Theoretical Physics Student Seminar 2008-2009 Alternative Theories of Gravity: Loop Quantum Gravity and Loop Quantum Cosmology

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Abstract

Quantizing gravity and unifying it with the other fundamental interactions has been the focus of theoretical physics research in the last couple of decades. Among different attempts, numerous theories have been suggested, some representing a more conservative approach, while others introduce new underlying fundamental theories, which reduce to known physics in certain limits. Loop Quantum Gravity (LQG) represents the former approach, being a conservative canonical Dirac quantization of general relativity. This review will attempt to describe the path that led to LQG in its current form and will later discuss the applications of this theory in FLRW background, which goes under the name of Loop Quantum Cosmology (LQC).

Contents

1 Introduction

Ever since the second decade of the twentieth century, general relativity changed the way we view gravity and replaced the Newtonian picture with a theory of geometry of spacetime. This new approach explained a number of problems with the preexisting theory and made a number of predictions, some of which, like gravitational radiation, we still need to experimentally verify. Over years, the confidence in the theory has been strengthened by wealth of experimental evidence.

The theory, however, also contributed a new fundamental problem, that of singularity. The Penrose-Hawking singularity theorems state that from an arbitrary distribution of mass a singularity must exist be it in a future or the past. Some obvious examples include black holes, or the Big Bang, which for an expanding universe, initially must have been a point with an infinite mass density from a classical theory point of view.

The existence of singularities was one of the reasons why physicists set out to find a quantum theory of gravity, hoping that the singularity problem will be fixed, alike the Bremsstrahlung caused atom stability problem has been solved by a quantum theory of the atom. Moreover such a theory could explain some of the CMB anisotropies that are observed, as well as the need for an introduction of the inflationary era in the Universe.

This review is divided into seven sections. Following the introduction, the second section presents the rational behind the necessity for gravity quantization. Subsequently the third section provides a framework, in which a quantum theory of gravity could be written. However, once quantized, this formalism does not solve the singularity problem. This will call for a reformulation of the general relativity in terms of the gauge formalism, which will be done is the fourth section, and the fifth section describes the quantization scheme that leads to what is now known as Loop Quantum Gravity. The sixth section focuses on the LQG application to the Friedmann-Lemaître-Robertson-Walker metric, where also the outcome will be compared to the results of a similar application to the quantum theory from the third section. As always this review will end with a conclusion.

2 Why do we need to quantize gravity?

Many believe that general relativity could just remain a classical theory and that there is no need for quantizing it. However there exist a number of problems with the current theory, that could be resolved if the theory was indeed quantized. In this section, we will address this issue, focusing on the structure of the Einstein's equations, and the interplay between the general relativity singularities with the QFT divergences.

2.1 Quantum Field Theory – General Relativity Interplay

On a number of occasions in quantum field theory the one-loop Feynman diagrams produce momentum integrals which are divergent. In a number of cases these infinities can be removed provided that our field theory is *renormalisable*, however if this is not the case, i.e. for nonrenormalisable theories, hen our theory requires additional counter-terms of order increasing with the order of the perturbation theory. On the other hand virtual particles "travelling" in those loops have spatial extend of Compton radius $\lambda = \frac{\hbar}{p}$ and mass $m \approx \frac{E}{c^2}$. If their momentum increases, their spatial extent decreases and as it approaches the Schwarzshield radius, $\lambda \to R_s = \frac{G_n E}{c^4}$, then virtual particle turns into a decaying black hole. From a phenomenological point of view this is rather problematic, because the particle might change its physical properties, e.g. an electroweakly interacting electron can radiate all kinds of particles via Hawking radiation, which are neither observed nor predicted by the minimal Standard Model. For this reason proponents of quantum gravity believe that quantized gravity would provide a hope for a cut-off length scale and fix the QFT divergences as well as keep the black holes from appearing – analogously to a solution to the atom stability problem, where the ground state energy level is introduced beyond which an electron cannot fall in the Bohr atomic model. This argument, however, is incomplete, as it assumes that we have a working quantum theory of off-shell particles responding to a Plank-scale size black hole.

2.2 Problems with Einstein's Equations

General Relativity is a theory of the geometrical structure of spacetime described by a metric tensor $g_{\mu\nu}$. Given a distribution of matter described by the stress-energy tensor $T_{\mu\nu}$, and can use Einstein's equations to find the spacetime's geometry. These equations read

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{Geometry} - \text{classical}} = \underbrace{\frac{8\pi G_n}{c^3} T_{\mu\nu}(g)}_{\text{Matter} - \text{Gauge Fields in QFT}}, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor and $R = R^{\mu}{}_{\mu}$ is its trace, also known as the Ricci scalar. One can argue that the left hand side and the right hand side of the Einstein equations are not consistent, due to the fact that while the left hand side is described by a classical theory, while the right hand side is described by the matter content of the spacetime in consideration. Thus far, quantum gauge field theories have been extremely successful in describing matter and its (non-gravitational) interactions, i.e. the three fundamental forces: electromagnetism, the weak and the strong nuclear force. Since we are only capable to do interacting quantum field theories in perturbation theory, these fields are subject to quantum fluctuation, which makes the left hand side of (1) distinct from the right hand side i.e. formally functions of space-time and operators on a Hilbert space are two different objects. The obvious intuitive solution to this problem would be then to simply consider the vacuum expectation value of the fields enclosed by the stress energy tensor with respect to some arbitrary background g_0, thus changing the form of the above equation to its "quantized" version

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_n}{c^3} \langle\, T_{\mu\nu}(g_0) \rangle,$$

where g_0 is an arbitrary background metric, which we can initially choose to be for example Minkowski spacetime $\eta_{\mu\nu}$. Obviously in general the notion of the vacuum depends on the choice of g_0. Due to vacuum fluctuations (e.g. particle-antiparticle pair creation-annihilations), the right hand side is non-vanishing, yielding a space-time solution g_1. One could attempt to iterate this process, hoping that the result would converge. Flanagan and Wald [1],however, have argued that generically this approach leads to "run - away" solutions.

Alternatively, we could require that we should promote our geometrical tensors to operators acting on some state:

$$\left(\hat{R}_{\mu\nu} - \frac{1}{2} \hat{g}_{\mu\nu} \hat{R} \right) |\Psi\rangle = \frac{8\pi G_n}{c^3} \hat{T}_{\mu\nu}(\hat{g}) |\Psi\rangle$$

and this is the path that we will initially try to follow i.e. we will canonically quantize general relativity. However, before we attempt to do that we will first take a closer look at to what extent General Relativity and Quantum Mechanics are compatible with each other.

3 Creating a "Quantizable" Structure of General Relativity

3.1 The Problem of Time in Quantum Gravity

Ever since the development of special and later general relativity, our understanding of what "time" means has changed forever. The Newtonian notion of time as an absolute has been replaced with time being just another coordinate that can be exchanged with space upon coordinate transformations. Quantum theory, however, still relies on the Newtonian view, where time is understood as the distinct evolution variable, therefore one of the first problems that one faces, when trying to merge the two theories together, is that of time. In this section we will look at this issue in more detail.

In quantum mechanics time plays two fundamental roles. Firstly, it allows us to determine the choice of canonical positions and momenta, since when we have an evolution parameter we can define momenta as derivatives with respect to *time* derivatives of positions, i.e. $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$. Without the fixed notion of what particular time direction we pick we are incapable to define canonical momenta, and thus we encounter a major obstacle on our way to write down a quantum theory in operator formalism. Note that this is different to Lorentz invariance in quantum field theories, as the symmetry group of general relativity allows for any local coordinate transformations. Secondly, time allows us to fix the normalisation of a wave function, which requires some specific point in time when we do that. Assuming unitary evolution this normalisation will not change, however we need a fixed point in time, that cannot be altered by coordinate transformations.

This double role appears to stand in a direct conflict with the role of time in GR, where time no longer describes the evolution, but is rather one of the coordinates on a $(3+1)$ dimensional manifold in a diffeomorphism invariant theory. Since there is no preferred choice of what time actually *means*, the spacetime metric can be recast in any form up to the researchers convenience. i.e. in GR jargon this means that "there is no preferred time slicing". With this in mind we clearly see that unless we choose some time direction ourselves, we will not be able to define canonical momenta or fix normalisation of our wave functions, as under coordinate transformations our time-parameter might be interchanged with any of the space directions.

