GENERAL RELATIVITY

Tutorial problem set 6, 01.11.2013.

SOLUTIONS

■ **PROBLEM 1** Killing vectors.

(a) Show that the commutator of two Killing vectors is a Killing vector. Show that a linear combination with constant coefficients of two Killing vectors is a Killing vector.

(b) Show that Killing vectors satisfy the following identities

$$\text{(i)} \qquad \nabla_\mu \nabla_\nu K^\rho = R^\rho{}_{\nu\mu\sigma} K^\sigma \ , \tag{1.1}$$

$$\text{(ii)} \qquad \nabla_\mu \nabla_\nu K^\mu = R_{\nu\mu} K^\mu \ , \tag{1.2}$$

$$\text{(iii)} \qquad K^\alpha \nabla_\alpha R = 0 \ . \tag{1.3}$$

Hints: For (i) first show that for any vector V^ρ

$$[\nabla_\mu, \nabla_\sigma] V_\rho = R_{\rho\nu\mu\sigma} V^\nu \ , \tag{1.4}$$

and make use of Killing's equation and properties of the Riemann tensor. For (iii) make use of one of the Bianchi identities from last time and the fact that a 2-tensor can always be decomposed into a symmetric and an antisymmetric part.

(a) Let $K = K^\alpha \partial_\alpha$ and $L = L^\alpha \partial_\alpha$ be two Killing vectors,

$$\nabla_{(\alpha} K_{\beta)} = 0 \ , \qquad \nabla_{(\alpha} L_{\beta)} = 0 \ . \tag{1.5}$$

Their linear combination with constant coefficients,

$$P = P^\alpha \partial_\alpha = aK + bL = (aK^\alpha + bL^\alpha)\partial_\alpha \ , \tag{1.6}$$

is trivially a Killing vector,

$$\nabla_{(\alpha} P_{\beta)} = a\nabla_{(\alpha} K_{\beta)} + b\nabla_{(\alpha} L_{\beta)} = 0 \ . \tag{1.7}$$

The commutator of two Killing vectors is

$$M = [K, L] = [K^\alpha \partial_\alpha, L^\beta \partial_\beta] = K^\alpha (\partial_\alpha L^\beta)\partial_\beta - L^\beta (\partial_\beta K^\alpha)\partial_\alpha \ . \tag{1.8}$$

With some relabeling of indices, and using $\partial_\alpha L^\beta = \nabla_\alpha L^\beta - \Gamma^\beta_{\alpha\sigma} L^\sigma$, we can write it as

$$M = M^\alpha \partial_\alpha = \Big[K^\beta(\nabla_\beta L^\alpha) - L^\beta(\nabla_\beta K^\alpha)\Big]\partial_\alpha \; . \tag{1.9}$$

Now we need to check that the Killing's equation is satisfied for M_α.

$$\begin{aligned}\nabla_\alpha M_\beta + \nabla_\beta M_\alpha &= \nabla_\alpha\Big[K^\gamma(\nabla_\gamma L_\beta) - L^\gamma(\nabla_\gamma K_\beta)\Big] + \nabla_\beta\Big[K^\gamma(\nabla_\gamma L_\alpha) - L^\gamma(\nabla_\gamma K_\alpha)\Big] \\ &= \underline{(\nabla_\alpha K^\gamma)\nabla_\gamma L_\beta} + K^\gamma \nabla_\alpha \nabla_\gamma L_\beta - \underline{(\nabla_\alpha L^\gamma)\nabla_\gamma K_\beta} - L^\gamma \nabla_\alpha \nabla_\gamma K_\beta \\ &\quad + \underline{(\nabla_\beta K^\gamma)\nabla_\gamma L_\alpha} + K^\gamma \nabla_\beta \nabla_\gamma L_\alpha - \underline{(\nabla_\beta L^\gamma)\nabla_\gamma K_\alpha} - L^\gamma \nabla_\beta \nabla_\gamma K_\alpha \; .\end{aligned} \tag{1.10}$$

