Lecture notes on "Topics in Non-equilibrium QFT"

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I. INTRODUCTION

In this minicourse we shall discuss some basics of non-equilibrium (quantum) physics. For simplicity, to a large extent we shall develop the relevant concepts in quantum mechanical systems, and then indicate how our results generalis to a quantum field theoretic setting. In the four lectures we shall cover the following topics:

- The density operator and entropy
- Propagators and two point functions
- The Schwinger-Keldysh or in-in formalism
- Aplications

If time permits, I will in also discuss the Boltzmann, Langevin and Fokker-Planck equations and briefly mention their limitations and applications.

II. DENSITY OPERATOR

A. Dyanmical equations in quantum mechanics

In this section we shall mostly focus on the properties of the density operator for general Gaussian states. The general gaussian states approximate well physical states in many areas of physics. This is so because, the Gaussian nature of a state is preserved by the evoltion generated by any quadratic Hamiltonians, and quadratic Hamiltonians generate dynamics that is often good approximation to the true dynamics. Indeed, most of interactions in Nature are (in some well defined sense) weak, and it is interactions and interactions only that generate non-Gaussianities [8]. For this reason we shall focus our attention to studying the gaussian density operator.

A quantum mechanical (ket) state $|\psi, t\rangle$ obeys a Schrödinger equation,

$$\imath\hbar\frac{d}{dt}|\psi, t\rangle = \hat{H}(t)|\psi, t\rangle\,, \tag{1}$$

where $\hat{H}(t)$ denotes a (time dependent) Hamiltonian and $|\psi, t\rangle$ is some state that is an element of the Hilbert space that is relevant for the corresponding problem. The Schrödinger equation (1) tells us nothing but that $\hat{H}/(\imath\hbar)$ generates time translations (just like the momentum operator $\hat{p}/(-\imath\hbar) \to \partial/\partial x$ generates translations in the x direction) in the sense that for an infinitezimal δt, from Eq. (1) we have $|\psi, t+\delta t\rangle = \{1 + [\delta t\hat{H}(t)/(\imath\hbar)](d/dt)\}|\psi, t\rangle$. Since the Hamiltonian is generally a Hermitean operator, $\hat{H}^\dagger = \hat{H}$, Eq. (1) implies that a bra state $\langle\psi, t| = (|\psi, t\rangle)^\dagger$ evolves according to,

$$-\imath\hbar\frac{d}{dt}\langle\psi, t| = \langle\psi, t|\hat{H}(t)\,. \tag{2}$$

A general solution of Eqs. (1–2) can be written in terms of the evolution operator as,

$$|\psi, t\rangle = \hat{U}(t, t_0)|\psi, t_0\rangle\,, \qquad \langle\psi, t| = \langle\psi, t|\hat{U}(t_0, t)\,, \tag{3}$$

where $\hat{U}(t_0, t) = [\hat{U}(t, t_0)]^\dagger$ and $\hat{U}(t, t_0)$ is given in terms of a time-ordered exponential (also known as the Dyson series),

$$\begin{aligned}\hat{U}(t, t_0) &= T\exp\left[-\frac{\imath}{\hbar}\int_{t_0}^{t} dt'\hat{H}(t')\right]\,,\\ &= 1 - \frac{\imath}{\hbar}\int_{t_0}^{t} dt'\hat{H}(t') + \Big(-\frac{\imath}{\hbar}\Big)^2\int_{t_0}^{t} dt'\hat{H}(t')\int_{t_0}^{t'} dt''\hat{H}(t'') + \cdots\,.\end{aligned} \tag{4}$$

In the Dyson series (4) there are no $1/n!$ of the usual exponential, and that the integrations are such that the times in the Hamiltonian factors at the left are larger than those at the right. In the limit of a time independent Hamiltonian one recovers the usual result, $\hat{U}(t, t_0) \to \exp[-(\imath/\hbar)\hat{H} \times (t - t_0)]$, where no time ordering is necessary, and where the factors $1/n!$ are recovered from an (incomplete) range of integration in (4), such that in this case the Hamiltonian factors at different times commute and the Dyson series reduces to the usual exponential function. Similarly, $\hat{U}(t_0, t)$ can be written in terms of an anti-time ordered exponential,