Related to time, is the issue of causality. In quantum field theory one requires that any observables operators at points separated by space-like intervals must commute. However is one allows that the metric itself is subject to quantum fluctuations, then the notion of space-, time- or light-like separations is no longer clear, and these fluctuations may exchange past and future [2]. For these reasons, in order for any progress to be made we are forced to choose a particular time slicing and only then quantize the theory.

3.2 The Arnowitt-Deser-Misner (ADM) Formalism

This problematic issue of the special role of time in quantum mechanics versus arbitrariness of time in general relativity has been resolved by Arnowitt, Deser and Misner, in what goes under the name ADM formalism [3]. It states, that given a manifold $\mathcal{M}$, with the topology $\mathbb{R} \times \sigma$, then, by fixing some time coordinate function, one can define constant time surfaces $t(x^i) = \text{const}$, and define a normal vector perpendicular to it $n_a = -N\partial_a t$, where N is allows us to normalise the vector. In what follows we will use the Latin alphabet $a, b, c \dots$ as arbitrary labels, $i, j, k, \dots$ for space coordinates, and $\mu, \nu, \sigma, \rho, \dots$ for spacetime coordinates.

The result is a foliation of a four-dimensional manifold $\mathcal{M}$ into a set of surfaces Σ on which one can write down a consistent quantum theory. Having picked a time coordinate, we have defined the way our slices look like, whose geometry can be described by the spatial metric q_{ab} given by

$$q_{ab} = g_{ab} - n_a n_b,$$

such that $q_{ab} n^b = 0$ and for a vector field tangent to Σ surface $V^a n_a = 0$ we get $q_{ab} V^a = g_{ab} V^a$. Now we can give a geometrical interpretation to N, a lapse function, which is just a measure of the separation between the constant time slices. With the coordinate transformation on the slice $x^i + dx^i$ and between the slices $x^i - N^i dt$ where N^i is a shift vector, we can write our invariant spacetime interval

$$ds^2 = N^2 dt^2 - q_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

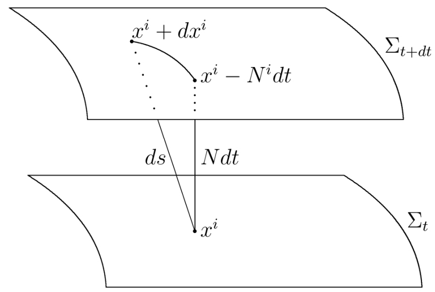

Figure 1: The representation of the ADM formalism and the corresponding line element in the foliated spacetime. It is straight forward to see that based on this diagram we obtain the form of the metric (2). Figure taken from [4].

such that we can just read off the form of an arbitrary spacetime metric under the ADM decomposition

$$g_{\mu\nu} = \begin{pmatrix} N^2 - q_{ij}N^iN^j & N_i \\ N_j & q_{ij} \end{pmatrix} \quad \text{and} \quad g^{\mu\nu} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N_i}{N^2} \\ -\frac{N_j}{N^2} & -q^{ij} + \frac{N^iN^j}{N^2} \end{pmatrix}.$$

Having the spatial metric q_{ab}, we can define a three-dimensional covariant derivative D_a, with the spatial metric compatible connection

$$D_e T^h_{fg} = q^a_e q^b_f q^c_g q^h_d \nabla_a T^d_{bc},$$

and three-dimensional extrinsic curvature K_{ab}

$$K_{ab} = q^c_a \nabla_c n_b,$$

which upon inserting definition of the normal vector n_b takes the form

$$K_{ij} = \frac{1}{2N}\left(\partial_t q_{ij} - D_i N_j - D_j N_i\right).$$

We needed to introduce all of these quantities, so that we can write down the four-dimensional Riemann tensor ${}^{(4)}R_{abcd}$ in terms of the extrinsic curvature K_{ab} and the Riemann tensor defined on the three-dimensional slice Σ_t denoted by ${}^{(3)}R_{abcd}$, such that

$$D_{[a}D_{b]}V_c = q^f_a q^d_b q^{e(4)}_c R^g_{fde} + K_{ac}K^g_b V_g - K_{bc}K^g_a V_g \equiv^{(3)} R^g_{abc} V_g \tag{2}$$

3.3 Action Formulation of General Relativity

In general any action can be written as a spacetime integral of a Lagrangian density $\mathcal{L} = \mathcal{L}(q, \dot{q})$, which can be rewritten in the Hamiltonian formulation using the Hamiltonian density $\mathcal{H} = \mathcal{H}(p, q)$ as

$$S = k \int_{\mathcal{M}} d^n x \mathcal{L} = k \int_{\mathcal{M}} d^n x \left(p\dot{q} - \mathcal{H}(p, q) + \sum_i \lambda_i C_i (p, q) \right),$$

where λ_i are the Lagrange multipliers, the non-dynamical the variables that introduce constraints $C_i (p, q)$ into our theory, as a result of equations of motion for λ_i. Let us now switch our attention to the Einstein-Hilbert action, given by

$$S_{EH} = \frac{1}{\kappa} \int_{\mathcal{M}} d^4 x \sqrt{g}^{(4)} R.$$

Let us now implement the ADM formalism in this action. We can write the curvature scalar as

$${}^{(4)}R =^{(3)} R - K_{ab} K^{ab} + K^2 + 2\nabla_a \left(n^b \nabla_b n^a - n^a \nabla_b n^b \right),$$

which makes the Einstein-Hilbert action become

$$S_{EH} = \frac{1}{16\pi G_N} \int d^4 x N \sqrt{q} \left({}^{(3)}R - K_{ab} K^{ab} + K^2 \right),$$

where the derivative terms from (3) in (3) become the boundary terms and do not contribute to the dynamics of the action. Choosing q_{ab} as the canonical position variable, we can define the canonical momentum in terms of the derivative of the Lagrangian density with respect to the time derivative of canonical position

$$\pi^{ab} = \frac{\partial \mathcal{L}}{\partial (\partial_t q_{ab})} = \frac{1}{16\pi G_N} \left(q^{ab} K - K^{ab} \right),$$

where we now automatically obtain the standardized Poisson bracket between the canonical positions and momenta

$$\{q_{ij}(x), \pi^{kl}(x')\}|_{t=t'} = \delta^k_{(i} \delta^l_{j)} \tilde{\delta}^{(3)} (x - x').$$

Note that here the subscript (ij) denotes the symmetrisation of the tensors, which comes about from the q_{ab}, and thus π_{ab}, being a symmetric tensor. Additionally note that here, following the convention in the literature the tilde sign denotes a densitised delta function, such that $\int d^3 x \sqrt{q} \tilde{\delta}^{(3)} (x - x') = 1$, such that the factor of $\sqrt{q}$ appears here, rather than in the definition of the canonical momentum. The action in the Hamiltonian formalism in terms of the canonical position and momentum becomes

$$S = \int d^4 x \pi^{ab} \partial_t q_{ab} - \underbrace{N^a \mathcal{H}_a - N\mathcal{H}}_{\text{constraints}},$$

where the terms $P^j_{N_i} \dot{N}_j$ and $P_N \dot{N}$ are missing, because the Lagrangian in its initial form did not contain any time derivatives of the shift vector or the lapse function, which automatically leads to their associated canonical momenta being zero. Additionally note that as a result of that N^a and N act as Lagrange multipliers, which introduce two constraints into the action: the momentum constraint

$$\mathcal{H}_a = -2D_b\pi^b_a = 0,$$

and the Hamiltonian constraint

$$\mathcal{H} = \frac{16\pi G_N}{\sqrt{q}}\left(\pi^{ab}\pi_{ab} - 1/2\pi^2\right) - \frac{\sqrt{q}}{16\pi G_N}{}^{(3)}R = 0,$$

also frequently denoted in literature by $\mathcal{H}_\perp$.It is easy to check that these are first class constraints, i.e. ones that obey

