GENERAL RELATIVITY

Tutorial problem set 7, 21.10.2016.

■ **PROBLEM 19** Riemann tensor.

Riemann tensor (with all indices down) is defined to be

$$R_{\lambda\sigma\mu\nu} = g_{\lambda\rho} R^{\rho}{}_{\sigma\mu\nu} = g_{\lambda\rho}\Big(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} - \Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\sigma}\Big) \ . \tag{19.1}$$

(a) Prove that Riemann tensor is antisymmetric in its first two indices,

$$R_{\lambda\sigma\mu\nu} = -R_{\sigma\lambda\mu\nu} \ , \tag{19.2}$$

and that it is antisymmetric in its last two indices,

$$R_{\lambda\sigma\mu\nu} = -R_{\lambda\sigma\nu\mu} \ . \tag{19.3}$$

Furthermore, prove that is is symmetric under the exchange of left-hand pair and right-hand pair of indices,

$$R_{\lambda\sigma\mu\nu} = R_{\mu\nu\lambda\sigma} \ . \tag{19.4}$$

Finally, prove the identity

$$R_{\lambda\sigma\mu\nu} + R_{\lambda\mu\nu\sigma} + R_{\lambda\nu\sigma\mu} = 0 \ . \tag{19.5}$$

(The last identity is equivalent to writing $R_{\lambda[\sigma\mu\nu]}$, can you see why?)

(b) Using the symmetries of Riemann tensor derived in (a) show that Riemann tensor in D dimensions has $D^2(D^2-1)/12$ independent components.

(c) In $D = 2$ dimensions Riemann tensor has only one independent component. Convince yourself that we can write it in the following form

$$R_{\lambda\sigma\mu\nu} = f(x)\Big(g_{\lambda\mu}g_{\sigma\nu} - g_{\lambda\nu}g_{\sigma\mu}\Big) \ , \tag{19.6}$$

where $f(x)$ is a scalar function of coordinates. Determine what f is by requireing that the appropriate contractions of the Riemann tensor yield the Ricci scalar.

(d) Show the Bianchi identities,

$$\nabla_\alpha R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\alpha} + \nabla_\nu R_{\rho\sigma\alpha\mu} = 0 \ , \tag{19.7}$$

$$\nabla^\alpha R_{\alpha\beta} - \frac{1}{2}\nabla_\beta R = 0 \ . \tag{19.8}$$

■ PROBLEM 20 Geodesic deviation.

Consider a pair of nearby freely falling particles that travel on trajectories $x^\mu(\tau)$ and $x^\mu(\tau) + \delta x^\mu(\tau)$. Their equations of motion are

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(x)\,\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\ , \tag{20.1}$$

$$\frac{d^2(x^\mu + \delta x^\mu)}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(x+\delta x)\,\frac{d(x^\alpha + \delta x^\alpha)}{d\tau}\frac{d(x^\beta + \delta x^\beta)}{d\tau} = 0\ . \tag{20.2}$$

(a) Show that the difference between these two equations to first order in δx^μ is

$$\frac{d^2(\delta x^\mu)}{d\tau^2} + \frac{\partial \Gamma^\mu_{\alpha\beta}}{\partial x^\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\delta x^\nu + 2\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d(\delta x^\beta)}{d\tau} = 0\ . \tag{20.3}$$

(b) Show that this can be rewritten in terms of covariant derivatives along the curve $x^\mu(\tau)$,

$$\frac{D^2(\delta x^\mu)}{d\tau^2} = -R^\mu{}_{\alpha\nu\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\delta x^\nu\ . \tag{20.4}$$

Remember the definition of the directional derivative along a curve $x^\alpha(\tau)$,

$$\frac{D}{d\tau} \equiv \frac{dx^\alpha}{d\tau}\nabla_\alpha\ . \tag{20.5}$$

■ PROBLEM 21 Weyl tensor.

The Weyl tensor in D dimensions is defined to be

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{(D-2)}\Big(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\Big) + \frac{2}{(D-1)(D-2)}\,g_{\rho[\mu}g_{\nu]\sigma}R\ . \tag{21.1}$$

(a) Determine the number of independent components of the Weyl tensor in D dimensions.

Hint: Show that it has the same symmetries as the Riemann tensor, and in addition $C^\alpha{}_{\mu\alpha\nu} = 0$.

(b) Show that for $D \geq 4$ the Weyl tensor satisfies the following version of Bianchi identity,

$$\nabla_\rho C^\rho{}_{\sigma\mu\nu} = 2\,\frac{D-3}{D-2}\left(\nabla_{[\mu}R_{\nu]\sigma} + \frac{1}{2(D-1)}g_{\sigma[\mu}\nabla_{\nu]}R\right)\ . \tag{21.2}$$

Hint: Use the Bianchi identities from problem 1.

(c) Can you argue why the expression above is not defined for $D < 3$?