Because of the Killing's equation the underlined terms cancel. The remaining terms we can rewrite using $\nabla_\mu \nabla_\nu V_\rho = [\nabla_\mu, \nabla_\nu]V_\rho + \nabla_\nu \nabla_\mu V_\rho$,

$$\begin{aligned}\nabla_\alpha M_\beta + \nabla_\beta M_\alpha &= K^\gamma[\nabla_\alpha, \nabla_\gamma]L_\beta + \underline{K^\gamma \nabla_\gamma \nabla_\alpha L_\beta} - L^\gamma[\nabla_\alpha, \nabla_\gamma]K_\beta - \underline{L^\gamma \nabla_\gamma, \nabla_\alpha K_\beta} \\ &\quad + K^\gamma[\nabla_\beta, \nabla_\gamma]L_\alpha + \underline{K^\gamma \nabla_\gamma \nabla_\beta L_\alpha} - L^\gamma[\nabla_\beta, \nabla_\gamma]K_\alpha - \underline{L^\gamma \nabla_\gamma, \nabla_\beta K_\alpha} \; .\end{aligned} \tag{1.11}$$

Underlined terms cancel because of the Killing's equation. On the remaining terms one can use identity (1.4).

$$\nabla_\alpha M_\beta + \nabla_\beta M_\alpha = \underline{R_{\beta\delta\alpha\gamma}K^\gamma L^\delta} - \underline{R_{\beta\delta\alpha\gamma}K^\delta L^\gamma} + \underline{R_{\alpha\delta\beta\gamma}K^\gamma L^\delta} - \underline{R_{\alpha\delta\beta\gamma}K^\delta L^\gamma} \; . \tag{1.12}$$

By relabeling dummy indices and using the symmetry of the Riemann tensor ($R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$) one can see that the underlined terms cancel.

(b) Identity (1.4) is obtained in a straightforward way by applying the definition of a covariant derivative, relabeling some dummy indices, canceling some terms, and grouping others to form a Riemann tensor.

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]V^\sigma &= \nabla_\mu \nabla_\nu V^\sigma - \nabla_\nu \nabla_\mu V^\sigma \\ &= \partial_\mu(\nabla_\nu V^\sigma) - \underline{\cancel{\Gamma^\alpha_{\mu\nu}\nabla_\alpha V^\sigma}} + \Gamma^\sigma_{\mu\alpha}\nabla_\nu V^\alpha - \partial_\nu(\nabla_\mu V^\rho) + \underline{\cancel{\Gamma^\alpha_{\nu\mu}\nabla_\alpha V^\sigma}} - \Gamma^\sigma_{\nu\alpha}\nabla_\mu V^\alpha \\ &= \underline{\cancel{\partial_\mu \partial_\nu V^\sigma}} + \underline{\partial_\mu(\Gamma^\sigma_{\nu\alpha}V^\alpha)} + \underline{\Gamma^\sigma_{\mu\alpha}\partial_\nu V^\alpha} + \Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\nu\beta}V^\beta \\ &\quad - \underline{\cancel{\partial_\nu \partial_\mu V^\sigma}} - \underline{\partial_\nu(\Gamma^\rho_{\mu\alpha}V^\alpha)} - \underline{\Gamma^\sigma_{\nu\alpha}\partial_\mu V^\alpha} - \Gamma^\sigma_{\nu\alpha}\Gamma^\alpha_{\mu\beta}V^\beta \\ &= \Big[\partial_\mu \Gamma^\sigma_{\nu\beta} - \partial_\nu \Gamma^\sigma_{\mu\beta} + \Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\nu\beta} - \Gamma^\sigma_{\nu\alpha}\Gamma^\alpha_{\mu\beta}\Big]V^\beta = R^\sigma{}_{\beta\mu\nu}V^\beta \; .\end{aligned} \tag{1.13}$$

By contracting this result with $g_{\rho\sigma}$, and using metric compatibility, we get identity (1.4).