$$\{C_i, C_j\} = \sum_k f_{i,j,k}C_k, \tag{3}$$

where C_i denotes a constraint and $f_{i,j,k}$ are some smooth functions, and where the equation implies that the constraints Poisson commute on the constrained subspace. In general in any constrained Hamiltonian system, the first class constraints generate gauge transformations [4], which can be seen by their action on the field variables. For example consider a generator of the kind

$$G[\xi_i] = \int d^3x\xi_i\mathcal{H}^i = -2\int d^3x\xi_i D_j\pi^{ij} = \int d^3x\left(D_j\xi_i + D_i\xi_j\right)\pi^{ij}, \tag{4}$$

then it is easy to verify that

$$\{G[\xi_i], q_{jk}\} = -\left(D_j\xi_k + D_k\xi_j\right) \equiv -\mathcal{L}_\xi q_{jk}, \tag{5}$$

where the last step is just a definition of the Lie derivative, which denotes the change of an object under infinitesimal coordinate transformation. From which it is easy to see that $\mathcal{H}_a$ generates surface deformations of Σ_t, while in a similar fashion $\mathcal{H}$ generates the time translations $\Sigma_t \to \Sigma_{t+\xi_0}$. Together these constraints generate transformations that mimic the diffeomorphism invariance of our theory, however due to the split, this algebra is rather convoluted and goes under the name of surface deformation algebra ([5] and [6]).

3.4 Dirac Quantization of ADM General Relativity

There are numerous ways that one can quantize a theory, depending on the further working convenience. For instance if one wants to work with manifest unitarity, one tends to choose canonical quantization, for manifest Lorentz invariance one usually picks path integral quantization. Of the two methods we will consider the former one, but even then one still remain with the issue of the ordering of constraints versus quantizing. One can choose to constrain and then quantize the system, which is known as the reduced phase space quantization, or vice-versa, known as Dirac quantization. Since the reduced phase space quantization scheme produces highly non-linear operators, from now on we will only focus on the Dirac quantization. We begin by defining an auxiliary Hilbert

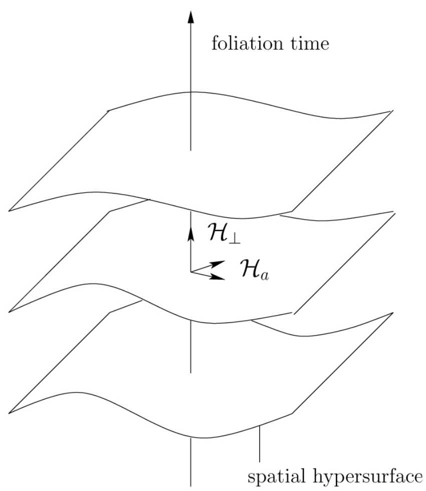

Figure 2: Transformations generated by the constraints. The Hamiltonian constraint $\mathcal{H}_\perp$ generates translation between hypersurfaces and the momentum constraint $\mathcal{H}_a$ generates surface deformations. Figure taken from [7].

space $H^{(aux)}$ consisting of functionals $\Psi[q]$ of positions q_{ij}. Let us promote the Poisson brackets to commutators and thus represent the momenta operators $\hat{\pi}_{kl}$ by

$$\hat{\pi}_{kl} = -i\frac{\delta}{\delta q^{kl}},$$

which is not different to the standard formulation of quantum mechanics. The next step is to implement the constraints into the system. Classically we required that the constraints have to be zero on shell, but now in the operator formalism of quantum mechanics we turn the constraints into operators acting on the Hilbert space $H^{(aux)}$. Then the states that are annihilated by the constraint operators will be considered to be the physical ones, which will thus span the new (smaller) physical Hilbert space $H^{(phys)}$.

Following the steps above and choosing the representation for the momentum operator (6) and requiring that the Hamiltonian constraint (3)in a form of an operator annihilates all physical states one obtains a Schrödinger-like functional equation

$$\left(16\pi G_N G_{ijkl}\frac{\delta}{\delta q_{ij}}\frac{\delta}{\delta q_{kl}} + \frac{\sqrt{q}}{16\pi G_n}\ {}^{(3)}R + \mathcal{H}_{\text{matter}}\right)\Psi[q] = 0,$$

known as the Wheeler-deWitt equation, and where $G_{ijkl} = \frac{1}{2\sqrt{q}}\left(q_{ik}q_{jl} + q_{il}q_{jk} - q_{ij}q_{kl}\right)$ is the so-called deWitt supermetric.

Unfortunately this equation is difficult to solve even for the simplest of the non-trivial systems, for a couple of reasons. Firstly, the canonical quantization formalism suffers from ordering ambiguities. Van Hove's theorem states that there exists no unique quantization map, which would consistently reproduce the Poisson algebra in terms commutator algebra of elements $\widehat{p^n q^m}$ when $n + m \geq 3$. Wheeler-deWitt equation upon quantization obtains an element of the kind $\widehat{p^2 q^2}$, which cannot be represented in terms of individual position and momentum variables. Since there exists no unique quantum theory, having chosen some conventional ordering, does not necessary mean that the theory we obtain corresponds to the true theory of quantum gravity. Secondly, it is unclear once the equation is solved, what kind of boundary conditions should be implemented [4]. Thirdly, the operator contains two functional derivatves with respect to the slice metric q_{ij} at the same point. Their combined action on the functional of q_{ij} will typically leave a factor of $\delta^{(3)}(0)$. Consider the functional in a harmonic approximation:

$$\Psi[q] \propto \exp\left(-\iint \mathrm{d}\vec{x}\,\mathrm{d}\vec{y}\; q_{ij}(\vec{x}) K^{ijkl}\left(\vec{x}-\vec{y}\right) q_{kl}(\vec{y})\right), \tag{6}$$

then

$$\frac{\delta}{\delta q_{ab}(\vec{u})}\frac{\delta}{\delta q_{cd}(\vec{u})}\Psi[q] \propto \left(\left(\int \mathrm{d}\vec{y} K^{abkl}\left(\vec{u}-\vec{y}\right) q_{kl}(\vec{y})\right)^2 - K^{abcd}\left(\vec{0}\right)\right)\Psi[q], \tag{7}$$

for which reason Wheeler-deWitt equation needs to be regularised. Some attempts have been taken to solve this equation, but due the reasons outlined above this approach has been deemed unsatisfactory and an alternatives were sought again. The successful attempt came with the idea of reformulation of general relativity in a form that is closer to well-known and understood gauge theories, in this way one has a similar kinematics, but different (and more difficult) dynamic framework.

4 Gauge Theory Formulation of General Relativity

The existence of gauge symmetries generated by the momentum and the Hamiltonian constraints suggested that one should follow the path taken by quantization of gauge theories. In order to achieve that goal one rewrites the action in terms of vielbeins e_a^J and their corresponding connection.

A vielbein denotes a set of unit vectors $\{e_\mu^1, e_\mu^2, \cdots e_\mu^I\}$, that define a tangent frame at a certain point on the manifold, where from now on we use the convention that capital Latin letters run over manifold's dimension (in this case $I = 0, 1, 2, 3$, μ's are still the spacetime indices, and a term "vielbein" will be replaced with "tetrad" or "vierbein"). Additionally this set $e_\mu^I(x)$ obeys the following relationship

$$e_\mu^I e_\nu^J \eta_{IJ} = g_{\mu\nu},$$

where η_{IJ} is used to raise and lower indices. Similarly to the metric formulation, we can introduce a dual tetrad $e^\mu_I(x)$, such that

$$e^I_\mu e^\mu_I(x) = g^\mu_\mu = \mathrm{Tr}[g_{\mu\nu}].$$

Clearly when parallel transported, both individual vector components making up the frame as well as its orientation would change, The former ones are changed by the Christoffel connection $\Gamma^\sigma_{\mu\nu}$, while for the later one needs to introduce a *spin connection* $\omega^I_{\mu J}$ that would define the way the frame's orientation changes. In general for V^I denoting an arbitrary element with a vielbein index and an arbitrary number of spacetime indices, then when parallel transported we require that

$$\nabla_\mu V^I = \partial_\mu V^I + \omega^I_{\mu J} V^J,$$

where the Christoffel symbol part has been omitted for presentation purposes. Requiring that $\omega^I_{\mu J}$ is the *tetrad compatible* spin connection, then from

$$\nabla_\mu e^I_\nu = \partial_\mu e^I_\nu - \Gamma^\rho_{\nu\mu} e^I_\rho + \omega^I_{J\mu} e^J_\nu = 0, \tag{8}$$

we can obtain its form, by making symmetrising this equation and making use of the symmetry property of the Christoffel connection, we can write $\nabla_\mu e^I_\nu - \nabla_\nu e^I_\mu = 0$ in the form

$$\partial_\mu e^I_\nu - \partial_\nu e^I_\mu + \omega^I_{J\mu} e^J_\nu - \omega^I_{J\nu} e^J_\mu = 0. \tag{9}$$

