GENERAL RELATIVITY

Tutorial problem set 8, 28.10.2016.

■ PROBLEM 22 Killing vectors.

(a) Show that the commutator of two Killing vectors is a Killing vector. Show that a linear combination with constant coefficients of two Killing vectors is a Killing vector.

(b) Show that Killing vectors satisfy the following identities

$$\text{(i)} \qquad \nabla_\mu \nabla_\nu K^\rho = R^\rho{}_{\nu\mu\sigma} K^\sigma \ , \tag{22.1}$$

$$\text{(ii)} \qquad \nabla_\mu \nabla_\nu K^\mu = R_{\nu\mu} K^\mu \ , \tag{22.2}$$

$$\text{(iii)} \qquad K^\alpha \nabla_\alpha R = 0 \ . \tag{22.3}$$

Hints: For (i) first show that for any vector V^ρ

$$[\nabla_\mu, \nabla_\sigma] V_\rho = R_{\rho\nu\mu\sigma} V^\nu \ , \tag{22.4}$$

and make use of Killing's equation and a properties of the Riemann tensor. For (iii) make use of one of Bianchi identities from last time and the fact that the 2-tensor can always be decomposed into a symmetric and an antisymmetric part.

■ PROBLEM 23 Killing vectors on a 2-sphere.

The two-sphere is given by

$$ds^2 = d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \ . \tag{23.1}$$

(a) Solve the Killing's equation to find that the Killing vectors of a 2-sphere are

$$K_1 = \cos\varphi \, \partial_\vartheta - \cot\vartheta \sin\varphi \, \partial_\varphi \ , \tag{23.2}$$

$$K_2 = -\sin\varphi \, \partial_\vartheta - \cot\vartheta \cos\varphi \, \partial_\varphi \ , \tag{23.3}$$

$$K_3 = \partial_\varphi \ . \tag{23.4}$$

(b) Show that the commutators of these three Killing vectors satisfy the following commutation relations

$$[K_1, K_2] = K_3 \ , \tag{23.5}$$

$$[K_2, K_3] = K_1 \ , \tag{23.6}$$

$$[K_3, K_1] = K_2 \ . \tag{23.7}$$

Check that they satisfy the algebra of Lie groups. Can you see which algebra is it?

■ **PROBLEM 24** Maximally symmetric spaces.

The metric of a *de Sitter space* is given by

$$ds^2 = \frac{1}{\cos^2(H\eta)}\Big[- d\eta^2 + \frac{d\chi^2}{H^2} + \frac{\sin^2(\chi)}{H^2}(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\Big] \tag{24.1}$$

(a) Calculate the Riemann tensor, the Ricci tensor and the Ricci scalar.

Note: For a diagonal metric Christoffel symbols can be calculated using the following simplified formulas

$$\begin{aligned} &\Gamma^{\gamma}_{\alpha\beta} = 0 \ , \qquad \Gamma^{\beta}_{\alpha\alpha} = -\frac{1}{2g_{\beta\beta}}\partial_\beta g_{\alpha\alpha} \ , \\ &\Gamma^{\beta}_{\alpha\beta} = \partial_\alpha\left(\ln\sqrt{|g_{\beta\beta}|}\right) \ , \qquad \Gamma^{\alpha}_{\alpha\alpha} = \partial_\alpha\left(\ln\sqrt{|g_{\alpha\alpha}|}\right) \ , \end{aligned} \tag{24.2}$$

where $\alpha \neq \beta \neq \gamma \neq \alpha$ and there is *no* summation over indices.

(b) Show that

$$R_{\rho\sigma\mu\nu} = \frac{R}{D(D-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \ , \qquad (D=4) \ . \tag{24.3}$$

A space for which this last equation is satisfied is called a maximally symmetric space. In such a space, the curvature is the same everywhere and the same in every direction. Hence, if one knows the curvature in one point of the space, it is known everywhere.