$$\begin{aligned}\hat{U}(t, t_0) &= \bar{T}\exp\left[\frac{\imath}{\hbar}\int_{t_0}^{t} dt'\hat{H}(t')\right]\,,\\ &= 1 + \frac{\imath}{\hbar}\int_{t_0}^{t} dt'\hat{H}(t') + \Big(\frac{\imath}{\hbar}\Big)^2\int_{t_0}^{t'} dt''\hat{H}(t'')\int_{t_0}^{t} dt'\hat{H}(t') + \cdots\,,\end{aligned} \tag{5}$$

where now the Hamiltonian factors that correspond to a later time are now pushed to the right. One can easily check by inspection that (3–5) constitute the solutions to (1–2) with the boundary condition $\hat{U}(t_0, t_0) = 0$. Furthermore, the (anti-)time ordering occurring in the above solutions are the principal reason for the Feynman (or time-ordered) propagator to be the propagator of choice for calculation in quantum field theory such as those of transition amplitudes.

B. The density operator

We can now introduce a pure density operator corresponding to a general (pure) state $|\psi, t\rangle$ as

$$\hat{\rho}_{\rm pure}(t) = |\psi, t\rangle\langle\psi, t|\,. \tag{6}$$

From (1–2) it immediately follows that the (Schrödinger picture) density operator $\hat{\rho}_{\rm pure}(t)$ satisfies the following von Neumann-Liouville equation,

$$\imath\hbar\frac{d}{dt}\hat{\rho}_{\rm pure}(t) = \hat{H}_S(t)\hat{\rho}_{\rm pure}(t) - \hat{\rho}_{\rm pure}(t)\hat{H}_S(t) \equiv [\hat{H}_S(t), \hat{\rho}_{\rm pure}(t)]\,, \tag{7}$$

where also the Hamiltonian $\hat{H}_S(t)$ is in Schrödinger picture. We now postulate that a general density operator $\hat{\rho}(t)$ (which can represent either a mixed or a pure state) obeys the same von Neumann-Liouville equation,

$$\imath\hbar\frac{d}{dt}\hat{\rho}(t) = [\hat{H}_S(t), \hat{\rho}(t)]\,, \tag{8}$$

and in general we have,

$${\rm Tr}[\hat{\rho}] = 1\,. \tag{9}$$

This normalization condition is preserved under the evolution (prove it!) and it is just saying that the probability for a particle to be in any state of the Hilbert space is equal to unity. Consequently, for any pure state ${\rm Tr}[\rho^n] = 1 (\forall n \in \mathbb{N})$ (show that!), where $\mathbb{N}$ is the set of non-negative integers. In fact one can take the following as the definition of a mixed state and of a pure state,

$$\begin{aligned} \texttt{mixed state}: &\quad {\rm Tr}[\rho^n] < 1 \quad (\forall n > 1) \\ \texttt{pure state}: &\quad {\rm Tr}[\rho^n] = 1 \quad (\forall n > 1)\,. \end{aligned} \tag{10}$$

This can be proven as follows. At any given time there must exist a basis $|i\rangle$ with respect to which $\hat{\rho}$ is diagonal (of course this basis is generally different at different times). In this basis $\hat{\rho} = \sum_i \rho_i |i\rangle\langle i|$, where $\rho_i = \langle i|\hat{\rho}|i\rangle$ and $\rho_i \leq 1$ $(\forall i)$. Now, if $\hat{\rho}$ is pure, then only one of the elements ρ_i (say j) does not vanish and $\rho_i = \delta_{ji}$, *i.e.* $\delta_j = 1$ (here for simplicity we neglect possible complications due to a non-discrete nature of indices i). If, on the other hand, $\hat{\rho}$ is mixed, then $\rho_i < 1 (\forall i)$. This follows from $\sum_i \rho_i = 1$ and from the fact that at least two elements ρ_i do not vanish. Now, $\rho_i < 1 (\forall i)$ and $\sum_i \rho_i = 1$, immediately imply that $\mathrm{Tr}[\hat{\rho}^n] = \sum_i \rho_i^n < 1 (\forall n > 1)$. Since $\mathrm{Tr}[\hat{\rho}^n]$ is independent of basis, this must be true in genral, completing the proof. An analogous consideration shows that, when $0 < n < 1$, $\mathrm{Tr}[\hat{\rho}^n] > 1$.