In this formulation now the Einstein-Hilbert action becomes

$$S = \frac{1}{16\pi G_N} \int d^4x |e| e^{\mu I} e^{\nu J} R_{\mu\nu IJ},$$

where it is not difficult to see where these terms came from. Firstly, from (8) we can see that $\sqrt{g}$ is simply reduced by the determinant of the tetrad, $|e|$, and secondly, using the fact that we can always trade a tetrad index for a spacetime index and vice-verse, we can write $R_{\mu\nu IJ} = R_{\mu\nu\rho\sigma} e^\rho_I e^\sigma_J$, which when we make sure that everything is properly contracted, we recover the spacetime integral of the Ricci scalar.

We can now fix our time direction and limit ourselves to the description of a particular time slice Σ. This is equivalent to introducing the gauge $e^0_{\tilde{I}} = 0$, where now $\tilde{I} = 1, 2, 3$. Note that upon imposition of this gauge, our gauge symmetry group has been reduced from $SO(3,1)$ to $SO(3)$. To make that explicit let us introduce an $SO(3)$ [1] connection $\Gamma^{\tilde{I}}_i$, such that

$$\Gamma^{\tilde{I}}_i = \frac{i}{2} \varepsilon^{0\tilde{I}\tilde{J}\tilde{K}} \omega_{i\tilde{J}\tilde{K}}.$$

Additionally it is not difficult to check that $\omega^{0\tilde{I}}_i = K^{\tilde{I}}_i = e^{j\tilde{I}} K_{ij}$, where K_{ij} is the extrinsic curvature defined previously. Historically the next step would

[1] Note that the $so(3)$ and the $su(2)$ Lie algebras have the same commutation relations, so frequently instead of an $SO(3)$ Lie group, its double cover, $SU(2)$, is chosen instead

involve a number of different linear combinations of the two connections above, however we will skip this discussion and jump to the more general definition of the field

$$A_i^{\tilde{I}}(\gamma) = \Gamma_i^{\tilde{I}} + \gamma K_i^{\tilde{I}},,$$

where $\gamma \in \mathbb{C}/\{0\}$ is the so-called Immirzi parameter introduced by Immirzi [8] and Barbero [9]. Historically this parameter was being chosen complex or real for different purposes. Now-a-days, however, there is a growing consensus to work with $\gamma \in \mathbb{R}$, and this is the choice that we will follow from now on. Additionally, not only the nature (complex or real), but also the actual value of this parameter will be consequential to the physical results later. In the classical theory is a bit mysterious, however in the quantum theory we will see it act as the scaling factor of the spectrum of our geometrical operators.

The second field that will be relevant to us is the densitised triad field transforming under the vector representation of $SU(2)$ and defined as

$$E_{\tilde{I}}^i = \sqrt{q} e_{\tilde{I}}^i.$$

Having introduced these fields we can rewrite the action as

$$S = \frac{1}{16\pi G_N} \int dt \int d^3x \left(\frac{1}{\gamma} A_i^{\tilde{I}} \frac{\partial}{\partial t} E_{\tilde{I}}^i - i A_{0\tilde{I}} G^{\tilde{I}} + i N^i V_i + \frac{N}{2\sqrt{q}} H + h.c. \right),$$

where *h.c.* denotes the Hermitian conjugate terms, and where the last three terms introduce the Gauss, vector and Hamiltonian constraints with their respective Lagrange multipliers $A_{0\tilde{I}}$, the shift vector N^i and the lapse function N. Their mathematical form reads

$$\begin{aligned} H &= \varepsilon^{\tilde{I}\tilde{J}\tilde{K}} E_{\tilde{I}}^i E_{\tilde{J}}^j F_{ij\tilde{K}} - 2\frac{1+\gamma^2}{\gamma^2} E_{[\tilde{J}}^i E_{\tilde{I}]}^j \left(A_i^{\tilde{I}}(\gamma) - \Gamma_i^{\tilde{I}} \right) \left(A_j^{\tilde{J}}(\gamma) - \Gamma_j^{\tilde{J}} \right) \quad (10) \\ G^{\tilde{I}} &= \partial_j E^{j\tilde{I}} + \varepsilon_{\tilde{J}\tilde{K}}^{\tilde{I}} A_j^{\tilde{J}} E^{j\tilde{K}} \equiv D_j E^{j\tilde{I}}, \\ V_i &= E_{\tilde{I}}^j F_{ij}^{\tilde{I}}, \end{aligned}$$

where

$$F_{ij}^{\tilde{I}} = \partial_i A_j^{\tilde{I}} - \partial_j A_i^{\tilde{I}} + \varepsilon^{\tilde{I}\tilde{J}\tilde{K}} A_{i\tilde{J}} A_{j\tilde{K}}$$

is a field strength tensor for an $SO(3)$ (or $SU(2)$) field. Alike the previous result, the vector and the Hamiltonian constraint will mimic the diffeomorphism invariance of the theory, such that the vector constraint generates the surface deformations of the time slice, and the Hamiltonian constraint is a generator of time translations between the time slices. The awaited virtue of this method is the appearance of the Gauss constraint which generates the $SO(3)$ gauge transformation, and the fact that when it requires that the densitised triad $E^{i\tilde{J}}$ does not change under parallel transport using the $SO(3)$ connection will be of great importance to us later on. One can also check whether this approach is equivalent to the ADM GR from the third section.

Theorem
For a phase space coordinatized by $\left(A_a^J, E_K^b\right)$ with the Poisson structure

$$\begin{aligned}\{E^a_{\tilde{J}}(x), E^b_{\tilde{K}}(y)\} &= 0 \quad (11)\\ \{A_a^{\tilde{J}}(x), A_b^{\tilde{K}}(y)\} &= 0 \quad (12)\\ \text{and}\ \ \{E^a_{\tilde{J}}(x), A^b_{\tilde{K}}(y)\} &= 8\pi G_N\gamma\delta^{ab}\delta_{\tilde{J}\tilde{K}}\delta(x-y) \quad (13)\end{aligned}$$

and the above constraints, upon solution of the Gauss constraint and determining the Dirac observables with respect to it, one again obtains the ADM phase space $(q_{ij}(x), \pi_{kl}(y))$ with the constraints $\mathcal{H}$ and $\mathcal{H}^i$.

The proof to this theorem is very laborious and tedious, since it can be found in [10], it will be omitted in this review. It is important to note however that some authors like to define the densitized triad field E divided by the Barbero-Immirzi constant, thus making it drop out of the Poisson bracket relations.

