Gravitational Waves in Astrophysics

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January 21, 2009

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Chapter 1

Introduction

The search for gravitational waves is one which has not, up until this point, given any conclusive results, however it seems likely that the necessary technology will be available within the next 15 years to actually detect this gravitational radiation. As such, it is one of the most exciting fronts today of astrophysics and cosmology, as well as being another excellent test of Einstein's theory of general relativity.

In 1905, Einstein outlined the theory of special relativity in which he expounded the new notion that space and time are not absolutes; rather, they are what are measured with rulers and clocks, respectively. Combining the theory of special relativity with the gravitational force led to the theory of general relativity in 1915. However, one of the conditions that gravity must satisfy in order to be compatible with special relativity is causality. This is analogous to the case of electrodynamics, where light waves set the standard of the concept of causality, i.e. two distant observers cannot communicate faster than the speed of light. Similarly, gravity must have such a causal structure and thus we expect there to be some type of "gravitational radiation".

Realizing this, it was Einstein who first produced work regarding this new form of radiation. His final result was the famous "quadrupole formula" which will be derived later on. Its role in general relativity is comparable to that of the dipole formula for electromagnetic radiation, in the sense that gravitational waves are produced by accelerating masses and electromagnetic waves are produced by accelerating charges.

From the quadrupole formula, it can be observed that gravitational waves are weak and difficult to produce in any detectable regime. In fact, detection would require waves which are produced by large masses moving at relativistic speeds. Such objects cannot be produced in laboratories and limit us those in the sky, astrophysical sources; hence, the title and focus of this report.

There is already significant indication that gravitational waves do exist, based on indirect measurements of orbital decay. The most well-known example of this is the Hulse-Taylor binary, which decays in very good approximation to the theoretical prediction based on energy lost due to gravitational radiation.

The discovery of gravitational waves by direct detection would not just be the unfolding of a new form of radiation (which, in itself, should not be taken lightly), but it would usher in a new era of detection possibilities: gravitational waves would allow us to probe deeper into the universe than has ever been possible. Before all these aspects are discussed however, it is imperative to first understand how gravitational waves arise from the theory of general relativity.

Chapter 2

Theory of Gravitational Waves

2.1 Linearized Gravity

Henceforth the signature (−,+,+,+) shall be adopted, such that the Minkowski metric is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.1}$$

As per standardized notation, Latin indices on objects will represent the spatial coordinates, while Greek indices will cover spacetime coordinates. Frequently throughout this report, quantities will be expressed in units where $G = c = 1$, where G is Newton's constant and c is the speed of light. The most straightforward way to obtain gravitational waves is to work in the theory of *linearized gravity*. Within this framework, it is assumed that spacetime, described by the metric tensor, is approximately flat. In other words, it is decomposed into the flat Minkowski metric $\eta_{\mu\nu}$ and some contribution $h_{\mu\nu}$.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{2.2}$$

Since the spacetime is approximately flat, this contribution must be small. As a result, in calculating physically significant quantities only terms up to linear order in $h_{\mu\nu}$ will be kept

$$||h_{\mu\nu}|| << 1. \tag{2.3}$$

The Christoffel connections in their most general form are

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}\right).$$

In linearized gravity, these reduce to a remarkably simple form

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} \eta^{\rho\sigma} \left(\partial_{\mu} h_{\sigma\nu} + \partial_{\nu} h_{\sigma\mu} - \partial_{\sigma} h_{\mu\nu}\right).$$

The Riemann tensor as a result of this approximation also reduces to a compact expression

$$\begin{aligned} R^{\rho}_{\sigma\mu\nu} &= \partial_{\mu}\Gamma^{\rho}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu}\lambda\Gamma^{\lambda}_{\mu\sigma} \\ &= \frac{1}{2}\left(\partial_{\mu}\partial_{\sigma}h^{\rho}_{\nu} + \partial_{\nu}\partial^{\rho}h_{\sigma\mu} - \partial_{\mu}\partial^{\rho}h_{\sigma\nu} - \partial_{\nu}\partial_{\sigma}h^{\rho}_{\mu}\right). \end{aligned}$$

In the above expression, the two terms with products of Christoffel symbols have vanished since they are of order $h^2_{\mu\nu}$. With the Riemann tensor in hand, the Ricci tensor can be calculated straightforwardly to produce

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \frac{1}{2}\left(\partial_{\rho}\partial_{\nu}h^{\rho}_{\mu} + \partial^{\rho}\partial_{\mu}h_{\rho\nu} - \Box h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h\right). \tag{2.4}$$

Finally, by performing the trace over the Ricci tensor, the Ricci scalar is found

$$R = R^{\mu}_{\mu} = \left(\partial_{\nu}\partial^{\mu}h^{\nu}{}_{\mu} - \Box h\right). \tag{2.5}$$

Combining these results, an unwieldy expression is found for the Einstein tensor

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\ &= \frac{1}{2}\left(\partial_{\rho}\partial_{\nu}h^{\rho}_{\mu} + \partial^{\rho}\partial_{\mu}h_{\rho\nu} - \Box h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h - \eta_{\mu\nu}\partial_{\rho}\partial^{\sigma}h^{\rho}_{\sigma} + \eta_{\mu\nu}\Box h\right). \end{aligned}$$

This is not a simple expression to work with; it would be suitable to reduce the Einstein equation to something more useful. One clever way of doing this is to cease working with $h_{\mu\nu}$ and instead work with an expression that is known as the *trace-reversed metric*[1]. It is defined as

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{2.6}$$

and its name stems from the fact that its trace is the opposite of the original metric

$$\bar{h}^{\mu}_{\mu} = -h^{\mu}_{\mu}. \tag{2.7}$$

Introducing this change into the Einstein equation (2.6), the total number of terms is reduced from six to four

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\nu\bar{h}^{\rho}_{\mu} + \partial^\rho\partial_\mu\bar{h}_{\nu\rho} - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial^\sigma\bar{h}^{\rho}_{\sigma}\right). \tag{2.8}$$

The resultant expression still seems to be complex to deal with in a straightforward manner however. In order to make it even simpler, the gauge degrees of freedom must be excluded properly (or, in the terminology of general relativity, an appropriate coordinate system needs to be chosen). In other words, the gauge needs to be fixed[3]. Consider a general infinitesimal coordinate transformation

$$x'^{\mu} = x^\mu + \xi^\mu. \tag{2.9}$$

Here ξ^μ is an arbitrary vector field whose magnitude is sufficiently small such that terms of second order or higher are negligible. Under such a coordinate transformation, the metric changes

$$g'_{\mu\nu} = \eta_{\mu\nu} - \partial_\nu\xi_\mu - \partial_\mu\xi\nu + h_{\mu\nu}. \tag{2.10}$$

Obviously, this expression is not equivalent to (2.2). Rather, there are now additional terms which shall be associated with $h_{\mu\nu}$ to be the transformed perturbation field

$$h'_{\mu\nu} = h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}, \tag{2.11}$$

[1]We shall use the words metric and metric perturbation interchangeably when it is clear what quantity is implied.