The von Neumann-Liouville equation (7) has a different sign (see Problem 1) than the Heisenberg operator equation, which is the evolution equation for any Hermitean operator,

$$\frac{d}{dt}\hat{\mathcal{O}}_H(t) = \frac{\imath}{\hbar}[\hat{H}_H(t), \hat{\mathcal{O}}_H(t)] + \frac{\partial}{\partial t}\hat{\mathcal{O}}_H(t)\,. \tag{11}$$

As opposed to the von Neumann equation (7) the Heisenberg equation is written in Heisenberg picture, which also means that the Hamiltonian is in Heisenberg picture. The partial derivative in the last term in (11) acts on the operator $\mathcal{O}_S$ in Schrödinger picture, and thus it vanishes if $\mathcal{O}_S$ is time independent (a time dependence of $\mathcal{O}_S$ can emerge *e.g.* through time dependent backgrounds or if $\mathcal{O}_S$ contains couplings to time dependent sources). It is easy to represent general solutions of Eqs. (8–11) in terms of the evolution operator (3–5) as,

$$\hat{\rho}(t) = \hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}(t_0,t)\,, \qquad \hat{\mathcal{O}}_H(t) = \hat{U}(t_0,t)\hat{\mathcal{O}}_S(t)\hat{U}(t,t_0)\,. \tag{12}$$

Both $\hat{U}(t,t_0)$ and $\hat{U}(t_0,t)$ appear in the solutions for $\hat{\rho}(t)$ and $\hat{\mathcal{O}}_H(t)$, such as both time ordering and anti-time ordering is involved in constructing a general evolution of the density operators and of observables (which are represented by Hermitean operators). This will be used in section IV, where we discuss the Schwinger-Keldysh, or in-in formalism. The solutions (12) generate identical (picture independent) evolution of physical observables. Indeed, a physical observable $\mathcal{O}(t)$ is given in Schrödinger picture by,

$$\mathcal{O}(t) = \mathrm{Tr}\left[\hat{\rho}(t)\mathcal{O}_S(t)\right]\,, \tag{13}$$

while in Heisenberg picture the relevant formula is,

$$\mathcal{O}(t) = \mathrm{Tr}\left[\hat{\rho}(t_0)\mathcal{O}_H(t)\right]\,. \tag{14}$$

The results (13) and (14) are obviously identical to each other (because of (12) and because operators under a trace can be cyclically moved without changing the result).

An important example is a thermal density operator, which is equal to

$$\hat{\rho}_{\mathrm{th}} = \frac{\mathrm{e}^{-\beta\hat{H}}}{\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}}\right]} \tag{15}$$

Obviously, $\hat{\rho}_{\rm th}$ is meaningfully defined only for time independent Hamiltonians, and $\hat{\rho}_{\rm th}$ is itself independent of time. This must be so because thermal states are time translation invariant. In many applications however, (15) is taken to be the density operator of a thermal state, even if the Hamiltonian is time dependent. This is justified provided the rate of the interactions Γ that are responsible for thermalisation is much larger than the rate of change of the Hamiltonian, *i.e.* when $\Gamma \gg (d/dt)\ln(H(t))$. This is known as adiabatic limit, and it is often used for approximate calculations in cosmology (thermal state of the plasma in the early Universe setting) and in condensed matter systems (when one adiabatically changes the temperature through a phase transition; the opposite limit is known as quench and the corresponding evolution cannot be approximated by the thermal density operator (15)).