5 Loop Quantum Gravity

5.1 Quantizing the Connection Formulation, Holonomies and Wilson Loops

Having introduced the fields above we are at a doorstep to Loop Quantum Gravity. Working in the connection A bases, we will define a space of functionals $\Psi[A]$, and promote the Poisson bracket to a commutator,

$$\begin{aligned}\{E^a_J(x), A^b_K(y)\} &= 8\pi G_n\gamma\delta^{ab}\delta_{JK}\delta^{(3)}(x-y)\\ \to \left[\hat{E}^a_J(x), \hat{A}^b_K(y)\right] &= 8i\hbar\pi G_N\gamma\delta^{ab}\delta_{JK}\delta^{(3)}(x-y), \quad (14)\end{aligned}$$

then we can choose the following representation of the E^i_I field

$$E^i_I = -8\pi i\gamma G_N\frac{\delta}{\delta A^I_i} \quad (15)$$

Now instead of defining a complete Hilbert space, we will immediately try to reduce it to the physically relevant one. Since the Dirac scheme of canonical quantization of constrained theories requires that physical states vanish under the action of a constraint operator, then in this case, from the presence of the Gauss constraint, we require that physically relevant quantum states need to be invariant under $SO(3)$ or $SU(2)$ gauge transformations, depending on the initial gauge group choice. These are simply the states that *vanish under the action of the gauge covariant derivative* $D_\mu = \partial_\mu + A_\mu$. Let us define the parallel transport equation on the gauge group manifold for a curve $\alpha : [0,1] \to \Sigma$ as

$$\frac{dx^\mu}{ds}D_\mu V_\nu = \frac{dV_\nu}{ds} + \underbrace{\frac{dx^\mu}{ds}A_\mu}_{\equiv A(s)} V_\nu = 0,$$

where we have defined the field components along the path by $\frac{dx^{\mu}}{ds}A_{\mu} \equiv A\left(s\right)$. This is simply a first order ordinary differential equation, which has a general solution $V_{\nu}\left(s\right) = U\left(s,0\right)V_{\nu}\left(0\right)$ such that

$$V_{\mu}\left(s\right) = V_{\mu}(0) - \int_{0}^{s} ds_1 A\left(s_1\right)V_{\mu}\left(s_1\right). \tag{16}$$

Substituting this into the equation (16) again we get

$$V_{\mu}\left(s\right) = V_{\mu}(0) - \int_{0}^{s} ds_1 A\left(s_1\right)\left(V_{\mu}(0) - \int_{0}^{s_1} ds_2 A\left(s_2\right)V_{\mu}\left(s_2\right)\right).$$

We can iterate this process to write the closed solution to the parallel transport equation. To make the notation more compact let us define the path ordering operator

$$\mathcal{P}\left(\alpha\left(s_1\right)\beta\left(s_2\right)\right) = \begin{cases} \alpha\left(s_1\right)\beta\left(s_2\right) & s_1 > s_2 \\ \beta\left(s_2\right)\alpha\left(s_1\right) & s_1 < s_2 \end{cases}.$$

Using this we can write the iterating series as

$$U\left(s,0\right) = \sum_{n}\frac{(-1)^n}{n!}\mathcal{P}\left(\int_{0}^{s} ds_1 A\left(s\right)\right) = \mathcal{P}\left(e^{-\int_{0}^{s} ds' A\left(s'\right)}\right)$$

which gives us the solution to the parallel transport equation (16).

Under the gauge group the matrix transforms as

$$U(s,s') \rightarrow g(s)U(s,s')g^{-1}(s'),$$

where $g(s) \in SO(3)$ or $g(s) \in SU(2)$. Let us consider a path $\alpha : [0,1] \rightarrow \Sigma$ with closed ends i.e. $\alpha\left(0\right) = \alpha\left(1\right)$. Then $U_{\alpha}(0,1)$ transforms as

$$U_{\alpha}(0,1) \rightarrow g(0)U_{\alpha}(0,1)g^{-1}(1) = g(0)U_{\alpha}(0,1)g^{-1}(0).$$

Note now that by taking a trace of the above, we can use the cyclicity of the trace to show that it is invariant under the gauge transformations. Thus $\mathrm{Tr}[U_{\alpha}(0,1)]$, a.k.a. holonomy, or the Wilson loop, being a gauge invariant function of the field A_{μ}, is an element that we can use to construct our quantum states. The appearance of the Wilson loops is the reason why this theory goes under the name of Loop Quantum Gravity.

We have now established the scope of our physical Hilbert space. These will include all of the objects composed of Wilson loops and their linear combinations with the $SU(2)$ generators, and densitised tetrads invariant under the action of the gauge group.

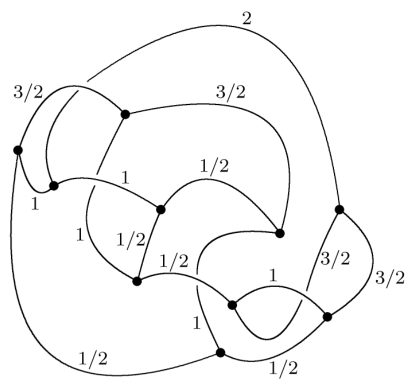

Figure 3: An example of a spin network, the oriented intersection number eigenstate. Note that for trivalent vertices, the intertwiners are simply given by the Clebsh-Gordon coefficients. Figure taken from [4].

5.2 Spin Networks

It was quickly realised that Wilson loops are just the simplest gauge invariant structures that one could use. These have been generalised to graphs, so-called spin networks, such that each spin network determines a gauge invariant functional $\Psi[A]$ of the connection. We can build a number of these spin networks using the following steps

- draw an arbitrary set of points and connect the points with an arbitrary number of lines/edges
- label each line with a half-integer spin s_1 labelling an irreducible representation of $SU(2)$
- to each edge labelled s_1 assign a holonomy of A in representation s_1
- to each vertex at which particular spins meet, assign an intertwiner, that is an invariant tensor in the tensor product of the representations of the spins in question. For example, for a trivalent diagrams (see Figure 3), the intertwiners will be the corresponding Clebsh-Gordon coefficients.

Ever since the 1910s, gravity has been considered as a theory of geometry of spacetime, our operators should be measuring geometrical quantities such as lengths, areas and volumes. Similarly to the states, the operators need to be gauge invariant objects and in general these will be given by

$$\begin{aligned}
\mathcal{T}[\alpha] &= -\mathrm{Tr}[U_\alpha(0,1)], \\
\mathcal{T}^i[\alpha](s) &= -\mathrm{Tr}[U_\alpha(s,s)E^i(s)], \\
&\dots \\
\mathcal{T}^{i_1\dots i_n}[\alpha](s_1,\dots,s_n) &= -\mathrm{Tr}[U_\alpha(s_1,s_n)E^{i_n}(s_n)\dots U_\alpha(s_2,s_1)E^{i_1}(s_1)],
\end{aligned} \tag{17}$$

whose geometrical interpretation will be the focus of the remainder of this section. The first operator is just a holonomy and in line with the representation of E based on the commutation relation, the first operator of our interest will be $\mathcal{T}^i[\alpha]$. Let us choose a direction perpendicular to a little patch of surface parametrized by σ_1 and σ_2, such that

$$E^3_{\tilde{I}} = \varepsilon_{ijk}\frac{\partial x^i}{\partial\sigma_1}\frac{\partial x^j}{\partial\sigma_2}E^k_{\tilde{I}},$$

then, we can write

$$\begin{aligned}
E_{\tilde{I}} &= \int_S d\sigma^1 d\sigma^2 \varepsilon_{ijk}\frac{\partial x^i}{\partial\sigma_1}\frac{\partial x^j}{\partial\sigma_2}E^k_I, \\
E_{\tilde{I}} &= -8\pi\gamma G_N \int_S d\sigma^1 d\sigma^2 \varepsilon_{ijk}\frac{\partial x^i}{\partial\sigma_1}\frac{\partial x^j}{\partial\sigma_2}\frac{\delta}{\delta A^{\tilde{I}}_k},
\end{aligned}$$

where S denotes a small surface around E^k_I. Let us focus on the last piece; i.e. act on a holonomy with a functional derivative $\frac{\delta}{\delta A^{\tilde{I}}_k}$

$$\frac{\delta}{\delta A^{\tilde{I}}_k}\exp\left(-\int_{s_1}^{s_2} ds A^{\tilde{J}}_i \frac{dx^i}{ds}J_{\tilde{J}}\right) = \int_{s_1}^{s_2} ds\, U(s_1,s)\,\frac{dx^k}{ds}J_{\tilde{I}}U(s_1,s_2)\,\delta(x-x(s)) \tag{18}$$

thus is x does not lie on the curve $x(s)$ then the operator returns zero, otherwise it returns the angular momentum eigenvalue associated with a particular edge. Thus we can write

$$\begin{aligned}
E_{\tilde{I}}U_\alpha(s_1,s_2) &= -8\pi\gamma G_N \int \underbrace{d\sigma^1 d\sigma^2 ds \varepsilon_{ijk}\frac{\partial x^i}{\partial\sigma_1}\frac{\partial x^j}{\partial\sigma_2}\frac{dx^k}{ds}}_{d^3\vec{x}}\delta(x-x(s))U(s_1,s)\,J_{\tilde{I}}U(s_1,s_2) \\
&= \mp 8\pi\gamma G_N \int d^3\vec{x}\delta(x-x(s))U(s_1,s)\,J_{\tilde{I}}U(s_1,s_2),
\end{aligned} \tag{19}$$

where the $\mp$ sign comes from the orientation of the (x^1,x^2,x^3) relative to (σ^1,σ^2,s) coordinate system, and for that reason expression (19) is called oriented intersection number.