where the parenthetical notation implies the usual symmetry of the pair of indices[2]. Using this notation, the transformed metric corresponds to a form which resembles earlier results

$$g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}. \tag{2.12}$$

However, previously it was shown that working with the trace-reverse metric has its advantages, and thus it is necessary to know how it acts under these transformations

$$\begin{aligned} \bar{h}'_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h' \\ &= \bar{h}_{\mu\nu} - 2\partial_{(\nu}\xi_{\mu)} + \eta_{\mu\nu}\partial^{\rho}\xi_{\rho}, \end{aligned} \tag{2.13}$$

which is similar but has the introduction of an additional term. The first step in gauge-fixing comes from introducing the *de Donder* (or harmonic) gauge, which is analogous to the Lorenz gauge in electrodynamics

$$\partial^{\mu}\bar{h}_{\mu\nu} = 0. \tag{2.14}$$

To show that this is always possible, consider acting on the transformed metric (2.13) with such a derivative

$$\partial^{\mu}\bar{h}'_{\mu\nu} = \partial^{\mu}\bar{h}_{\mu\nu} - \partial^{\mu}\partial_{\nu}\xi_{\mu} - \Box\xi_{\nu} + \partial_{\nu}\partial^{\sigma}\xi_{\sigma} \tag{2.15}$$

$$= \partial^{\mu}\bar{h}_{\mu\nu} - \Box\xi_{\nu}. \tag{2.16}$$

From this, it can be seen that any metric perturbation can be put into the de Donder gauge by a suitable transformation which satisfies

$$\Box\xi_{\nu} = \partial^{\mu}\bar{h}_{\mu\nu}. \tag{2.17}$$

The de Donder gauge fixes some of our gauge degrees of freedom. As a result, the initially 10 independent components of the symmetric metric perturbation $h_{\mu\nu}$ have been reduced to six due to the four conditions that this gauge imposes. The process of gauge fixing is not complete by employing the Lorenz gauge. Consider that for

[2] $\partial_{(\mu}A_{\nu)} = \frac{1}{2}\left(\partial_{\mu}A_{\nu} + \partial_{\nu}A_{\mu}\right)$.

$$\Box\xi_\nu = \partial^\mu \bar{h}_{\mu\nu}, \tag{2.18}$$

the most general solution is given by

$$\xi_\nu = \xi_\nu^h + \int d^4y G(x^\rho - y^\rho)\partial^\mu \bar{h}_{\mu\nu}(y^\rho), \tag{2.19}$$

where ξ_ν^h is the homogeneous solution satisfying $\Box\xi_\nu^h = 0$. Thus, by fixing the de Donder gauge, we have only fixed the inhomogeneous solution and still have the freedom to fix the homogeneous components (we can add an arbitrary amount of homogenous equations to (2.17) and it will remain unaffected), which correspond to another four degrees of freedom. These homogenous solutions will be fixed later by a more thorough treatment of gauge fixing in which it becomes clear that they are indeed gauge artifacts (see Section 2.3).

Beforehand, the de Donder gauge is applied to the reduced Einstein equation (2.8). Consequently, the Einstein equation simplifies greatly

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} \Rightarrow \Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{2.20}$$

This important expression will be used throughout much of this report. It follows from this expression that the conservation of the energy momentum tensor reduces to $\partial_\mu T^{\mu\nu} = 0$ in the linearized theory[3]. Having shown the form of the necessary general relativistic quantities in the linearized theory, it is time to proceed to the consideration of solutions in such a theory.

2.2 Vacuum Solutions

A natural starting point would be to consider vacuum solutions, i.e. solutions corresponding to those emitted by a source at some point in time which has since stopped emitting. As a result, the energy-momentum tensor is zero and the Einstein equation (2.20) becomes

$$\Box\bar{h}_{\mu\nu} = 0. \tag{2.21}$$

[3]The momentum energy tensor always satisfies conservation under a contracted covariant derivative. However, the Christoffel symbol term drops out since it is second order in $h_{\mu\nu}$.

This is nothing more than the homogeneous wave equation! Similar to the case of electromagnetism, the wave equation admits a class of homogeneous solutions which are superpositions of plane waves

$$\bar{h}_{\mu\nu}(\vec{x},t) = \mathrm{Re} \int d^3k \, \left(A_{\mu\nu}(\vec{k}) e^{i(\vec{k}\cdot\vec{x}-\omega t)} + B_{\mu\nu}(\vec{k}) e^{i(\vec{k}\cdot\vec{x}+\omega t)} \right), \tag{2.22}$$

where $|k| = \omega$ and the complex Fourier coefficients $A_{\mu\nu}(\vec{k})$ and $B_{\mu\nu}(\vec{k})$ depend on the wave vector $\vec{k}$. From the de Donder gauge condition, they are imposed with the constraint $k^\mu A_{\mu\nu} = 0 = A_{\mu\nu}k^\nu$ (and similarly for $B_{\mu\nu}$), where k^μ is an arbitrary four-dimensional wave vector and the second equality follows from the symmetry of $A_{\mu\nu}$. Furthermore, we take the real part of the solution on the (complex) right hand side of the equation since the metric on the left hand side is real. The plane waves that have been produced here are gravitational waves.

2.3 Fixing the Gauge

To find an explicit form for the metric perturbation, the gauge-fixing process must be completed, since the de Donder gauge does not remove all unphysical degrees of freedom from the theory. For this subject, I will closely follow along the lines of Carroll[1], who gives a clearer and more rigorous treatment of this process than much of the literature.

One begins by renaming the elements of the original (not trace-reversed) metric

$$\begin{aligned} h_{00} &= -2\Phi, \\ h_{0i} &= w_i, \\ h_{ij} &= 2s_{ij} - 2\Psi\delta_{ij}. \end{aligned} \tag{2.23}$$

Here, Ψ is the trace of h_{ij} and s_{ij} is the traceless part

$$\begin{aligned} \Psi &= -\frac{1}{6}\delta^{ij}h_{ij}, \\ s_{ij} &= \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij} \right). \end{aligned} \tag{2.24}$$

This requires a recalculation of the Christoffel symbols in terms of these new variables

$$\begin{aligned}
\Gamma^0_{00} &= \partial_0\Phi, \\
\Gamma^i_{00} &= \partial_i\Phi + \partial_0 w_i, \\
\Gamma^0_{j0} &= \partial_j\Phi, \\
\Gamma^i_{j0} &= \frac{1}{2}\partial_j w_i - \frac{1}{2}\partial_i w_j + \frac{1}{2}\partial_0 h_{ij}, \\
\Gamma^0_{jk} &= -\frac{1}{2}\partial_j w_k - \frac{1}{2}\partial_k w_j + \frac{1}{2}\partial_0 h_{jk}, \\
\Gamma^i_{jk} &= \frac{1}{2}\partial_j h_{ki} + \frac{1}{2}\partial_k h_{ji} - \frac{1}{2}\partial_i h_{jk}.
\end{aligned} \tag{2.25}$$

Note that we have not yet rewritten h_{ij} in terms of Ψ and s_{ij}, since they will differ only later on when we take traces. Next, we find the components of the Riemann tensor

$$\begin{aligned}
R_{0j0l} &= \partial_j\partial_l\Phi + \frac{1}{2}\partial_0\partial_j w_l + \frac{1}{2}\partial_0\partial_l w_j - \frac{1}{2}\partial_0\partial_0 h_{jl}, \\
R_{0jkl} &= \frac{1}{2}\partial_j\partial_k w_l - \frac{1}{2}\partial_j\partial_l w_k - \frac{1}{2}\partial_0\partial_k h_{lj} + \frac{1}{2}\partial_0\partial_l h_{kj}, \\
R_{ijkl} &= \frac{1}{2}\partial_j\partial_k h_{li} - \frac{1}{2}\partial_j\partial_l h_{ki} - \frac{1}{2}\partial_i\partial_k h_{lj} + \frac{1}{2}\partial_i\partial_l h_{kj}.
\end{aligned} \tag{2.26}$$