Now, to prove identity (1.1) we start with the Killing's equation,

$$\nabla_\nu V_\rho + \nabla_\rho V_\nu = 0 \; , \tag{1.14}$$

and take a covariant derivative of it

$$\nabla_\mu \nabla_\nu V_\rho + \nabla_\mu \nabla_\rho V_\nu = 0 \ . \tag{1.15}$$

We can relabel the indices of this equation and it is still valid. Therefore, we can add and subtract them with any combination of indices, and the result is still zero. So, we add and subtract four of these equations with indices permuted in a convenient way

$$\begin{aligned} 0 = &\Big[\nabla_\mu \nabla_\nu V_\rho + \underline{\nabla_\mu \nabla_\rho V_\nu}\Big] - \Big[\underline{\nabla_\rho \nabla_\mu V_\nu} + \underline{\nabla_\rho \nabla_\nu V_\mu}\Big] \\ &+ \Big[\underline{\nabla_\nu \nabla_\rho V_\mu} + \underline{\nabla_\nu \nabla_\mu V_\rho}\Big] - \Big[\underline{\nabla_\mu \nabla_\nu V_\rho} + \nabla_\mu \nabla_\rho V_\nu\Big] \ , \end{aligned} \tag{1.16}$$

so that we can group some terms in commutators,

$$0 = \nabla_\mu \nabla_\nu V_\rho - \nabla_\mu \nabla_\rho V_\nu + [\nabla_\mu, \nabla_\rho] V_\nu - [\nabla_\rho, \nabla_\nu] V_\mu + [\nabla_\nu, \nabla_\mu] V_\rho \ . \tag{1.17}$$

First two terms are the same because of the Killing's equation, and on the commutators we use identity (1.4),

$$2\nabla_\mu \nabla_\nu V_\rho = \Big[- R_{\nu\sigma\mu\rho} + R_{\mu\sigma\rho\nu} - R_{\rho\sigma\nu\mu} \Big] V^\sigma \ . \tag{1.18}$$

Using the symmetries of the Riemann tensor we can rearrange their sum,

$$\nabla_\mu \nabla_\nu V_\rho = \frac{1}{2}\Big[- R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} - R_{\sigma\rho\mu\nu} \Big] V^\sigma \ . \tag{1.19}$$

Since we know that

$$R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} + R_{\sigma\rho\mu\nu} = 0 \ , \tag{1.20}$$

the final result is

$$\nabla_\mu \nabla_\nu V_\rho = R_{\sigma\mu\nu\rho} V^\sigma = R_{\rho\nu\mu\sigma} V^\sigma \ . \tag{1.21}$$

By contracting with $g^{\alpha\rho}$ we can raise the apropriate index to get exactly (1.1).

Identity (1.2) follows directly from identity (1.1) by contracting it with ${g^\mu}_\rho$.

To prove identity (1.3) we first use the following Bianchi identity

$$\nabla_\alpha R = 2\nabla^\beta R_{\alpha\beta} \ , \tag{1.22}$$

and reorder some derivatives using the product rule,

$$K^\alpha \nabla_\alpha R = 2K^\alpha \nabla^\beta R_{\alpha\beta} = 2\nabla^\beta (R_{\alpha\beta} K^\alpha) - 2R_{\alpha\beta} \nabla^\beta K^\alpha \ . \tag{1.23}$$

The last term on the right is zero, since $R_{\alpha\beta}$ is symmetric in its indices, and $\nabla^\beta K^\alpha$ is antisymmetric in its indices because of the Killing's equation. For the remaining term we use identity (1.2) to write it as

$$K^\alpha \nabla_\alpha R = 2\nabla^\beta \nabla_\alpha \nabla_\beta K^\alpha = 2\nabla_\beta \nabla_\alpha \nabla^\beta K^\alpha \ . \tag{1.24}$$

Since $\nabla^\beta K^\alpha$ is antisymmetric in its indices (because of Killing's equation), the expression can be written as

$$K^\alpha \nabla_\alpha R = [\nabla_\beta, \nabla_\alpha] \nabla^{[\beta} K^{\alpha]} \ , \tag{1.25}$$

which is left to you to prove that it is zero.

