GENERAL RELATIVITY

Tutorial problem set 9, 04.11.2016.

■ PROBLEM 25 Riemann normal coordinates.

By definition of a Riemann manifold, at a given point p there exists a coordinate system with respect to which the first partial derivatives of the metric components vanish (local flatness). In these coordinates (*Riemann normal coordinates* with respect to the point p), which by definition are the coordinates of a local inertial frame, the equation of a geodesic through the point p is $x^\mu = a^\mu s$, where $a^\mu = \frac{dx^\mu}{ds}\big|_p$ and s is the curve parameter.

(a) Derive (from the geodesic equation) that

$$\Gamma^\alpha_{\mu\nu}\big|_p = 0 \ , \tag{25.1}$$

and that

$$\partial_{(\beta}\Gamma^\alpha_{\mu\nu)}\big|_p = 0 \ . \tag{25.2}$$

(b) Use identity (25.2) to show that the metric around point p can be written as

$$g_{\mu\nu} = g_{\mu\nu}\big|_p + \frac{1}{3}\, R_{\alpha\mu\nu\beta}\big|_p\, x^\alpha x^\beta + \mathcal{O}(x^3) \ . \tag{25.3}$$

where $x^\alpha|_p = 0$.

Hints: First prove that (25.2) implies

$$\partial_\nu \Gamma^\mu_{\alpha\beta} = -\frac{1}{3}\Big(R^\mu{}_{\beta\alpha\nu} + R^\mu{}_{\alpha\beta\nu}\Big) \ , \tag{25.4}$$

and that this implies

$$\partial_\alpha\partial_\beta g_{\mu\nu} = -\frac{1}{3}\Big(R_{\mu\alpha\nu\beta} + R_{\mu\beta\nu\alpha}\Big) \ . \tag{25.5}$$

■ PROBLEM 26 Energy-momentum tensor for a scalar field.

The action for the scalar field non-minimally coupled to curvature is

$$S_\phi = \int d^4x\,\sqrt{-g}\,\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) - \frac{1}{2}\xi R\phi^2\right) \ , \tag{26.1}$$

where ξ is the so-called non-minimal coupling. This is one of the models used in inflationary theories, which goes under the name *Higgs inflation.*

(a) Derive the equation of motion for the scalar field.

(b) Calculate the energy-momentum tensor for the minimally coupled scalar field (ξ=0).

(c) Calculate the additional contribution to the energy-momentum tensor coming from the non-minimal coupling of the scalar.

Recall that the definition of the energy-momentum tensor is

$$T_{\mu\nu}(x) = \frac{-2}{\sqrt{-g}(x)} \frac{\delta S_\phi}{\delta g^{\mu\nu}(x)} \ . \tag{26.2}$$