C. Evolution Generated by Quadratic Hamiltonian

Let us consider evolution of a particle in one spatial dimension generated by the following general quadratic Hamiltonian (in Heisenberg picture),

$$\hat{H}(t) = \frac{1}{2}\left[A(t)\hat{q}^2 + B(t)\hat{p}^2 + C(t)\{\hat{q},\hat{p}\} + 2D(t)\hat{q} + 2E(t)\hat{p}(t)\right] , \tag{16}$$

Due to the Hermiticity of the Hamiltonian, $A(t), B(t), C(t), D(t)$ and $E(t)$ are real functions of time. Note that the term $\imath F(t)[\hat{q},\hat{p}] = -\hbar F(t)$ does not constitute an independent contribution to $\hat{H}$. For example, in the case of a simple harmonic oscillator (SHO), $A = m\omega^2$, $B = 1/m$, $C = D = E = 0$, and m and ω are the (time independent) particle's mass and ω is the frequency. Another example is a charged particle that couples to an external electromagnetic field $A^\mu(t,\vec{x}) = (\phi_C, \vec{A})$. In this case the interaction time dependent and Lorentz invariant,

$$\hat{H}_{\rm int}(t) = \frac{e}{c}\left[\gamma\vec{v}\cdot\vec{A} - \phi_C\gamma\right] , \tag{17}$$

where $\vec{v} = d\vec{x}/dt$, $\gamma = (1-\vec{v}^2/c^2)^{-1/2}$, c is the speed of light and e the electric charge. In the non-relativistic limit and in one dimension the coupling (17) reduces to a bi-linear coupling, $\hat{H}_{\rm int}(t) \to -(e/c)(dx/dt)A$, such that in this case $E(t) \approx -eA/(mc)$.

In this subsection we discuss what kind of evolution is generated by the Hamiltonian (16). In particular we shall see how an initial Gaussian state changes due to the evolution induced by (16). The meaning of a general Gaussian state is made more precise below. The Hamiltonian (16) implies the following Heisenberg equations,

$$\begin{aligned}\dot{\hat{q}} &= \widehat{\partial_p H} = C\hat{q} + B\hat{p} + E \\ \dot{\hat{p}} &= -\widehat{\partial_q H} = -A\hat{q} - C\hat{p} - D\,,\end{aligned} \tag{18}$$

where here and in the rest of the lecture notes, $\dot{} = d/dt$. Let us now shift $\hat{q}$ and $\hat{p}$ as follows,

$$\hat{q} = \Delta\hat{q} + \bar{q}\,, \qquad \hat{p} = \Delta\hat{p} + \bar{p} \tag{19}$$

where

$$\bar{q}(t) = \langle\hat{q}\rangle \equiv \mathrm{Tr}[\hat{\rho}_S\hat{q}]\,, \qquad \bar{p}(t) = \langle\hat{p}\rangle \equiv \mathrm{Tr}[\hat{\rho}_S\hat{p}] \tag{20}$$

are the expectation values of $\hat{q}$ and $\hat{p}$, which obey the classical equations of motion and $\hat{\rho}_S$ is the density operator in Schrödinger picture. When $\hat{\rho}_S$ is pure, the following simplified equations, Eqs. (20) reduce to the usual formulae, *e.g.* $\mathrm{Tr}[\hat{\rho}_S\hat{q}] \rightarrow \langle\psi, t_0|\hat{q}(t)|\psi, t_0\rangle$. Indeed, inserting (19) into (18) yields

$$\frac{d\Delta\hat{q}}{dt} = C\Delta\hat{q} + B\Delta\hat{p} \tag{21}$$

$$\frac{d\Delta\hat{p}}{dt} = -A\Delta\hat{q} - C\Delta\hat{p}\,, \tag{22}$$

where we assumed,

$$\begin{aligned}\dot{\bar{q}} &= C\bar{q} + B\bar{p} + E \\ \dot{\bar{p}} &= -A\bar{q} - C\bar{p} - D\,.\end{aligned} \tag{23}$$

We have thus shown that the operators for fluctuations $\Delta\hat{q}$ and $\Delta\hat{p}$ obey simplified equations, with D and $E(t)$ set to zero. Conversely, the role of the terms $D(t)\hat{q}+E(t)\hat{p}(t)$ in the Hamiltonian (16) is to shift the expectation value of $\hat{q}$ and $\hat{p}$. When $\hat{q}$ and $\hat{p}$ are operators whose fluctuations are Gaussian (the meaning of which is made more precise below), then the linear terms in the Hamiltonian generate a displacement (shift) in $\hat{q}$ and $\hat{p}$. The quantum states describing motion of a quantum particle with non-vanishing $\bar{q}$ and $\bar{p}$ are known as *coherent states*, and were originally introduced by Glauber [1, 2] to study the effects of a charged distribution on the photon (vacuum) state, and their properties are summarized in Appendix A. Essentially, a coherent state is defined as the eigenstate of the annihilation operator $\hat{a}$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\,, \tag{24}$$