■ **PROBLEM 2** Killing vectors on a 2-sphere.

The two-sphere is given by

$$ds^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2 \ . \tag{2.1}$$

(a) Solve the Killing's equation to find that the Killing vectors of a 2-sphere are

$$K_1 = \cos\varphi\,\partial_\vartheta - \cot\vartheta\sin\varphi\,\partial_\varphi \ , \tag{2.2}$$
$$K_2 = -\sin\varphi\,\partial_\vartheta - \cot\vartheta\cos\varphi\,\partial_\varphi \ , \tag{2.3}$$
$$K_3 = \partial_\varphi \ . \tag{2.4}$$

(b) Show that the commutators of these three Killing vectors satisfy the following commutation relations

$$[K_1, K_2] = K_3 \ , \tag{2.5}$$
$$[K_2, K_3] = K_1 \ , \tag{2.6}$$
$$[K_3, K_1] = K_2 \ . \tag{2.7}$$

Check that they satisfy the aproperties of Lie algebra. Can you see which algebra is it?

(a) To write down the Killing's equations we first need to calculate the Christoffel symbols. The metric components are

$$g_{\vartheta\vartheta} = 1 = g^{\vartheta\vartheta} \ , \qquad g_{\varphi\varphi} = \sin^2\vartheta = \frac{1}{g^{\varphi\varphi}} \ , \qquad g_{\vartheta\varphi} = g_{\varphi\vartheta} = g^{\vartheta\varphi} = g^{\varphi\vartheta} = 0 \ , \tag{2.8}$$

and the Christoffel symbols are

$$\begin{aligned} &\Gamma^\vartheta_{\vartheta\vartheta} = 0 \ , \qquad \Gamma^\varphi_{\varphi\varphi} = 0 \ , \qquad \Gamma^\varphi_{\vartheta\vartheta} = 0 \ , \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta \ , \\ &\Gamma^\vartheta_{\vartheta\varphi} = \Gamma^\vartheta_{\varphi\vartheta} = 0 \ , \qquad \Gamma^\varphi_{\varphi\vartheta} = \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta \ . \end{aligned} \tag{2.9}$$

The Killing's equations are then

$$0 = \nabla_{(\vartheta}K_{\vartheta)} = \nabla_\vartheta K_\vartheta = \partial_\vartheta K_\vartheta - \Gamma^\alpha_{\vartheta\vartheta}K_\alpha = \partial_\vartheta K_\vartheta \ , \tag{2.10}$$

$$0 = \nabla_{(\varphi}K_{\varphi)} = \nabla_\varphi K_\varphi = \partial_\varphi K_\varphi - \Gamma^\alpha_{\varphi\varphi}K_\alpha = \partial_\varphi K_\varphi + \sin\vartheta\cos\vartheta K_\vartheta \ , \tag{2.11}$$

$$0 = \nabla_\vartheta K_\varphi + \nabla_\varphi K_\vartheta = \partial_\vartheta K_\varphi - \Gamma^\alpha_{\vartheta\varphi}K_\alpha + \partial_\varphi K_\vartheta - \Gamma^\alpha_{\varphi\vartheta}K_\alpha = \partial_\vartheta K_\varphi + \partial_\varphi K_\vartheta - 2\cot\vartheta K_\varphi \ . \tag{2.12}$$

Now there is a matter of solving these equations. K_ϑ and K_φ components of the Killing vector are generally functions of coordinates ϑ and φ,

$$K_\vartheta = K_\vartheta(\vartheta,\varphi) \ , \qquad K_\varphi = K_\varphi(\vartheta,\varphi) \ . \tag{2.13}$$

From the first Killing equation (2.10) we have that K_ϑ does not depend on ϑ,

$$\partial_\vartheta K_\vartheta \qquad \Rightarrow \qquad K_\vartheta(\vartheta,\varphi) = K_\vartheta(\varphi) \ . \tag{2.14}$$

Then it is natural to try to construct a differential equation just for K_ϑ.

