentic field given by the vector potential A^μ. In this problem, we would like to calculate the equation of motion for this particle, in the presence of the electromagnetic vector potential A^μ.

(a) Write down the variation of the action $\delta_x S = 0$ with respect to x^ν for a neutral particle ($e = 0$). You should first derive

$$\delta \mathrm{d}\tau = -\frac{\mathrm{d}x_\mu \mathrm{d}(\delta x^\mu)}{c^2 d\tau} \,. \tag{12.3}$$

(b) Let us now define the following quantity

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\sqrt{1 - \frac{\vec{v}^2}{c^2}} dt} = \left(\frac{c}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}, \frac{\vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \right). \tag{12.4}$$

Evaluate $u_\mu u^\mu$!

(c) Now, Taylor expand $A_\mu(x^\nu + \delta x^\nu)$ around x_ν to find δA_μ. Furthermore, piecing things together, derive the differential equation that governs the motion of our charged particle whose action is given by (12.2). Give the physical interpretation of the (time-like and space-like) terms obtained by the variation of the second term in Eq. (12.2).

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[1]Note that $A^\mu \to A_\mu = \eta_{\mu\nu} A^\nu = (-\varphi, \vec{A})$.