The other terms of the Riemann tensor are related to these by the usual symmetries. Moreover, from these we find the Ricci tensor where h_{ij} will be written in terms of s_{ij} and Ψ as described before

$$\begin{aligned}
R_{00} &= \nabla^2\Phi + \partial_0\partial_k w^k + 3\partial_0^2\Psi, \\
R_{0j} &= -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s^k_j, \\
R_{ij} &= -\partial_i\partial_j\left(\Phi - \Psi\right) - \frac{1}{2}\partial_0\partial_i w_j - \frac{1}{2}\partial_0\partial_j w_i + \Box\Psi\delta_{ij} - \Box s_{ij} + \partial_k\partial_i s^k_j + \partial_k\partial_j s^k_i.
\end{aligned}$$

Here $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the usual flat, 3D space Laplacian. With the necessary quantities in hand, the Einstein tensors can be found from the Einstein equation

$$
\begin{aligned}
G_{00} &= 2\nabla^2\Psi + \partial_k\partial_l s^{kl}, \\
G_{0j} &= -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s_j^k, \\
G_{ij} &= \left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)(\Phi - \Psi) + \delta_{ij}\partial_0\partial_k w^k - \frac{1}{2}\partial_0\partial_i w_j - \frac{1}{2}\partial_0\partial_j w_i \\
&+ 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} + \partial_k\partial_i s_j^k + \partial_k\partial_j s_i^k - \delta_{ij}\partial_k\partial_l s^{kl}.
\end{aligned} \tag{2.27}
$$

By working with the relation between the Einstein tensor and the energy-momentum tensor $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, we can decipher which degrees of freedom are actual physical degrees of freedom and which ones are not. Starting with the first component G_{00} we find

$$
\nabla^2\Psi = 4\pi G T_{00} - \frac{1}{2}\partial_k\partial_l s^{kl}. \tag{2.28}
$$

There are no derivatives with respect to time in this equation. In other words, if we know T_{00} and s_{ij} at any moment in time, then we know what Ψ is. As a result, Ψ is not a real propagating degree of freedom, as it is determined by the other two quantities. Having eliminated one variable, let us move on to the next Einstein tensor component G_{0j} which produces the relation

$$
\left(\delta_{jk}\nabla^2 - \partial_j\partial_k\right) w^k = -16\pi G T_{0j} + 4\partial_0\partial_j\Psi + 2\partial_0\partial_k s_j^k. \tag{2.29}
$$

Here, from the reasoning given above, it is clear that w^k is also not a propagating degree of freedom, as it can be determined from s_{ij} and $T_{\mu\nu}$, which also determine Ψ in this equation. Moving along, the last equation is given by

$$
\begin{aligned}
\left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)\Phi &= 8\pi G T_{ij} + \left(\delta_{ij}\nabla^2 - \partial_i\partial_j - 2\delta_{ij}\partial_0^2\right)\Psi - \delta_{ij}\partial_0\partial_k w^k \\
&+ \frac{1}{2}\partial_0\partial_i w_j + \frac{1}{2}\partial_0\partial_j w_i + \Box s_{ij} - \partial_k\partial_i s_j^k - \partial_k\partial_j s_i^k - \delta_{ij}\partial_k\partial_l s^{jl}.
\end{aligned}
$$

Once again, we observe that there are no time derivatives acting on Φ, and hence it can also be determined from the other fields. The only degree of freedom we are left with then is s_{ij}. In a bit, we shall see that this part describes gravitational waves. In any case, all the other functions in $h_{\mu\nu}$ can be determined from s_{ij} and the energy-momentum tensor.

Previously, when we derived the de Donder gauge, we had derived the transformation of the metric perturbation under an infinitesimal coordinate transformation

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{2.30}$$

Having decomposed our metric perturbations into different fields, we now find the change that such a transformation produces on them

$$\begin{aligned} \Phi &\to \Phi - \partial_0 \xi^0, \\ w_i &\to w_i - \partial_0 \xi^i + \partial_i \xi^0, \\ \Psi &\to \Psi + \frac{1}{3}\partial_i \xi^i, \\ s_{ij} &\to s_{ij} - \frac{1}{2}\partial_i \xi_j - \frac{1}{2}\partial_j \xi_i + \frac{1}{3}\partial_k \xi^k \delta_{ij}. \end{aligned} \tag{2.31}$$

From here, we can begin the gauge-fixing process. We shall choose the *transversal gauge* which is closely related to the Coulomb gauge ($\partial_i A^i$) you may be familiar with from electromagnetism. In this gauge, we fix s_{ij} to be transverse

$$\partial^i s_{ij} = 0. \tag{2.32}$$

This institutes the following requirement on our coordinate transformation

$$\nabla^2 \xi^j + \frac{1}{3}\partial_j \partial_i \xi^i = -2\partial_i s^{ij}. \tag{2.33}$$

At this point, the component ξ^0 is still arbitrary and we can use it to make the vector perturbation transverse

$$\partial_i w^i = 0, \tag{2.34}$$

in which ξ^0 then satisfies

$$\nabla^2 \xi^0 = \partial_i w^i + \partial_0 \partial_i \xi^i. \tag{2.35}$$

These conditions specify the transverse gauge and the Einstein tensor's components are reduced in this gauge

$$\begin{aligned} G_{00} &= 2\nabla^2\Psi, \\ G_{0j} &= -\frac{1}{2}\nabla^2 w_j + 2\partial_0\partial_j\Psi, \\ G_{ij} &= \left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)(\Phi - \Psi) - \frac{1}{2}\partial_0\partial_i w_j - \frac{1}{2}\partial_0\partial_j w_i + 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij}. \end{aligned} \tag{2.36}$$

Coming back to our earlier discussion of the metric perturbation in vacuum space, for the G_{00} equation we obtain

$$\nabla^2\Psi = 0. \tag{2.37}$$

If we assume that the boundaries are well-behaved, then the solution to this differential equation is given by[4] $\Psi = c + f(t)$, where c is a constant and $f(t)$ is some arbitrary function dependent on time. This constant can be set to zero, since it is simply a reference point to which we perform all other measurements (this is similar to how all energies in quantum field theory are measured with respect to an infinite vacuum energy).
The time-dependent function we can also set to zero, since we are interested in a mass source in a finite volume; hence, the field emitted by the source must vanish asymptotically, such that there can be no contribution from a purely time-dependent field (since this does not vanish far from the source) and thus $f(t) = 0$. Using this value for Ψ, the G_{0j} equation in the vacuum simplifies

$$\nabla^2 w_i = 0. \tag{2.38}$$

Given the previous arguments, the solution to this equation can be set to $w_i = 0$. Finally, if we look at the trace G_i^i of the G_{ij} equation, and thus separate the Φ and s_{ij} fields

$$\nabla^2\Phi = 0. \tag{2.39}$$

[4]The most general solution for the spatially dependent part is in fact $\Psi(x) = ax + c$, in which a and c are constants. However, in order for the function not to diverge at the boundaries, the linear term must be zero. A source which is not finite at infinite boundaries must be infinite itself, and we do not expect there to be infinite mass sources according to observational evidence.