Having done all that, we are ready to answer what could an area operator look like, and what are its eigenvalues. Analogously to the ADM decomposition of our manifold $\mathcal{M}$, we can slice our volumes into surfaces with a metric $h_{\hat{i}\hat{j}}$ and "space lapses" $\tilde{N}^{\hat{j}}$

$$(\mathcal{M}, g_{\mu\nu}) \xrightarrow[ADM]{} (\mathcal{M}', q_{ij}) \to \left(\mathcal{M}'', h_{\hat{i}\hat{j}}\right), \tag{20}$$

where $\hat{i}, \hat{j} = 1, 2$. Then our metric q_{ij} becomes

$$q_{ij} = \begin{pmatrix} h_{\hat{i}\hat{j}} & h_{\hat{i}\hat{j}}\tilde{N}^{\hat{i}} \\ h_{\hat{i}\hat{j}}\tilde{N}^{\hat{j}} & \tilde{N}^2 + h_{\hat{i}\hat{j}}\tilde{N}^{\hat{i}}\tilde{N}^{\hat{j}} \end{pmatrix} \quad \text{and} \quad q^{ij} \begin{pmatrix} h^{\hat{i}\hat{j}} + \frac{\tilde{N}^{\hat{i}}\tilde{N}^{\hat{j}}}{\tilde{N}^2} & -\frac{\tilde{N}^{\hat{i}}}{\tilde{N}^2} \\ -\frac{\tilde{N}^{\hat{j}}}{\tilde{N}^2} & \frac{1}{\tilde{N}^2} \end{pmatrix}, \tag{21}$$

such that $\sqrt{\det q} = \tilde{N}\sqrt{\det h}, \frac{1}{\tilde{N}} = \sqrt{q_{33}}$ and $\det h = q_{11}q_{22} - q_{12}^2$. Note the difference in signs compared to the ADM decomposition, as here we are slicing $\mathcal{M}'$, which is a Euclidean manifold. In general area observable of a small surface S denoted by $\mathcal{A}_S$ can be written as the following integral

$$\begin{aligned} \mathcal{A}_S &= \int_S dx^1 dx^2 \sqrt{\det h_{\hat{i}\hat{j}}} \\ &= \int_S dx^1 dx^2 \frac{1}{\tilde{N}}\sqrt{q} = \int_S dx^1 dx^2 \sqrt{q_{33}}\sqrt{q} \\ &= \int_S dx^1 dx^2 \sqrt{\sqrt{q}e^3_{\tilde{I}}\sqrt{q}e^{3\tilde{I}}} = \int_S dx^1 dx^2 \sqrt{E^3_I E^{3I}}, \end{aligned} \tag{22}$$

where E^3_I denotes the set of components of the densitised triad perpendicular to the measured surface. Thus when we quantize it we require that in its operator form it becomes

$$\hat{\mathcal{A}}_S = \int dx^1 dx^2 \sqrt{\hat{E}^3_{\tilde{I}}\hat{E}^{3\tilde{I}}} = \int dx^1 dx^2 \sqrt{\mathcal{T}^3_{\tilde{I}}[\alpha](s)\mathcal{T}^{3\tilde{I}}[\alpha](s)}. \tag{23}$$

Using the result (19) we get that the eigenvalue of $\hat{\mathcal{A}}_S$ is given by $8\pi\gamma G_N\sqrt{j(j+1)}$, where we get the values of j from the measurements of the spin assigned to a particular holonomy line that intersects the little patch of surface $S(\sigma_1, \sigma_2)$. Clearly the spin networks are thus eigenfunctions of the area operator, and area is quantized as a result of this treatment! Additionally note that the Immirzi parameter enters as a scaling factor of the quantum area spectrum, which indicates that one obtains different theories of quantum gravity with different choices of γ. One could try to find the precise value of the Barbero-Immirzi parameter γ, and so far the predictions based on the black hole entropy calculations state that $\gamma = \frac{\ln 2}{\sqrt{3}\pi}$ or $\gamma = \frac{\ln 3}{\sqrt{8}\pi}$, depending on the gauge group choice used in LQG [11].

Similar analysis can be done for the volume operator, but the calculations are much more elaborate and these will be later considered in a simplified spacetime models. It is important to note, that there is no direct operator for quantized lengths, but instead one can measure a quantum of volume of a cylinder and divide it by a quantum of its cross-sectional surface area, to find a way of obtaining a quantum of length.

6 Quantum Cosmology

Einstein's equations can only be solved for a limited number of cases, let alone their quantized version. That is why it is always a good idea to test the predictions on a model with numerous symmetries and simplifications. The FLRW metric, describing an isotropic and homogenous universe seems to be a perfect candidate as, not only it depends only on a single time dependent variable $a(t)$. Moreover, it also allows one to generate the results which would be profound to our cosmological interpretations at small scales, i.e. at times very close to Big Bang, some of which one may hope to be able to verity one day. In the ADM formalism we can write our line element as

$$ds^2 = -N\left(t\right)^2 dt^2 + a\left(t\right)^2 d\Omega_3^2,$$

where the lapse function $N(t)$ measures the change of time coordinate with proper time. Setting N(t)=1, we recover Friedmann time.We can also now write down the action for this simple spacetime

$$S = S_{grav} + S_{matter} = \frac{3V_0}{\kappa}\int dtN\left(-\frac{a\dot{a}^2}{N^2} + ka - \frac{\Lambda a^3}{3}\right) + \frac{V_0}{2}\int dtNa^3\left(\frac{\dot{\phi}^2}{N^2} - 2V\left(\phi\right)\right) \tag{24}$$

where $\kappa = 8\pi G_N$, Λ is the cosmological constant, and where we have assumed the simplest matter content of the Universe given by a scalar field.

6.1 Dirac Quantization of the FLRW Spacetime — the Wheeler-DeWitt equation of the Universe

Based on this action we can easily write down the canonical momenta

$$\pi_a = \frac{\partial\mathcal{L}}{\partial\dot{a}} = -\frac{6a\dot{a}}{\kappa^2 N},\ \pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{a^3\dot{\phi}}{N},\ \pi_N = \frac{\partial\mathcal{L}}{\partial\dot{N}} = 0 \tag{25}$$

and write down the Hamiltonian density,

$$\mathcal{H} = \pi_a\dot{a} + \pi_\phi\dot{\phi} + \pi_N\dot{N} - \mathcal{L} = \tag{26}$$

$$= -\frac{\kappa^2}{12a}\pi_a^2 + \frac{1}{2}\frac{\pi_\phi^2}{a^3} + a^3\frac{\Lambda}{\kappa^2} + a^3 V - \frac{3ka}{\kappa^2}, \tag{27}$$

just like we did previously. It is easy to check that upon the vanishing of the Hamiltonian constraint we recover the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{\kappa^2}{3}\left(\frac{\dot{\phi}^2}{2} + V\left(\phi\right)\right) + \frac{\Lambda}{3} - \frac{k}{a^2}, \tag{28}$$

therefore, not surprisingly, in the classical theory we recover the equations of motion. Once we quantize the theory in the Wheeler-DeWitt sense we end up with the following quantum constraint equation

$$\frac{3}{2}\left(-\frac{1}{9}l_p^4 a^{-1}\frac{\partial}{\partial a}a^{-1}\frac{\partial}{\partial a}\right)\left[a\psi\left(a,\phi\right)\right] = 8\pi G_N \hat{H}_\phi\left(a\right)\frac{1}{a}\left[a\psi\left(a,\phi\right)\right]. \tag{29}$$