where, for a SHO, $\hat{a} = \sqrt{m\omega/(2\hbar)}\,[\hat{q} + \imath\hat{p}/(m\omega)]$ and $\alpha = \sqrt{m\omega/(2\hbar)}\,[\bar{q} + \imath\bar{p}/(m\omega)]$. Glauber observed that the canonical coupling of a photon field $\hat{A}^\mu(t,\vec{x}) = \int d^3x e^{\imath\vec{k}\cdot\vec{x}}\,\hat{A}^\mu(t,\vec{k}\,)$ to a (classical) distribution of moving charges $j^\mu(t,\vec{x}) = (c\rho,\vec{j})(t,\vec{x}\,) = \int d^3x e^{\imath\vec{k}\cdot\vec{x}}\,j^\mu(t,\vec{k}\,)$,

$$\hat{H}_{\rm int}(t) = \frac{1}{c^2}\int d^4x j_\mu A^\mu = \frac{1}{c^2}\int \frac{d^3k}{(2\pi\hbar)^3}\Big[-\hat{A}^0(t,\vec{k}\,)\rho(t,\vec{k}\,) + \hat{A}^i(t,\vec{k}\,)j^i(t,\vec{k}\,)\Big] \tag{25}$$

is such to generate a light in a coherent state, making coherent states a very natural description for photons in laser beams, and thus has a very large number of applications in quantum optics. Of course, since electromagnetism is a field theory, one can represent a coherent photon state as a product of the momenta of a coherent state for a photon of a given polarization $p = \pm$ and momentum, $|\alpha_{\vec{k},p}(t)\rangle$, where $\hat{A}^\mu(t,\vec{k}\,) = \sum_{p=\pm}\epsilon^\mu_p(\vec{k})\hat{A}^p(t,\vec{k})$, where $\epsilon^\mu_p(\vec{k})$ are the two photon polarization vectors. Of course, as light propagates through a medium (such as air, water or various types of transparent materials such as glasses), the beam photons will scatter off particles in the medium, thus decohering the light beam. These type of effects cannot be described within the formalism of coherent states, but they *can* be described by the more general Gaussian states which also contain information about the phase decoherence between individual photons in the beam. The Gaussian density operator introduced below is suitable for the description of photons in a partially decohered laser beam.

An interesting question is what kind of physical effects can be produced by the remaining terms (A,B,C) in Eqs. (21–22). These time dependent terms can in general produce effect of *squeezing*, and for that the theory of squeezed states has been developed (for a review see Refs. [3, 4] and for an interesting application to cosmology consult [5]). In general one can say that squeezed states are those which exhibit correlations between $\hat{x}$ and $\hat{p}$, *i.e.* those for which the correlator $\langle\{\Delta\hat{q},\Delta\hat{p}\}\rangle$ does not vanish. From Eqs. (26–28) below it follows that squeezing is dynamically produced if $A(t)\langle(\Delta\hat{q})^2\rangle \neq B(t)\langle(\Delta\hat{p})^2\rangle$, which will be quite generally the case when either A, B or C are time dependent. Some basic properties of *squeezed states* are presented in Appendix B, and their applications range from quantum optics (when laser beams propagate through transparent, anisotropic media) to early cosmology (where they are used to describe (scalar and tensor) cosmological perturbations, see *e.g.* [5]). Squeezed states are characterized by the squeeze factor $r(t)$ (roughly speaking $e^{r(t)}$ measures the size of quantum fluctuations in $\Delta\hat{q}$ and in $\Delta\hat{p}$) and the squeeze phase $\phi(t)$ (which is the time dependent phase along which the squeezing occurs). Squeezed states are pure states, which means that (at each moment in time) there exists an axis which defines $\Delta\hat{q}'$ and $\Delta\hat{p}'$ and along which $\Delta q'\Delta p' = \hbar/2$. Squeezed coherent states represent the most general Gaussian pure states. In order to characterise the most general Gaussian state we need another parameter, which characterises state impurity, and that we discuss next.