At this point, the solution to the above equation should be obvious. The leftover piece of the G_{ij} equation in the vacuum becomes a wave equation for the traceless tensor

$$\Box s_{ij} = 0. \tag{2.40}$$

A fortunate byproduct of these equations is that our decomposition shows the metric perturbation is now also traceless. In fact, it can be represented in matrix form by

$$h_{\mu\nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2s_{ij} & \\ 0 & & & \end{pmatrix}, \tag{2.41}$$

where $h_{\mu\nu}^{TT}$ is the metric perturbation in the *transverse traceless gauge*. The equation of motion is a familiar one

$$\Box h_{\mu\nu}^{TT} = 0, \tag{2.42}$$

and you may recognize this result from earlier, when we obtained the same result for the trace-reversed metric perturbation. As one can easily verify, in the traceless transverse gauge

$$h_{\mu\nu}^{TT} = \bar{h}_{\mu\nu}^{TT}. \tag{2.43}$$

Thus we have arrived at the same result as before, however now we have much more to work with: an explicit form of the matrix. There are several components that are zero, which will make the steps from here on out much simpler. In most of the literature, the imposition of the transverse traceless gauge follows quickly after the de Donder gauge (note: this is actually a strange way to proceed, since the transverse traceless gauge *automatically* imposes the de Donder gauge) in an effort to obtain (2.41) by claiming they are still allowed to impose additional constraints on ξ^μ without justifying rigorously why so. In any case, the decomposition above should make this clear. In summary, we present the properties of the traceless transverse gauge one more time

$$\begin{aligned} h_{0\nu}^{TT} &= 0, \\ \eta^{\mu\nu} h_{\mu\nu}^{TT} &= 0, \\ \partial_\mu h_{TT}^{\mu\nu} &= 0. \end{aligned} \tag{2.44}$$

(2.45)

2.4 Vacuum Solutions Revisited

One of the solutions to the equation of motion (2.42) is that of a plane wave

$$h_{\mu\nu}^{TT} = C_{\mu\nu} \exp(ik_\sigma x^\sigma), \tag{2.46}$$

where $C_{\mu\nu}$ is a constant, symmetric tensor which, in line with the above properties, is traceless, transverse, and purely spatial

$$\begin{aligned} C_{0\nu} &= 0, \\ \eta^{\mu\nu} C_{\mu\nu} &= 0 \\ k^\mu C_{\mu\nu} = k^\nu C_{\mu\nu} &= 0. \end{aligned}$$

Plugging the plane wave solution into the differential equation (2.42) will give the usual condition $k^2 = 0$ for nontrivial metric perturbations. In other words, gravitational waves propagate at the speed of light. In the plane wave equation (2.46), the left hand side $h_{\mu\nu}^{TT}$ is real, while the right side is complex. Thus, we keep in mind that at the end of our calculations we should take the real part of the plane wave.

Moreover, from the transversality condition of the metric perturbation, one can easily verify (by taking a derivative on both sides) that $k_\mu C^{\mu\nu} = 0$, which shows that the waves are transversal. Finally, the plane wave solution given is obviously not the most general solution, which actually consists of a superposition of such plane waves. This general solution was given previously in (2.22), where $A_{\mu\nu}(\vec{k})$ is the generalization of the $C_{\mu\nu}$ matrix to all possible values of $\vec{k}$.

For purposes of illustration, we shall restrict ourselves to a single plane wave, in particular one which is propagating in the z-direction, such that its wave vector is given by

$$k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega). \tag{2.47}$$

Here the second equality follows from the fact that k^μ is a null vector. For this scenario, the transversality condition $k^\mu C_{\mu\nu} = \omega C_{0\nu} + \omega C_{3\nu} = 0$, along with $C_{0\nu}$, ensures

$$C_{3\nu} = 0. \tag{2.48}$$

Combining this with the properties that this constant matrix must be traceless and symmetric, the most general form is given by

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.49}$$

2.5 The Effect of Gravitational Waves

The plane wave is then completely described by its frequency ω and the two independent components of this matrix. Let us now acquire an intuition for how such a wave affects matter. We begin by considering the most simple example: a non-relativistic, freely falling particle. Since it is moving non-relativistically, the velocity components can be neglected[5] and plugging this into the geodesic equation, one readily finds using our previously derived Christoffel symbols (2.25)

$$\begin{aligned} \frac{d^2x^i}{d\tau^2} &= -\left(\Gamma^i_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right) \\ &= -\left(\Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}\right) \Rightarrow \frac{d^2x^i}{dt^2} = 0. \end{aligned} \tag{2.50}$$

So it seems naively that our particle remains stationary! However, this is not the case, since all the geodesic equation tells us is that the coordinate

[5] In the relativistic case, the velocity components cannot be neglected and the last three Christoffel symbols in (2.25) will play a role. As a result, there will be dependence on $\partial_0 h_{ij}$ in this setting and the particle will not remain stationary.

location of the slowly moving body is left unaffected in the transverse traceless gauge and any hardened practitioner of general relativity knows that one should look at *coordinate invariant* observables. An example would be the proper distance between two freely falling particles. Let us consider two such particles at $z = 0$ separated on the x-axis by the coordinate distance L_c. Their proper distance L is

$$L = \int_0^{L_c} dx \sqrt{g_{xx}} = \int_0^{L_c} dx \sqrt{1 + h_{xx}^{TT}(t, z = 0)}, \tag{2.51}$$

where $h_{xx}^{TT}(t, z = 0)$ is the xx (or 11) component of the traceless transverse metric. Since this metric is small, we can use a Taylor expansion to first order for the square root and then simply integrate since we have no x-dependence[6]

$$L \approx \int_0^{L_c} dx \left(1 + \frac{1}{2} h_{xx}^{TT}(t, z = 0)\right) = L_c \left(1 + \frac{1}{2} h_{xx}^{TT}(t, z = 0)\right). \tag{2.52}$$

At this point, it is useful to introduce some name changes for reasons which will become clear shortly. We will refer to the components C_{11} and $C_{22} = -C_{11}$ as h_+ and $-h_+$ respectively. Moreover, we also introduce $C_{12} = C_{21} = h_\times$. Taking the real part of the plane wave solution, such that our exponential becomes a cosine function, the full metric becomes

$$\begin{aligned} ds^2 &= -c^2dt^2 + dz^2 + dy^2 \left[1 - h_+ \cos\left[\omega\left(t - \frac{z}{c}\right)\right]\right] \\ &+ dx^2 \left[1 + h_+ \cos\left[\omega\left(t - \frac{z}{c}\right)\right]\right] + 2dxdyh_\times \cos\left[\omega\left(t - \frac{z}{c}\right)\right], \end{aligned}$$

and the previous calculation for two particles at $z = 0$ becomes more explicit

$$\begin{aligned} L &= \int_0^{L_c} dx \sqrt{1 + h_+ \cos\left[\omega\left(t - \frac{z}{c}\right)\right]} \\ &\approx L_c \left(1 + \frac{h_+}{2} \cos\left[\omega t\right]\right). \end{aligned}$$

[6] Since there is no local dependence on x, we see that we in fact obtain a global description of the change in proper distance.

