GENERAL RELATIVITY

Tutorial problem set 6, 14.10.2016.

■ PROBLEM 16 .

In Euclidean 3-dimensional space, we can define the so-called *paraboloidal coordiantes* (u, v, ϕ) by

$$x = uv\cos\phi \ , \qquad y = uv\sin\phi \ , \qquad z = \frac{1}{2}(u^2 - v^2) \ , \tag{16.1}$$

where the ranges of coordinates are $u \in [0, \infty\rangle$, $v \in [0, \infty\rangle$, $\phi \in [0, 2\pi\rangle$.

(a) Find the coordinate transformation matrix between paraboloidal and Cartesian coordinates $\partial x^\alpha / \partial \tilde{x}^\beta$ and the inverse transformation. Are there any singular points in the map?

(b) Find the basis vectors in paraboloidal coordinates in terms of Cartesian basis vectors.

(c) Find the metric and inverse metric in paraboloidal coordinates.

(d) Calculate the Christoffel symbols in paraboloidal coordinates.

(e) Express the divergence $\nabla_\mu V^\mu$ and Laplacian $\nabla_\mu \nabla^\mu f$ in paraboloidal coordinates.

■ PROBLEM 17 Geodesics in Rindler space-time.

Let (x^0, x) denote Minkowski coordinates in an inertial system **I**, and let (w^0, w) denote the Rindler coordinates of a system of reference **R** with constant acceleration g relative to I. The two systems are related by

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{gw^0}{c^2}\right) - 1\right] + w\cosh\left(\frac{gw^0}{c^2}\right) \ , \tag{17.1}$$

$$x^0 = \frac{c^2}{g}\sinh\left(\frac{gw^0}{c^2}\right) + w\sinh\left(\frac{gw^0}{c^2}\right) \ . \tag{17.2}$$

You have derived in Homework 1, Problem 1, that the invariant element in Rindler coordinates is

$$ds^2 = dw^2 - \left(1 + \frac{gw}{c^2}\right)^2 (dw^0)^2 \tag{17.3}$$

(a) Write down the equation of motion for a particle in free fall as a second order differential equation in w^0.

(b) Show that the solution for a particle starting at $w = \overline{w}$ at time $w^0 = 0$ with velocity zero is

$$w\left(w^0\right) = \frac{c^2}{g}\left[\left(1 + \frac{g\overline{w}}{c^2}\right)\frac{1}{\cosh\left(\frac{gw^0}{c^2}\right)} - 1\right] \ . \tag{17.4}$$

(c) Calculate the velocity of particle as a function of w^0 ($v = cdw/dw^0$). Find the maximum velocity of the particle and compare with the velocity of light. What happens for $w^0 \to \infty$.

(d) Calculate the proper time of the particle as a function of w^0 and $\overline{w}$.

■ PROBLEM 18 Parallel transport on a sphere.

A vector V^μ is said to be parallel transported in the direction ν if its covariant derivative along that direction is zero,

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\sigma} V^\sigma = 0 \ . \tag{18.1}$$

Consider now a 2-dimensional sphere of unit radius,

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \ . \tag{18.2}$$

In this space vector V has just two components – V^θ and V^ϕ.

(a) Compute the Christoffel symbols and show that the parallel transport equations along the ϕ-coordinate (θ =const.) are

$$\partial_\phi V^\theta - \sin\theta\cos\theta \, V^\phi = 0 \ , \tag{18.3}$$

$$\partial_\phi V^\phi + \cot\theta \, V^\theta = 0 \ . \tag{18.4}$$

The two equations above can be seen as a system of two coupled ordinary differential equation of first order (with constant coefficients, since θ is fixed). We solve them by first constructing a second-order ordinary differential equation for one of the functions (in this case it can be done for both). Derive the following equation,

$$\partial_\phi^2 V^\theta = -\cos^2\theta \, V^\theta \ . \tag{18.5}$$

After solving this equation for V^θ, the solution for V^ϕ can be found from either (18.3) or (18.4).

(b) Show that the full solution is

$$V^\theta = A\cos(\phi\cos\theta) + B\sin(\phi\cos\theta) \ , \tag{18.6}$$

$$V^\phi = -\frac{A\sin(\phi\cos\theta)}{\sin\theta} + \frac{B\cos(\phi\cos\theta)}{\sin\theta} \ , \tag{18.7}$$

where A and B are real constants (they determine the initial value of the vectors).

Next we want to understand how a vector is parallel transported along a triangle on a sphere. We are interested in the case where one of the sides of the triangle lies in the equator, subtending an angle α with the center of the sphere, and the other two sides connect the north pole to the equator (see Figure 1). The initial vector V at point X_1 is pointing in the ϕ-direction, namely $V^\phi = 1$ and $V^\theta = 0$. In the following, consider these three points on a sphere,

$$X_1 = (1, 0, 0) \ , \qquad X_2 = (\cos\alpha, \sin\alpha, 0) \ , \qquad X_3 = (0, 0, 1) \ . \tag{18.8}$$

(c) Verify that the vector V remains unchanged under parallel transport from X_1 to X_2, i.e. that it remains in the ϕ direction. Make use of the solutions (18.6) and (18.7) you found.

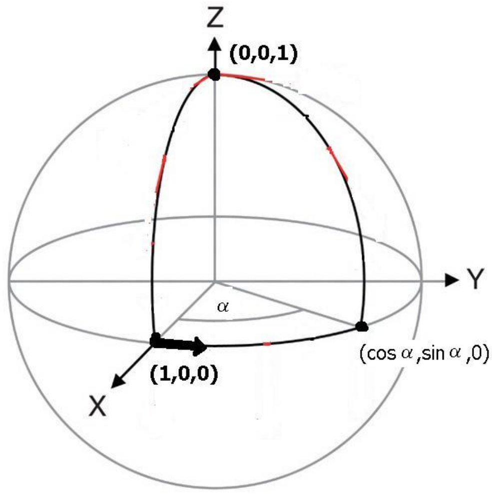

Figure 1: Triangle on a sphere

(d) Find how the vector is transported from X_2 to X_3. In this case you can exploit the symmetries of the system: you can rotate your frame in such a way that the X_2X_3 section lies on the equator. In this frame the vector V has just the component V^θ.

(e) Finally, using a similar trick, find how the vector changes if it is transported from X_3 back to X_1. What happens if $\alpha = \frac{\pi}{2}$? Verify whether this result is compatible with what you expect from intuition (see Figure 1.)

For (d) and (e) it is enough to sketch and explain in words how the vector is parallel transported.