Recall however that, this is only one of the many possible forms of the Wheeler-deWitt equation, since previously we have seen that, this equation was plagued by ordering ambiguities in the kinetic term. On the right hand side, $\hat{H}_{\phi}(a)$ denotes the scalar Hamiltonian of the matter content only. Overall this equation is something that one could call the Schrödinger equation for the scale factor of the Universe. Note that the order of the differential operator is different as well as the fact that the derivatives are not with respect to time, therefore one could argue that it may not describe the quantum evolution of the Universe. This however is not an issue as a can be understood as a time parameter in disguise, assuming that $a(t)$ is a bijective function of time. Additionally note, that from the form of this equation, as $a \to 0$, i.e. as we move closer to the "beginning" of the Universe, then the energy densities, all with a factor of a^{-1} in front, remain unbounded, and the Wheeler-deWitt equation cannot tell us what takes place on the other side of the $a = 0$ singularity. DeWitt attempted to resolve this problem with taking $\psi(a = 0, \phi) = 0$, however it is not clear whether this interpretation, borrowed directly from quantum mechanics, makes any sense in the cosmological context.Moreover,we would also need appropriate fall-off conditions as we approach $a = 0$, and what are they is not clear either. and the problems at singularity persists.

Yet again, the naive, simplified formulation of quantum general relativity has been unsuccessful. However as we will see in the following subsection, Loop Quantum Gravity in the FLRW background will provide an alternative description, where the Big Bang singularity is no longer as problematic.

6.2 Loop Quantum Cosmology Predictions

Working in the FLRW background, isotropy and homogeneity of the spacetime are assumed. For this reason our operators can be written in the following simple form

$$A_a^{\tilde{I}} dx^a = c\omega^{\tilde{I}}, \quad \text{and} \quad E_{\tilde{I}}^i \frac{\partial}{\partial x^i} = pX_{\tilde{I}}, \tag{30}$$

where for spatially *flat* configurations $\omega^{\tilde{I}} = dx^{\tilde{I}}$ and $X_{\tilde{I}} = \frac{\partial}{\partial x^{\tilde{I}}}$; these are more complicated for curved FLRW spacetimes. Additionally c and p are functions of time only and can be recast in terms of previously encountered variables as

$$c = \frac{1}{2}\left(k - \gamma\dot{a}\right), \quad \text{and} \quad |p| = a^2, \tag{31}$$

which follows from the definition of the spin connection, the intrinsic curvature and the densitised triad. We can see that based on this we have the desired Poisson structure $\{c, p\} = \frac{\kappa^2}{3}$. It is important to note that p can have a negative as well as a positive sign. Additionally in classical theory we are required to choose one sign over the other, since $p = 0$ represents a degenerate triad, hence disconnecting the positive and the negative p domains. It is important to note however that this situation will be different once we quantize the theory.

Upon the above simplification, temporarily choosing a sign for p, our holonomy obtains a very simple form

$$U(s_1, s_2; c) = \exp(c\tau_{\tilde{I}}) = \cos\left(\frac{c}{2}\right) + 2\tau_{\tilde{I}} \sin\left(\frac{c}{2}\right). \tag{32}$$

We can now construct the states of our interest by acting with the holonomy matrix (being the creation operator) on the ground state $\mathbb{1}$, hence all of these states will be functions of variable c only. Due to the spacetime symmetry allows, we do not have to worry about the complications resulting from spin networks with arbitrary number of curves, and we can introduce an orthonormal basis in the connection representation [11], such that

$$\langle\, c|n\,\rangle = \frac{\exp(inc/2)}{\sqrt{2}\sin(c/2)}, \tag{33}$$

where $n \in \mathbb{Z}$. Let us define a state $|n\,\rangle$, which is an eigenstate of the oriented intersection number operator $\mathcal{I}^i$, in our simplified space-time. Then based on the previous analysis of the area operator and the definition of p, which when turned into an operator $\hat{p}$, we obtain

$$\hat{p}|n\,\rangle = \frac{1}{6}\gamma l_p^2 n|n\,\rangle \tag{34}$$

We can see that the spectrum is discrete, and that it is very different to the one found from the Wheeler-DeWitt quantization scheme. Recall that previously, the action of the operator was purely multiplicative and that the operator had a continuous spectrum; there was no direct information on what boundary conditions should be implemented, so the Schrödinger-like equation could not yield a discrete spectrum. Additionally note that in LQC, the classical singularity is annihilated by $\hat{p}$. This is because the $n = 0$ state has zero eigenvalue.

Following Bojowald [12], and [13] we can obtain the following expression for the volume spectrum

$$V_{(|n|-1)/2} = \left(\frac{1}{6}\gamma l_p^2\right)^{\frac{3}{2}} \sqrt{(|n|-1)\,|n|\,(|n|+1)}. \tag{35}$$

Note that for $n = 0, 1$ volume is degenerately zero, however for $n \geq 2$, it is non-zero, being minimum at exactly $n = 2$.

We have previously noticed that the inverse scale factor appeared as a quantum operator in the Wheeler-DeWitt formulation of quantum cosmology. There $\hat{a}^{-1}$ was an unbounded operator. The situation is similar in LQC, however here $a = 0$ is the admissible value at a single point, since now the scale factor has a discrete spectrum. The difference, however, is that now $\hat{a}^{-1}$ can have an admissible quantization, when they have the correct classical limit and are densely defined operators.

The expression is different for different choices of representation of the $SU(2)$ gauge group and it is simplest for $j = \frac{1}{2}$, where it takes the form

$$\widehat{a^{-1}}|n\,\rangle = \frac{16}{\gamma^2 l_p^4}\left(\sqrt{V_{\frac{|n|}{2}}} - \sqrt{V_{\frac{|n|}{2}-1}}\right)^2 |n\,\rangle \tag{36}$$

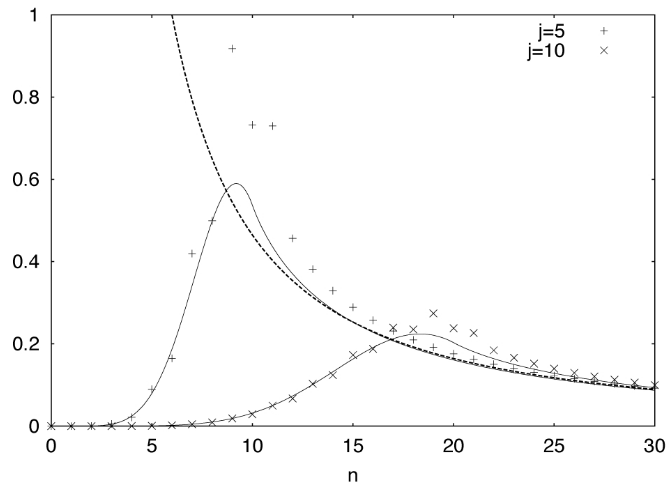

Figure 4: Eigenvalues of inverse scale factor operators for $j = 5$ and $j = 10$ compared to the classical behaviour (dashed). Figure taken from [11]

Note now that the maximum of the inverse scale factor is $(a^{-1})_{max} \propto \frac{1}{l_p}$ at the $n = 2$ peak. The exact value of this maximum obviously depends on the representation, and the Immirzi parameter choice. If we manage to restrict our spectrum to $n \geq 2$, then the singularity is wiped out. This can also be seen from the Figure 4, produced by Bojowald and Morales-Técotl [11], that neither the representation choice nor the Immirzi parameter value change the qualitative aspect of the theory. Moreover they argue that it is the choice of the sign of p that we had to make is responsible for making the eigenvalue of $\hat{a}^{-1}$ be zero at $n = 0$.