Thus we observe that, while the coordinate distance remains unchanged between the two particles, the proper distance oscillates with time around the initial state due to the small varying piece containing the cosine. So it seems that our gravitational wave does produce some type of effect after all! To get the full picture, it is necessary to look at four particles. Let us consider four freely falling particles in a ring which are displaced from each other on the x and y axes but all at $z = 0$. For convenience, we switch to the use of sine functions, which amounts to nothing more than shifting our time variable by a phase factor. Let us first investigate the case in which $h_\times = 0$. The small varying pieces are then given by

$$\delta x(t) = \frac{h_+}{2} x_0 \sin[\omega t], \quad \delta y(t) = -\frac{h_+}{2} y_0 \sin[\omega t],$$

where x_0 and y_0 are the original locations of the particles. Consequently, if you have two of the particles purely displaced on the x-axis ($y = 0$) and the other two purely along the y-axis, then first those on the x-axis will stretch (while the y-axis particles contract) in proper distance from each other, and then vice versa. In other words, they form a kind of a pulsating plus sign (Figure 2.1), and hence the notation h_+ and the accompanying name *plus polarization*.

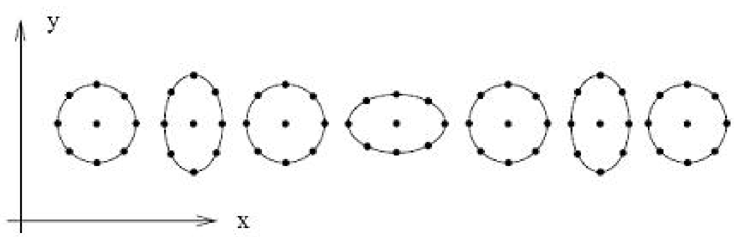

Figure 2.1: The plus polarization's effect on a ring of freely falling particles.[2]

Now we set h_+ to zero to see the effect that $h_\times$ induces. The variations are given by

$$\delta x(t) = \frac{h_\times}{2} y_0 \sin[\omega t], \quad \delta y(t) = \frac{h_\times}{2} x_0 \sin[\omega t]. \tag{2.53}$$

If we again consider the same ring of particles (though it may be useful to rotate this ring by 45 degrees), then it is now a pulsating cross and thus we have found the *cross polarization* (Figure 2.2) whose strength is determined by $h_\times$.

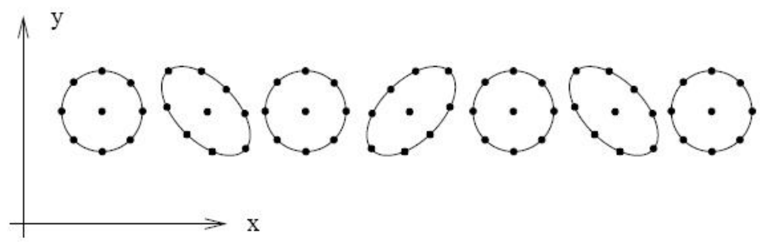

Figure 2.2: The cross polarization's effect is similar to that of the plus polarization, yet rotated by 45 degrees.[2]

So, to conclude, we have found the gravitational waves cause an oscillatory behavior in the proper distance of freely falling particles and that these waves are composed of two polarizations which differ by 45 degrees in orientation. At this point, we have a pretty good idea of how gravitational waves act in the vacuum, which behooves us to move on to the specifics of their generation.

Chapter 3

Measuring Gravitational Radiation

3.1 Generation of Gravitational Waves

Working in the vacuum, we found the traceless transverse gauge to be very useful, and in this gauge the trace-reversed and normal metric perturbations were equal to each other (since the trace was zero). However, when we have some matter in our spacetime then this is no longer as straightforward. As explained before, it is then simpler to work with the trace-reversed metric perturbation, such that our Einstein equation (2.20) becomes relatively simple. Furthermore, we can still impose the de Donder gauge and energy momentum conservation which were derived before[1]

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \quad \partial_\mu\bar{h}^{\mu\nu} = 0, \quad \partial_\mu T^{\mu\nu} = 0. \tag{3.1}$$

The Einstein equation above can readily be solved with the use of method of Green's functions. As a starting point, the retarded Green's function is introduced[5,7]

$$G(x - x') = -\frac{1}{4\pi}\frac{1}{|\vec{x} - \vec{x}'|}\delta\left(t - \frac{|\vec{x} - \vec{x}'|}{c} - t'\right), \tag{3.2}$$

which of course satisfies the defining relation for a Green's function, namely

[1]Factors of c are now reintroduced for the purpose of expressing the orders of magnitude of physically relevant quantities later in this chapter.

$\Box_x G(x - x') = \delta^{(4)}(x - x')$. Putting it to use, the solution for the trace-reversed metric perturbation can be written as

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x'\, G(x - x') T_{\mu\nu}(x'). \tag{3.3}$$

We still wish to work in the TT gauge, and this is possible if we introduce some projection operators. First we introduce the $P(\vec{n})$ operator, which removes any components parallel to $\vec{n}$ in a tensor

$$P_{ij}(\vec{n}) = \delta_{ij} - n_i n_j. \tag{3.4}$$

Next, we combine the P operators such that they also form a traceless operator that we shall call the Λ operator

$$\Lambda_{ij,kl}(\vec{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}. \tag{3.5}$$

Thus, by acting with Λ on the metric perturbation, we transform it into a traceless transverse metric perturbation for any arbitrary propagation direction $\vec{n}$ of the waves

$$h^{TT}_{ij} = \Lambda_{ij,kl} h_{kl}. \tag{3.6}$$

Continuing along with the general expression for the metric perturbation, we now transform it into the traceless transverse gauge using the above operators.

$$\bar{h}^{TT}_{ij}(t, \vec{x}) = \Lambda_{ij,kl}(\vec{n}) \frac{4G}{c^4} \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} T_{kl}\left(t - \frac{|\vec{x} - \vec{x}'|}{c}; \vec{x}'\right). \tag{3.7}$$

We want to examine the region far from the source, such that the distance to the source r is much larger than the characteristic source size d. This allows us to make the following first order approximation

$$|\vec{x} - \vec{x}'| = r - \vec{x}' \cdot \vec{n} + \mathcal{O}(d^2/r), \;\; r = ||\vec{x}||. \tag{3.8}$$

By inserting this identity into our expression, we can pull the inverse length dependence out of the integral, while the argument of the energy-momentum tensor changes accordingly (note that the integral's bounds are still over all of the source's points)

$$\bar{h}_{ij}^{TT}(t,\vec{x}) = \frac{4G}{c^4}\frac{1}{r}\Lambda_{ij,kl}(\vec{n})\int_{|x'|<d} d^3x'\, T_{kl}\left(t - \frac{r}{c} + \frac{\vec{x}'\cdot\vec{n}}{c}; \vec{x}'\right). \tag{3.9}$$