By inspecting the Heisenberg equations (21–22) for $\Delta\hat{q}$ and $\Delta\hat{p}$, one may observe that they can be equivalently recast as a set of *three* Heisenberg equations for the (composite, equal time) correlators $\hat{q}^2$, $\hat{p}^2$ and $\{\hat{q},\hat{p}\}$ (of course, the commutator $[\hat{q},\hat{p}]=\imath\hbar$ is trivial and needs no special consideration). These equations are,

$$\begin{aligned}\frac{d(\Delta\hat{q})^2}{dt} &= 2C(\Delta\hat{q})^) + B\{\Delta\hat{q},\Delta\hat{p}\} &\quad (26)\\ \frac{d(\Delta\hat{p})^2}{dt} &= -A\{\Delta\hat{q},\Delta\hat{p}\} - 2C(\Delta\hat{p})^2 &\quad (27)\\ \frac{d\{\Delta\hat{q},\Delta\hat{p}\}}{dt} &= -2A(\Delta\hat{q})^2 + 2B(\Delta\hat{p})^2\,. &\quad (28)\end{aligned}$$

The first equation is obtained by multiplying (21) from the left by $\hat{q}$ and from the right by $\hat{p}$ and adding the two; an analogous procedure gives the other two equations. Eqs. (26–28) are the Heiselberg picture operator equations, so they are identical to those written for the corresponding expectation values (which are obtained by multiplying with a time independent $\hat{\rho}_S(t)=\hat{\rho}_S(t_0)$ and taking a trace. At first it seems puzzling that we have started with *two* Heisenberg equations for the linear canonical operators and got an equivalent system of *three* equations for quadratic operators. So, one equation must be redundant. This is indeed so, and an inspection of Eqs. (26–28) shows that they contain a *conserved quantity*, namely,

$$\frac{d\Delta(t)^2}{dt}=0 \qquad \left(\frac{\hbar\Delta(t)}{2}\right)^2 = \langle(\Delta\hat{p})^2\rangle\langle(\Delta\hat{p})^2\rangle - \left[\left\langle\frac{1}{2}\{\Delta\hat{q},\Delta\hat{p}\}\right\rangle\right]^2 \qquad (29)$$

$\Delta(t)$ is known as the Guassian invariant of a (Gaussian) state. An interesting question is of course, what is the physical meaning of Δ. In order to answer this question, we have to introduce a new concept of (Gaussian) entropy for quantum systems. In passing we note that the form of $\Delta(t)$ in Eq. (29) suggest that it has something to do with the amount of fluctuations in a quantum system. Indeed, one can show that for a general Gaussian state, $\hbar\Delta/2$ is limited from below by $\hbar/2$. More precisely, one can show that for pure states $\Delta=1$ and for mixed states $\Delta>1$. Thus we can formulate the following *generalized uncertainty relation*,

$$\begin{aligned}\texttt{pure state}:\ \langle(\Delta\hat{p})^2\rangle\langle(\Delta\hat{p})^2\rangle - \left[\left\langle\tfrac{1}{2}\{\Delta\hat{q},\Delta\hat{p}\}\right\rangle\right]^2 &= \frac{\hbar^2}{4}\\ \texttt{mixed state}:\ \langle(\Delta\hat{p})^2\rangle\langle(\Delta\hat{p})^2\rangle - \left[\left\langle\tfrac{1}{2}\{\Delta\hat{q},\Delta\hat{p}\}\right\rangle\right]^2 &> \frac{\hbar^2}{4}\,. \qquad (30)\end{aligned}$$

Below we show that these relations are correct. But before we do that, we need to introduce the concept of entropy.

D. Entropy

Eq. (30) is helpful, in that it tells us that Δ can be used to parametrize state purity, and since entropy is a measure of (im-)purity of a state, it is reasonable to demand that a definition of entropy for quantum states should satisfy the following properties:

- entropy S should be a function of Δ, $S = S(\Delta)$;
- entropy of a pure state ($\Delta = 1$) should vanish;
- entropy of a mixed state ($\Delta > 1$) should be strictly positive;
- entropy should be a monotonically increasing function of Δ.