First we take a derivative ∂_φ of equation (2.12),

$$\partial_\vartheta(\partial_\varphi K_\varphi) + \partial_\varphi^2 K_\vartheta - 2\cot\vartheta\,(\partial_\varphi K_\varphi) = 0\ , \tag{2.15}$$

and then we substitute in

$$\partial_\varphi K_\varphi = -\sin\vartheta\cos\vartheta\, K_\vartheta\ , \tag{2.16}$$

from equation (2.11). This yields

$$\partial_\vartheta\Big[-\sin\vartheta\cos\vartheta\, K_\vartheta\Big] + \partial_\varphi^2 K_\vartheta - 2\cot\vartheta\Big[-\sin\vartheta\cos\vartheta\, K_\vartheta\Big] = 0\ , \tag{2.17}$$

which is just a harmonic oscillator equation,

$$\partial_\varphi^2 K_\vartheta + K_\vartheta = 0\ . \tag{2.18}$$

Therefore, a general solution for K_ϑ is

$$K_\vartheta = A\sin\varphi + B\cos\varphi\ , \tag{2.19}$$

where A and B are constants.

Now we still have to find K_φ. If we insert the solution (2.19) in Killing's equation (2.11) we get

$$\partial_\varphi K_\varphi = -\sin\vartheta\cos\vartheta\Big[A\sin\varphi + B\cos\varphi\Big]\ . \tag{2.20}$$

It is easy to see that the solution of this inhomogeneous differential equation is

$$K_\varphi = A\sin\vartheta\cos\vartheta\cos\varphi - B\sin\vartheta\cos\vartheta\sin\varphi + F(\vartheta)\ , \tag{2.21}$$

where $F(\vartheta)$ is a solution of the homogeneous equation $\partial_\varphi K_\varphi = 0$. Its dependence on ϑ can be determined from Killing's equation (2.12) in which we plug in solutions (2.19) and (2.21). What remains is the following equation

$$\partial_\vartheta F - 2\cot\vartheta F = 0\ . \tag{2.22}$$

Dividing this equation by $\sin^2\vartheta$ we can recognize that it has the form

$$\partial_\vartheta\left[\frac{F}{\sin^2\vartheta}\right] = 0\ , \tag{2.23}$$

which means its solution is

$$F(\vartheta) = C\sin^2\vartheta\ , \tag{2.24}$$

where C is a constant.

Therefore, we have the following general solution for Killing vectors

$$K_\vartheta = A\sin\varphi + B\cos\varphi\ , \tag{2.25}$$
$$K_\varphi = A\sin\vartheta\cos\vartheta\cos\varphi - B\sin\vartheta\cos\vartheta\sin\varphi + C\sin^2\vartheta\ . \tag{2.26}$$

Since we have three independent constants A, B, C, we have three independent Killing vectors. For $A = 0, B = 1, C = 0$ we have

$$(K_1)_\vartheta = \cos\varphi\ , \qquad (K_1)_\varphi = -\sin\vartheta\cos\vartheta\sin\varphi\ , \tag{2.27}$$

for $A=-1, B=0, C=0$ we have

$$(K_2)_\vartheta = -\sin\varphi \;, \qquad (K_2)_\varphi = -\sin\vartheta\cos\vartheta\cos\varphi \;, \tag{2.28}$$

and for $A=0, B=0, C=1$ we have

$$(K_3)_\vartheta = 0 \;, \qquad (K_3)_\varphi = \sin^2\vartheta \;. \tag{2.29}$$