At this point, we assume that the typical velocities inside the source are much smaller than the speed of light

$$\left|\frac{\vec{x}'\cdot\vec{n}}{c}\right|\partial_{t_R} << 1, \tag{3.10}$$

where we have introduced the retarded time $t_R = t - \frac{r}{c}$. In making this approximation, we can perform a Taylor expansion of the stress-energy tensor around $\left|\frac{\vec{x}'\cdot\vec{n}}{c}\right|$

$$\begin{aligned}\bar{h}_{ij}^{TT}(t,\vec{x}) \approx\ & \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\vec{n})\Big[\int d^3x T^{kl}(t_R,\vec{x}) \\ +\ & \frac{1}{c}n_m\frac{d}{dt_R}\int d^3x T^{kl}(t_R,\vec{x})x^m \\ +\ & \frac{1}{2c^2}n_m n_p\frac{d^2}{dt_R^2}\int d^3x T^{kl}(t_R,\vec{x})x^m x^p + \cdots\Big]\Big|_{t_R=t-r/c},\end{aligned} \tag{3.11}$$

where every consecutive term becomes increasingly smaller due to the slow motion of the source. To clean up this long equation and gain a physical perspective on it, one now introduces the momenta of mass density

$$\begin{aligned} M &= \frac{1}{c^2}\int d^3x\, T^{00}(t,\vec{x}), \\ M^i &= \frac{1}{c^2}\int d^3x\, T^{00}(t,\vec{x})x^i, \\ M^{ij} &= \frac{1}{c^2}\int d^3x\, T^{00}(t,\vec{x})x^i x^j. \end{aligned}$$

From the conservation law of linearized gravity $\partial_\mu T^{\mu\nu} = 0$ and setting $\nu = 0$, we can find from the resultant equation, $\partial_0 T^{00} + \partial_i T^{i0} = 0$, upon integrating over a volume which fully contains the source, the conservation of mass $\dot{M} = 0$. By integrating over such a volume, we can employ the divergence (or Stokes') theorem to set $\int d^3x\ \partial_i T^{i0} = 0$. Using analogous arguments, one can also prove the conservation of momentum $\ddot{M}^i = 0$. Thus, the zeroth and first momenta of mass density represent conserved quantities in our system, and as such we do not expect them to contribute to our metric.

In fact, we must look to the second momentum of mass density to find the leading order contribution[4]. By using the conservation relation and integrating by parts, once again over a volume containing the entire source, we can make the following rewriting

$$\begin{aligned} c\dot{M}^{ij} &= \int_V d^3x\, x^i x^j \partial_0 T^{00} = -\int_V d^3x\, x^i x^j \partial_k T^{0k} \\ &= \int_V d^3x\, \left(x^j T^{0i} + x^i T^{0j}\right). \end{aligned}$$

By taking another derivative with respect to time, the equation takes a simple form

$$\ddot{M}^{ij} = 2\int_V d^3x\, T^{ij}, \tag{3.12}$$

which is, up to a constant, just the original, leading order integral in the Taylor expanded equation (3.11) for the metric! In summary, the general expression for the metric perturbation (in the transverse traceless gauge) in a space-time with a slowly moving and distant source is given by

$$h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\vec{n})\ddot{M}^{kl}(t - \frac{r}{c}). \tag{3.13}$$

For sources with relativistic velocities, the higher order terms in (3.11) would have to be considered, corresponding to octopole, hexadecapole, and higher order moments.

3.2 Measurable quantities

With an expression for the metric at our disposal, we would like to divert our attention to experimentally measurable quantities. A very useful one would be the power radiated from a source (which is also of historical significance for the Hulse-Taylor binary). A full derivation of this would be quite lengthy, but let us at least get an idea of how this works. Naturally, since we are considering a space-time with a source massive enough to generate appreciable gravitational waves, a Minkowski background metric is no longer justified. Thus we must first have a formulation of linearized gravity around a generic curved background[4,5]. This can be done roughly as follows by first defining the total metric to be the background plus some small perturbation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{3.14}$$

Next one reproduces the same steps that were done before, keeping terms only first order in the perturbation. The conditions that keep the perturbation small can be described as follows: 1) the typical wavelength of a wave is much smaller than the typical scale of the background, and thus the wave can be treated as a small ripple, or 2) the background only contains frequencies up to some f_B which is much smaller than the frequencies of the waves and thus the background can be treated as static. This will lead to an Einstein's equation that looks as follows

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}\left(\bar{T}_{\mu\nu} + t_{\mu\nu}\right), \tag{3.15}$$

where the barred quantities refer to those which belong to the background and the tensor $t_{\mu\nu}$ is the effective stress energy (pseudo)tensor of the gravitational waves

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right). \tag{3.16}$$

The treatment is fairly straightforward so long one is careful to keep the background and perturbative contributions separate. An explicit calculation of this tensor far from the source produces the result

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\partial_\mu h^{TT}_{\alpha\beta}\partial_\nu h^{TT}_{\alpha\beta}. \tag{3.17}$$

In the literature, you will often find averaging brackets $\langle\rangle$ on the product of metrics which indicate an average over several wavelengths. This procedure is a consequence of the fact that $t_{\mu\nu}$ is not invariant under gauge transformations (we have circumvented this problem by going into the traceless transverse gauge, for which case it is invariant). By averaging over several wavelengths, they attempt to circumvent this problem, capturing enough curvature in a small region of space-time to describe a gauge-invariant measure. However, by performing such an average they break the covariance of

the theory. Thus it is best to avoid this approach.
For a plane wave, the corresponding effective energy component of the waves is then

$$t_{00} = \frac{c^2}{32\pi G}\dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT} = \frac{c^2}{16\pi G}\left(\dot{h}_+^2 + \dot{h}_\times^2\right), \tag{3.18}$$

and thus the gravitational wave flux per unit area is given by a relatively simple result

$$\frac{dE}{dtdA} = \frac{c^3}{16\pi G}\left(\dot{h}_+^2 + \dot{h}_\times^2\right). \tag{3.19}$$

For purposes of measurement with a detector on Earth, and assuming the source emits spherically symmetric, it is more useful to convert this quantity into units of power over solid angle by using $\Omega = \frac{kA}{r^2}$ where k is a proportionality constant that we take to be equal to one such that we work in the SI units of steradians

$$\frac{dP}{d\Omega} = \frac{r^2c^3}{32\pi G}\left(\dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\right) = \frac{G}{8\pi c^5}\Lambda_{kl,mp}(\vec{n})\left[\frac{d}{dt}\left(\ddot{Q}_{kl}\right)\frac{d}{dt}\left(\ddot{Q}_{mp}\right)\right]. \tag{3.20}$$

The right-most equation follows from plugging in the solution we found for $h_{\mu\nu}^{TT}$ before in terms of the M_{ij} and consequently tidying up the result with the introduction of Q_{ij}, the traceless quadrupole tensor, which is just the second momentum of mass density with its trace removed

$$Q_{ij} = M_{ij} - \frac{1}{3}\delta_{ij}M_{kk}. \tag{3.21}$$

We can perform the integration over the solid angle with the help of a couple of relations

$$\begin{aligned} \int \frac{d\Omega}{4\pi}n_in_j &= \frac{1}{3}\delta_{ij}, \\ \int \frac{d\Omega}{4\pi}n_in_jn_kn_l &= \frac{1}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \end{aligned} \tag{3.22}$$

Performing the integration, we then find the total power radiated by a source in the form of gravitational waves

$$P = \frac{G}{5c^5}\left[\frac{d}{dt}\left(\dddot{Q}_{ij}\right)\frac{d}{dt}\left(\dddot{Q}_{ij}\right)\right]. \tag{3.23}$$

In the literature, you will often see this expression refered to as the *Einstein quadrupole formula*. On a side note, the fact that energy is carried away from the source would imply that linear momentum and angular momentum are not conserved either. At leading order, the angular momentum radiated away is

$$\frac{dL^i}{dt} = \frac{2G}{5c^5}\epsilon^{ijk}\left[\ddot{Q}_{jl}\frac{d}{dt}\ddot{Q}_{lk}\right], \tag{3.24}$$

while the amount of linear momentum radiated is given by

$$\frac{dP^i}{dt} = -\frac{G}{8\pi c^5}\int d\omega\left(\frac{d}{dt}\ddot{Q}^{TT}_{jk}\right)\left(\partial^i\ddot{Q}^{TT}_{jk}\right). \tag{3.25}$$

This covers the quantities we would like to measure. It is now time to start with a very basic example and then move on to some real calculations!