In fact, such a definition exists. Namely, inspired by the Boltzmann's definition of entropy, $S = -f\ln(f)$ in terms of the (classical) phase space distribution function $f(q, p\,; t)$, von Neumann introduced the following definition of entropy for quantum systems,

$$S_{\rm vN} = -\langle \ln(\hat\rho)\rangle = -{\rm Tr}\left[\hat\rho\ln(\hat\rho)\right]\,. \tag{31}$$

We shall see that the von Neumann entropy of a Gaussian state satisfies all of the requirements mentioned above. While the definition (31) seems appealing, is is not as such very useful, since it is conserved, *i.e.* it carries no dynamical information about the system. This can be shown by making use of Eq. (8) and performing some simple operations which are legitimate under a trace,

$$\frac{d}{dt}S_{\rm vN} = -{\rm Tr}\left[\frac{d\hat\rho(t)}{dt}\ln(\hat\rho) + \frac{d\hat\rho(t)}{dt}\right] = \frac{\imath}{\hbar}{\rm Tr}\left\{[\hat H_S(t)\hat\rho(t) - \hat\rho(t)\hat H_S(t)](\ln(\hat\rho)+1)\right\} = 0\,, \tag{32}$$

such that $S_{\rm vN}$ =const. Since this holds for general systems, it must also hold for Gaussian systems of interest for us. The von neumann entropy has some of the right properties. Namely, when viewed in a diagonal basis $|i\rangle$, Eq. (31) implies, $S_{\rm vN} = \sum_i \rho_i \ln(1/\rho_i)$. Since all ρ_i are positive and less than unity, $0 \le \rho_i \le 1$, $S_{\rm vN} \ge 0$, and moreover $S_{\rm vN} = 0$ for a pure state (for which $\rho_i = \delta_{ji}$ for some j).

E. General gaussian states

The density operator of a *general Gaussian state* is of form,

$$\hat\rho(t) = \frac{1}{Z}\exp\left\{-\frac{1}{2}\left[\alpha(t)\hat q^2 + \beta(t)\{\hat q,\hat p\} + \gamma(t)\hat p^2 + \delta(t)\hat q + \eta(t)\hat p\right]\right\}\,, \tag{33}$$

where here $\hat{q}$ and $\hat{p}$ are the Schrödinger picture operators and $\alpha(t), \beta(t), \gamma(t), \delta(t)$ and $\eta(t)$ are real functions of time (the reality follows immediately from $\hat{\rho}^\dagger = \hat{\rho}$). The norm Z (also known as the partition function) is chosen such that $\mathrm{Tr}[\hat{\rho}] = 1$.

GAUSSIAN STATES:

[6] [7]

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[1] R. J. Glauber, "The Quantum theory of optical coherence," Phys. Rev. **130** (1963) 2529.

[2] R. J. Glauber, "Coherent and incoherent states of the radiation field," Phys. Rev. **131** (1963) 2766.

[3] C. M. Caves and B. L. Schumaker, "New formalism for two-photon quantum optics. 1. Quadrature phases and squeezed states," Phys. Rev. A **31** (1985) 3068.

[4] B. L. Schumaker and C. M. Caves, "New formalism for two-photon quantum optics. 2. Mathematical foundation and compact notation," Phys. Rev. A **31** (1985) 3093.

[5] A. Albrecht, P. Ferreira, M. Joyce and T. Prokopec, "Inflation and squeezed quantum states," Phys. Rev. D **50** (1994) 4807 [astro-ph/9303001].

[6] T. Prokopec, M. G. Schmidt, J. Weenink, M. G. Schmidt and J. Weenink, "The Gaussian entropy of fermionic systems," Annals Phys. **327** (2012) 3138 [arXiv:1204.4124 [hep-th]].

[7] J. F. Koksma, T. Prokopec and M. G. Schmidt, "Decoherence in Quantum Mechanics," Annals Phys. **326** (2011) 1548 [arXiv:1012.3701 [quant-ph]].

[8] Of course, there are important examples of strongly interacting physical theories, notable examples are the low-energy limit of QCD in high energy physics and the Hubbard model in condensed matter applications. Even in these cases Gaussian approximations may be good, provided one correctly identifies the dynamical variables of the strongly interacting theory. E.g. in QCD the low energy degrees of freedom are the (composite) mesons and baryons.