Note that these Killing vector components are written with indices down. The components given in (2.2)-(2.4) (which you can find in Carroll's book) are the ones with indices up,

$$K_n = (K_n)^\vartheta \partial_\vartheta + (K_n)^\varphi \partial_\varphi \;. \tag{2.30}$$

If we raise the indices on the components of Killing vectors we found here we reproduce the results given,

$$(K_1)^\vartheta = g^{\vartheta\vartheta}(K_1)_\vartheta = \cos\varphi \;, \tag{2.31}$$
$$(K_1)^\varphi = g^{\varphi\varphi}(K_1)_\varphi = -\cot\varphi\sin\varphi \;, \tag{2.32}$$
$$(K_2)^\vartheta = g^{\vartheta\vartheta}(K_2)_\vartheta = -\sin\varphi \;, \tag{2.33}$$
$$(K_2)^\varphi = g^{\varphi\varphi}(K_2)_\varphi = -\cot\varphi\cos\varphi \;, \tag{2.34}$$
$$(K_3)^\vartheta = g^{\vartheta\vartheta}(K_3)_\vartheta = 0 \;, \tag{2.35}$$
$$(K_3)^\varphi = g^{\varphi\varphi}(K_3)_\varphi = 1 \;. \tag{2.36}$$

Lesson for both the students and the assistants: Be careful whether index is up or down!

(b) For the first commutator we have

$$\begin{aligned}
[K_1, K_2] = &\Big[\cos\varphi\partial_\vartheta - \cot\vartheta\sin\varphi\partial_\varphi \;, -\sin\varphi\partial_\vartheta - \cot\vartheta\cos\varphi\partial_\varphi\Big] \\
= &\cos\varphi\partial_\vartheta\Big(-\sin\varphi\partial_\vartheta - \cot\vartheta\cos\varphi\partial_\varphi\Big) \\
&- \cot\vartheta\sin\varphi\partial_\varphi\Big(-\sin\varphi\partial_\vartheta - \cot\vartheta\cos\varphi\partial_\varphi\Big) \\
&+ \sin\varphi\partial_\vartheta\Big(\cos\varphi\partial_\vartheta - \cot\vartheta\sin\varphi\partial_\varphi\Big) \\
&+ \cot\vartheta\cos\varphi\partial_\varphi\Big(\cos\varphi\partial_\vartheta - \cot\vartheta\sin\varphi\partial_\varphi\Big) \\
= &-\cos\varphi\sin\varphi\partial_\vartheta^2 - \cos^2\varphi(\partial_\vartheta\cot\vartheta)\partial_\varphi - \cot\vartheta\cos^2\varphi\partial_\vartheta\partial_\varphi \\
&+ \cot\vartheta\sin\varphi\cos\varphi\partial_\vartheta + \cot\vartheta\sin^2\varphi\partial_\varphi\partial_\vartheta - \cot^2\vartheta\sin^2\varphi\partial_\varphi + \cot^2\vartheta\sin\varphi\cos\varphi\partial_\varphi^2 \\
&+ \sin\varphi\cos\varphi\partial_\vartheta^2 - \sin^2\varphi(\partial_\vartheta\cot\vartheta)\partial_\varphi - \cot\vartheta\sin^2\varphi\partial_\vartheta\partial_\varphi \\
&- \cot\vartheta\sin\varphi\cos\varphi\partial_\vartheta + \cot\vartheta\cos^2\varphi\partial_\varphi\partial_\vartheta - \cot^2\vartheta\cos^2\varphi\partial_\varphi - \cot^2\vartheta\sin\varphi\cos\varphi\partial_\varphi^2 \\
= &-\cot^2\vartheta\partial_\varphi - (\partial_\vartheta\cot\vartheta)\partial_\varphi = \partial_\varphi = K_3 \;.
\end{aligned} \tag{2.37}$$

And the remaining two are calculated analogously.