3.3 Application to a Simple Binary System

A very simple calculation would involve a binary system of two masses m_1 and m_2 which are both in a circular orbit. The total mass is given by $M = m_1 + m_2$ and the reduced mass by $\mu = \frac{m_1 m_2}{M}$[7]. In the center-of-mass frame, their relative coordinates are then given by

$$X(t) = R\cos(\omega t), \quad Y(t) = R\sin(\omega t), \quad Z(t) = 0,$$

where R is the relative distance between the two bodies. The second momentum of mass tensor in this case is $M^{ij} = \mu X^i X^j$ of which the non-zero components are

$$\begin{aligned} M_{11} &= \frac{1}{2}\mu R^2(1+\cos(2\omega t)), \\ M_{22} &= \frac{1}{2}\mu R^2(1-\cos(2\omega t)), \\ M_{12} &= \frac{1}{2}\mu R^2\sin(2\omega t). \end{aligned}$$

Plugging these in to the Einstein quadrupole formula we have just derived, the power radiated is

$$P = \frac{32}{5}\frac{G\mu^2 R^4 \omega^6}{c^5}, \tag{3.26}$$

and the power radiated per solid angle is

$$\frac{dP}{d\Omega} = \frac{G}{8\pi c^5}\left[\frac{d\dddot{Q}_{ij}}{dt}\frac{d\dddot{Q}^{ij}}{dt} - 2\frac{d\dddot{Q}_i^{j}}{dt}\frac{d\dddot{Q}^{ik}}{dt}n_j n_k + \frac{1}{2}\frac{d\dddot{Q}^{ij}}{dt}\frac{d\dddot{Q}^{kl}}{dt}n_i n_j n_k n_l\right], \tag{3.27}$$

where

$$\frac{d\dddot{Q}_{ij}}{dt} = 3\mu R^2\omega^3\begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.28}$$

For purposes of illustration, the two components of the metric perturbation tensor are as follows

$$\begin{aligned} h_+(t) &= \frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2\frac{(1+\cos^2\theta)}{2}\cos(2\omega t), \\ h_\times(t) &= \frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2\cos\theta\sin(2\omega t). \end{aligned} \tag{3.29}$$

A quick calculation of the orders of magnitude should convince you that these quantities are in fact small. This is the reason gravitational waves are so difficult to detect directly at the distance Earth is from any strongly generating sources. Another way of understanding this is to see how much power is radiated. A useful, close-to-home example would be the Sun-Jupiter binary system. Their masses, distance from each other, and rotational frequency of Jupiter around the Sun are

$$m_J = 2\times 10^{27}\ \text{kg},\ \ \mathrm{m_S} = 2\times 10^{30}\ \text{kg},$$
$$R = 7.8\times 10^{13}\ \text{cm},\ \ \omega = 1.68\times 10^{-7}\ \text{Hz}.$$

Upon plugging these values into our equation for the total power, we find an almost insignificant amount

$$P_{gw} = 5 \times 10^3 \,\mathrm{W}. \tag{3.30}$$

In fact, this is the same order of magnitude as the amount of electromagnetic radiation the Earth receives per square meter from the Sun. Indeed, compare the above number to the total power put out by the Sun in the form of electromagnetic radiation

$$P_{em} = 3.9 \times 10^{26} \,\mathrm{W}. \tag{3.31}$$

They differ by 23 orders of magnitude! It is safe to say that gravitational radiation is not a viable solution to the energy crisis in any case. A more interesting, and historically relevant, example of a binary system would be one with an eccentricity e. The relation for power radiation then differs by an eccentricity-dependent constant

$$P = \frac{32}{5}\frac{G^4\mu^2M^2}{a^5c^5}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \tag{3.32}$$

In 1974, Russell A. Hulse and Joseph H. Taylor Jr. found a binary system in which one of the stars was a pulsar and thus served as a reliable measure of rotational velocity of the system. Using the eccentricity, along with the values of the masses and semi-major axis

$$\begin{aligned} a &= 1.95 \times 10^{11}\,\mathrm{cm},\;\; \mathrm{m_1 = 1.441 M_{sun}}, \\ m_2 &= 1.383 M_{\mathrm{sun}},\;\; e = 0.617, \end{aligned}$$

they computed the power radiated as predicted by the theory of linearized gravity through measurements of the orbital frequency decay. The result is astonishingly accurate, at least accurate enough to have merited them the 1993 Nobel Prize in Physics. Their binary system stands as the best (indirect) evidence of the theory of gravitational radiation. Since then, several other binaries have been found which also fit the theory to high precision. The indirect evidence supporting the theory is convincing, still we have yet to detect gravitational waves directly. The next and final section will discuss the efforts that are ongoing in this search.

3.4 Experimental Hopes for the Future

Even for a binary system with large masses like the Hulse-Taylor system, the power radiated is still meager

$$P_{H-T} = 7.35 \times 10^{24}\ \mathrm{W}, \tag{3.33}$$

which is only about two percent of the EM radiation from the Sun; just another harsh reminder of how little of this weakly interacting gravitational radiation there is to detect. The weakness of the interaction has the advantage that it can come from very deep in space, but the disadvantage that it is highly difficult to detect as soon as it gets to Earth. Gravitational waves are produced by the bulk motion of large massive objects, hence we will not be able to find such radiation from an ordinary star in the foreseeable future.

Rather, possible sources include binary systems (preferably of neutron stars or black holes) and non-axially symmetric collapsing stars. In order to distinguish between the sources in a more systematic way, it is useful to classify four different bandgaps in which we might find gravitational waves[8]. The first band is known as the *high frequency band* and it covers the frequency range

$$1\ \mathrm{Hz} \leq \mathrm{f} \leq 10^4\ \mathrm{Hz}. \tag{3.34}$$

This is the frequency range in which all our current detectors operate. These detectors include LIGO, Virgo, GEO600, TAMA300, and ACIGA which are diffused all over the world[2]. The principle backbone of each of these is laser interferometry[9,10]. The detectors wish to detect the waves directly by observing a change in the proper distance between two test particles (which we have addressed previously). However, this change in distance would be too small to measure with any conventional apparatus.

