GENERAL RELATIVITY

Tutorial problem set 5, 07.10.2016.

■ PROBLEM 13 Causality in GR.

Consider the following three 3-dimensional space-time metrics,

$$\text{(i)} \qquad ds^2 = -c^2dt^2 + dr^2 + r^2 d\varphi^2 \ , \tag{13.1}$$

$$\text{(ii)} \qquad ds^2 = -c^2dt^2 + \frac{d\rho^2}{1-\kappa\rho^2} + \rho^2 d\varphi^2 \ , \qquad (\kappa > 0 \text{ or } \kappa < 0) \tag{13.2}$$

$$\text{(iii)} \qquad ds^2 = -c^2dt^2 + R^2 d\vartheta^2 + R^2 \sin^2(\vartheta) d\varphi^2 \ . \qquad (R > 0) \tag{13.3}$$

The ranges of coordinates are $\varphi \in [0, 2\pi\rangle$, $r \in [0, \infty\rangle$, $\vartheta \in [0, \pi\rangle$, $\rho \in [-\frac{1}{\sqrt{\kappa}}, \frac{1}{\sqrt{\kappa}}]$ for $\kappa > 0$ where $\rho = \frac{1}{\sqrt{\kappa}}$ and $\rho = -\frac{1}{\sqrt{\kappa}}$ are identified, $\rho \in [0, \infty]$ for $\kappa < 0$.

(a) Can you recognize what are cases (ii) for $\kappa > 0$ and cases (iii)? Can you parametrize a sphere in cylindrical coordinates?

(b) Derive what are the curves traced by light in these three spaces in given coordinates and sketch the light cones. Note that there is a rotation symmetry in all three metrics with respect to angle φ.

(c) Sketch the closed achronal surface S and the future and past domains of dependence $D^+(S)$ and $D^-(S)$, if for each of the cases S is defined to be

$$\text{(i)} \qquad r \in [0, R], \varphi \in [0, 2\pi\rangle \ ; \tag{13.4}$$

$$\text{(ii)} \qquad \kappa > 0 : \quad \rho \in [0, \rho_0], \varphi \in [0, 2\pi\rangle \qquad (0 < \rho_0 < \tfrac{1}{\sqrt{\kappa}}) \tag{13.5}$$

$$\kappa < 0 : \quad \rho \in [0, \rho_0], \varphi \in [0, 2\pi\rangle \ ; \tag{13.6}$$

$$\text{(iii)} \qquad \vartheta \in [0, \vartheta_0], \varphi \in [0, 2\pi\rangle \qquad (0 < \vartheta_0 < \pi) \ . \tag{13.7}$$

(d) Sketch the Cauchy surface for all the cases.

■ PROBLEM 14 Christoffel symbols and covariant derivatives I.

In the lecture you it was shown that Christoffel symbols compatible with the metric have the following form

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \Big(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \Big) \ . \tag{14.1}$$

(a) Show that Christoffel symbols are not tensors, i.e. show that under coordinate transformation $x \to \widetilde{x}$ they transform in the following manner

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} = \frac{\partial \widetilde{x}^\alpha}{\partial x^\beta} \frac{\partial x^\rho}{\partial \widetilde{x}^\mu} \frac{\partial x^\sigma}{\partial \widetilde{x}^\nu} \Gamma^{\beta}_{\rho\sigma} + \frac{\partial \widetilde{x}^\alpha}{\partial x^\beta} \frac{\partial^2 x^\beta}{\partial \widetilde{x}^\mu \partial \widetilde{x}^\nu} \ . \tag{14.2}$$

(Note that this is the reason why we are not careful about index placement with Christoffel symbols). Are they tensors under linear coordinate transformations?

(b) Using the result of problem (a) and the definition of the covariant derivative, show explicitly that the following two quantities transform as tensors

$$\text{(i)} \quad \nabla_\mu \omega_\nu \ , \qquad\qquad \text{(ii)} \quad \nabla_\mu V^\nu \ , \tag{14.3}$$

and that $\nabla_\mu V^\mu$ is a scalar quantity.

■ **PROBLEM 15** Integration measure in terms of the ϵ-tensor

Recall from last week that the ϵ-tensor is given by

$$\epsilon \equiv \sqrt{|g|} \ \frac{1}{n!} \tilde{\epsilon}_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n} \equiv \sqrt{|g|} d^n x \tag{15.1}$$

The volume of an n-manifold M is defined as the integral

$$V[M] \ \equiv \ \int_M \epsilon. \tag{15.2}$$

For a scalar field $\phi : M \to \mathbb{R}$, the integral over M is then defined as

$$I_M[\phi] \ \equiv \ \int_M \epsilon \ \phi \, . \tag{15.3}$$

(a) Consider the unit 2-sphere S^2. Calculate ϵ and the 'volume' $V[S^2]$.

Furthermore, given $\phi, \psi : S^2 \to \mathbb{R}$, $\phi(x, y, z) = x$ and $\psi(x, y, z) = x^2$, calculate the integrals $I_{S^2}[\phi]$ and $I_{S^2}[\psi]$.

(b) Now consider the torus T^2, with radii r and R ($r < R$). Calculate $V[T^2]$ and the integrals $I_{T^2}[\phi]$ and $I_{T^2}[\psi]$.

(c) Find the area of a cone, with deficit angle $2\pi\Delta$ and length L?

Hint: Show first that the metric can be written as, $ds^2 = d\rho^2 + (1 - \Delta)\rho^2 d\phi^2$, $(0 \leq \rho \leq L, 0 \leq \phi < 2\pi)$, where $0 \leq \Delta < 1$ is a constant. What is the physical meaning of Δ?