What do these results mean from a qualitative point of view. Firstly we notice that the scale factor as well as the volume have a discrete spectrum. When we look closely at the plot of the inverse scale factor against the oriented intersection number and compare it to the classical predictions for a, we notice that, as expected, the two differ *only* in the small volumes domain, thus our quantization has a correct classical limit. We have also noticed that the singularity problem is not removed yet, as at $n = 0$ the scale factor is still divergent, however there is a small detail that we have overlooked, namely, that in our Poisson structure it was not p that appeared but rather $|p|$. It is the issue of the appearance of absolute values that will later play an important role.

It appears that we can rewrite the classical expressions in a number of ways, depending on the representation chosen, and then the quantizations will not necessarily be equivalent. The quantitative features may change under these ambiguities, however the aforementioned qualitative features will not.

The results above hint at the fact that the quantum behaviour is less singular than the classical one. However, we can only be certain of the correctness of the theorem if it would be possible to extend the evolution through the singularity, at which point we will learn what lies beyond it and will eliminate all the worries about its presence.

6.3 Evolution in LQC

In the Wheeler-DeWitt of quantum cosmology we have noticed that the equation we obtain was very difficult to work with, besides the fact that did not remove the singularity. In order to make progress in the Loop Quantum Cosmology formulation we have to transform the connection representation into the triad representation, by a simple change of basis, where a state can be expanded in the triad eigenstates $|n\rangle$, such that $|\psi\rangle = \sum_n \psi(\xi)|n\rangle$, where ξ denotes all of the possible degrees of freedom. Since n are $\hat{p}$ eigenvalues (recall that it was directly related to the scalar factor a), they will now play a role of an internal time, and being discrete, unlike in the Wheeler-DeWitt quantization, the internal time will be discrete too. This is going to introduce a major change, as now differential equations will become just simple difference equations. Now we have

$$\begin{aligned}\langle\, c \,|\, U(c)\, |n\,\rangle &= \langle\, c \,|\, \cos\left(\frac{c}{2}\right) + 2\tau_i \sin\left(\frac{c}{2}\right)\, |n\,\rangle; \\ &= \frac{1}{2}\left(\langle\, c|n+1\,\rangle + \langle\, c|n-1\,\rangle\right) - \frac{1/2}{\tau}{}_{i}\left(\langle\, c|n+1\,\rangle - \langle\, c|n-1\,\rangle\right).\end{aligned} \tag{37}$$

Thus in triad representation holonomies change n by ± 1, since

$$\left(\cos\left(\frac{c}{2}\right)\psi\right)_n = \frac{1}{2}\sum_n \psi_n\left(|n+1\,\rangle + |n-1\,\rangle\right) = \frac{1}{2}\sum_n \psi_n\left(\psi_{n+1} + \psi_{n-1}\right)|n\,\rangle; \tag{38}$$

$$\left(\sin\left(\frac{c}{2}\right)\psi\right)_n = -\frac{i}{2}\sum_n \psi_n\left(|n+1\,\rangle - |n-1\,\rangle\right) = \frac{i}{2}\sum_n \psi_n\left(\psi_{n+1} - \psi_{n-1}\right)|n\,\rangle. \tag{39}$$

The constraint operators can now be rewritten in the following convenient form (for technical reasons it can only be derived for flat and positively curved space-times, i.e. $k = 0, 1$)

$$\left(V_{|n+4|/2} - V_{(|n+4|/2)-1}\right) e^{ik}\psi_{n+4}(\xi) - 2\left(2+\gamma^2 k^2\right)\left(V_{|n|/2} - V_{(|n|/2)-1}\right)\psi_n(\xi) \tag{40}$$

$$+\left(V_{|n-4|/2} - V_{(|n-4|/2)-1}\right) e^{-ik}\psi_{n-4}(\xi) = -\frac{8\pi}{3} G_N \gamma^3 l_P^2 \hat{H}_{matter}(n)\,\psi_n(\xi), \tag{41}$$

where we can see that the right hand side remained the right hand side of the Wheeler-DeWitt equation, however the scale factor dependent content inside the matter Hamiltonian now has changed its character due to the discrete spectrum of $\hat{a}$. The discreteness of this spectrum is also the reason why the evolution equation is no longer a differential, but rather a difference equation.

An interesting aspect of this equation is that in the large volume limit, i.e. far away from the singularity, at $n \gg 1$, and where $\psi_n(\xi)$ does not display rapid oscillations. We can then interpolate the discrete wave functions to a continuous one $\tilde{\psi}(p,\xi)$, where the $e^{\pm ik}$ factors have been absorbed into ψ's and redefined as $\tilde{\psi}$. Upon insertion into the difference equation and performing the Taylor expansion of $\tilde{\psi}_{n\pm} = \tilde{\psi}\left(p(n) \pm \frac{2}{3}\gamma l_P^2\right)$ in terms of $\frac{p}{\gamma l_P^2}$ we obtain the Wheeler-DeWitt equation in some ordering convention upon the identification $a = \sqrt{|p|}$,

$$\frac{1}{2}\left(\frac{4}{9}l_P^2\frac{\partial^2}{\partial p^2} - k\right)\tilde{\psi}(p,\xi) = -\frac{8\pi}{3}G_N\gamma^3 l_P^2 \hat{H}_{matter}(n)\,\tilde{\psi}_n(\xi). \tag{42}$$

Thus the difference equation has the expected large volume behaviour, which reproduces the equation that still suffers from the classical singularity. Additionally we notice that the Wheeler-DeWitt equation, however not fundamental, reliably describes the dynamics far away from the singularity, which can be used to study semi-classical approximations.

6.4 Evolution through the singularity

Looking back at the difference equation we can now study how the FLRW spacetime evolves through the singularity at the quantum scales. For large n the evolution is to a good extend described by the classical equations. We can then evolve backwards by the recurrence relation for ψ_{n-4}. This process can be carried out as long as $\left(V_{|n-4|/2} - V_{(|n-4|/2)-1}\right)$ is non-zero, however this is possible when $n = 4$. At this point we are about to find the wave function at $n = 0$ (given ψ_8), however we are never capable to finding it and we can evolve without any problems through the singularity. This is because for ψ_{-1} we need ψ_3 and ψ_7, and similarly for ψ_{-2} and ψ_{-3}. For ψ_{-4} however, we would need to know ψ_0 and ψ_4, however at this value ψ_0 drops out of the equation completely. Thus the value of ψ_{-4} is determined by ψ_4 only and the issue of ψ_0 does not cause any problems again. Therefore we see that the system evolves through the singularity; there exists a spacetime before the Big Bang and the Universe appears to be decreasing in size, up to a volume of the order of l_P^3 and expand into the Universe that we see today. Note that this prediction only holds for $k = 0, 1$, and could be taking place multiple times for the positively curved Universe, where the sequence of Big Bang and Big Crunch singularities is described by the process described above. Thus we see that the simplicity of the FLRW metric allowed us to develop the theory far enough to change our understanding of the Universe we live in.

One of the problems with this approach is that it might be difficult to verify experimentally. A number of authors argue that within a couple of decades we should be able to develop the technology that will measure Loop Quantum Gravitational corrections to various energy spectra we observe currently. The appearance, detectability of this correction, and their order of magnitude are still heavily debated.

7 Conclusion

In this concise review we have seen that Loop Quantum Gravity is a background independent, non perturbative treatment of gravity. Once the problem of time was solved by the Arnowitt-Deser-Misner formalism, we notice that the brute force Dirac quantization of the resulting theory still was far from finding a consistent theory, which would introduce a bounded from below geometrical spectrum of spacetime. Since this was one of the initial motivations behind introducing a quantum theory of gravity we needed to rewrite our general relativity in a connection formulation. When we attempt to quantize the resulting theory, we find that we require intrinsically $SU(2)$ gauge invariant structures. This were initially known as the Wilson loops, but were subsequently generalised to the so-called spin networks. These represented eigenfunctionals of the densitised triad field $E_i^{\tilde{I}}$, and were eigenstates of the quantum area operator, which had a discrete spectrum bounded from below.

Lastly we have noticed that by putting this formalism to a test on a highly symmetric FLRW spacetime, we have noticed that as a result time became quantized as well, and that even though the classical singularity persisted in the geometrical spectrum, at quantum level the system was capable of evolving through it, completely removing the need for Big Bang or Big Crunch.

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