Instead, one constructs two long arms which are vacuumed and contain a mirror at their far ends. Where the two arms meet there is a beam splitter, which takes laser light from a (much shorter) third arm and splits the beam such that each arm has one beam travelling through it. On their way back, the lasers enter a photodetector in a small fourth arm. If a wave would come through this detector, it would cause the mirrors to oscillate and, unless the

[2]A gravitational radiation antenna, the miniGRAIL, has also been constructed in the Netherlands at Leiden University. For more information on this nearby development, see http://www.minigrail.nl/.

incoming wave was at a very specific angle, they would be displaced by a different amount.

In the unperturbed mode, the two lasers are set such that when they meet again they completely destructively interfere. In other words, the photodetector does not detect anything. However, the influence of a wave would change the phase of the lasers (usually differently) such that photons would in fact be measured.

Since the interaction of the gravitational waves is so weak, additional partially reflective mirrors have been put in each arm so that any light travelling back from the mirror bounces back and forth on average 100 times. Thus, the phase change would get amplified by an effective two orders of magnitude, which makes a clear measurement more probable. So far, some claims of detection have been made, but none have been verified. With so many detectors around the globe, verification is quite straightforward and also allows for the reconstruction of the wave's propagation direction.

The high end of this frequency range is set by our calculations of what the largest massive bodies we know of would radiate. The low end, however, is set by experimental problems with mechanical coupling of the detector to ground vibrations at low frequencies and interference from human activity and atmospheric motions. Some of the sources we expect to radiate in this range are coalescing compact binaries, aymmetric stellar core collapses, and periodic emitters such as rotating neutron stars. Furthermore, it may be possible to detect a stochastic background which is produced by the culmination of emitters around us.

The other frequency band which is relevant for astrophysicists is the *low frequency band* which covers the frequency range

$$10^{-5}\,\mathrm{Hz} \leq \mathrm{f} \leq 1\,\mathrm{Hz}. \tag{3.35}$$

The upper limit on this range was discussed before, and thus there is no possibility of scanning this range with ground-based detectors. Instead, the solution is to use a detector which is in space far from Earth-related disturbances. No such detector has been launched yet, but there is steady development in getting LISA (the Laser Interferometer Space Antenna) up and running[6,9]. Currently, they are in the process of getting the funding to deploy such an ambitious project as well as launching LISA Pathfinder (circa 2010) which will test, on a smaller and cheaper scale, the technologies needed to construct LISA. If all goes according to schedule, LISA should be in space by 2020.

LISA works off of the same principle of laser interferometry as the ground-based detectors such as LIGO, however it possesses three arms (and thus six lasers) instead of two. This will not only allow it to detect gravitational waves, but also to triangulate them. Also, LISA is a much larger detector. Each of its arm's lengths is an approximate 5×10^6 km. They are set up in a triangle which is inclined 60 degrees with respect to Earth's plane of rotation while having a 20 degree lag with Earth itself. Thus it performs one revolution per year.

In theory, LISA will sit in one of the Lagrangian points: one of the five points in which the Earth and Sun's gravitational fields effectively cancel each other. Due to the other bodies in the Solar System, there will be microNewton thrusters on board to account for perturbations. Since these perturbations occur over a long scale of several weeks or months, they can be easily distinguished from the perturbations induced by a gravitational wave, while these are in fact much smaller.

The lower end of the range is determined by experimental bounds, such as the problem of diffraction of laser light in such long armlengths (on one passage through, the laser diffracts to over a distance of 20 km width). Nonetheless, there are many interesting sources to be found in this range. The most important one is certainly periodic emitters. While in the high frequency band the largest contributer to this group are isolated neutron stars, in the low frequency band we expect to observe primarily close proximity white dwarf binaries.

From optical observations, we know where some are and how much radiation they should be emitting (which is in this range), so LISA has to detect them otherwise something is terribly wrong. In fact, it is expected that these will act as calibrators for the detector.

Some other sources which are expected in this range are coalescing binary systems which contain black holes (which we will be able to detect to essentially the edge of the observable Universe), and LISA will aid the ground-based detectors in their search for stochastic backgrounds. Moreover, LISA may be able to detect some cosmological gravitational waves. For example, if the electroweak phase transition occured at some temperature $T \sim 100 - 1000$ GeV, then this would have generated waves in LISA's band. This follows from the fact that the peak frequency of gravitational waves produced at a phase transition is given by

$$f_{peak} \sim 100\mathrm{Hz}\left(\frac{\mathrm{T}}{10^5\mathrm{TeV}}\right). \tag{3.36}$$

The above equation has been derived under the assumption of a first-order phase transition with the collision of bubbles[3].

The other two bands, known as the *very low frequency* and *ultra low frequency* bands, are given by

$$10^{-9}\ \mathrm{Hz} \leq \mathrm{f} \leq 10^{-7}\ \mathrm{Hz}, \tag{3.37}$$
$$10^{-18}\ \mathrm{Hz} \leq \mathrm{f} \leq 10^{-13}\ \mathrm{Hz}. \tag{3.38}$$

Currently, there are no plans for detecting such waves[4], and their periodicity is of the order of months to decades or longer which makes detection difficult in any case. There is a possiblity to use millipulsars to detect waves in the very low frequency range, however not much progress has been made in this direction yet (due to the necessary length of detection). Moreover, the sources in the ultra low frequency band, as well as some in the very low frequency band, will correspond to cosmological sources to be addressed in another report and hence will be suppressed here.

[3]For a complete description of the calculation, consult *The stochastic gravity-wave background: sources and detection* by B. Allen, gr-qc/9604033.

[4]Although we do know indirectly from the Cosmic Microwave Background, specifically the BB polarization, that the lower limit of the ultra low band is set to 10^{-18} Hz. Moreover, since gravitational waves induce gradient as well as curl type polarization as opposed to scalar perturbations which only influence gradient type polarization, the detection of curl type polarization would confirm GW production by inflation.

Chapter 4

Conclusion

A linearized form of Einstein's theory of general relativity expects us to find gravitational radiation. So far, such radiation has not been detected directly, but indirect evidence as characterized by Hulse and Taylor's results seem insurmountable. Moreover, as efforts increase towards direct detection, the launch of LISA should be able to verify whether it is actually there with the use of our knowledge of close binary white dwarf systems. In fact, by LISA's launch we may have already found proof with ground-based detectors which are also constantly updating their technology and setup.

The real question though, is how this discovery will change things for us. Most obviously, we will have found a new form of radiation which not only unfolds another aspect of Nature, but also gives us a new tool at our disposal. The advantages of gravitational waves are numerous. For one, since it is weakly interacting, it can also travel from longer distances to us. Thus, if we get detections fully functioning, we can probe deeper into the Universe than ever before.

And while electromagnetic astronomy requires to focus on a small patch of sky to obtain a good image, GW astronomy is a nearly all-sky study. Then there is also the phase-coherence of waves from a source, as compared to the largely phase-incoherent EM radiation; this makes detection simpler once you have calibrated the detector correctly. Furthermore, above we have seen that the two polarizations scale in magnitude as $\frac{1}{r}$. This is different from the quantities we measure in EM radiation, such as the electric field, which typically scales as $\frac{1}{r^2}$. This implies that if you increase the sensitivity of a GW detector by a factor of two, you effectively increase its detection volume eightfold.

On a final note though, for a theoretical physicist perhaps the most valuable of all additions brought forth by this new form of radiation is that, yet again, Einstein's theory of general relativity has successfully predicted another phenomenon. This brings us one step closer to truly saying we understand Nature at large scales.

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