Loop Quantum Gravity (LQG) and Loop Quantum Cosmology (LQC)

The quest to removing inherent singularities by covariant quantization of general relativity.

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Presentation Outline

Why do we need to quantize gravity?

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The Arnowitt-Deser-Misner (ADM) Formalism

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Action formulation of General Relativity

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First Attempts to Quantize General Relativity

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Problems with Einstein's Equations

The left and the right hand side of the Einstein equations are not consistent

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\mathrm{Geometry - classical}} = \underbrace{\frac{8\pi G_n}{c^3} T_{\mu\nu}(g)}_{\mathrm{Matter - Gauge\,Fields\,in\,QFT}}$$

Functions of space-time and operators on a Hilbert space are two different things. Solution?

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where g_0 is a background metric.The notion of the vacuum depends on the choice of g_0. Due to vacuum fluctuations RHS is non-vanishing, yielding a solution g_1. Requirement:

$$\hat{R}_{\mu\nu} - \frac{1}{2} \hat{g}_{\mu\nu} \hat{R} = \frac{8\pi G_n}{c^3} \hat{T}_{\mu\nu}(\hat{g})$$

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- As $\lambda \to R_s = \frac{G_n E}{c^4}$, the virtual particle turns into a decaying black hole.

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Conclusion: There is a hope that quantized gravity would provide a cut-off on the momentum integrals.

Problem of time in Quantum Gravity

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In GR there is no preferred time slicing.
Additionally quantum fluctuations of the metric can exchange past and future - problems with causality.

Conventions used:

Lower case Latin letters denote spatial indices

$$i, j, k, \cdots = 1, 2, 3$$

or arbitrary indices, in which case I will use

$$a, b, c, d, \cdots$$

Capital Latin letters denote the tetrad indices

$$I, J, K, \cdots = 0, 1, 2, 3$$

with $\eta_{IJ} = \mathrm{diag}\left(1, -1, -1, -1\right)$, and finally the Greek indices denote the space-time components

$$\mu, \nu, \sigma, \rho, \cdots = 0, 1, 2, 3$$

The Arnowitt-Deser-Misner (ADM) Formalism

Pick your time coordinate n^a perpendicular to the some constant time surface $t(x^i) = \mathrm{const}$, such that $n_a = -N\partial_a t$.

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The result is a foliation of a four-dimensional manifold $\mathcal{M}$ into a set of surfaces Σ on which quantum evolution will take place.
The metric of our interest becomes

$$q_{ab} = g_{ab} - n_a n_b,$$

such that $q_{ab}n^b = 0$.

With the coordinate transformation on the slice $x^i + dx^i$ and between the slices $x^i - N^i dt$ where N^i is a shift vector and introducing N, a lapse function being just a measure of the separation between the slices. As a result our metric becomes

$$ds^2 = N^2 dt^2 - q_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

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so

$$g_{\mu\nu} = \begin{pmatrix} N^2 & N_i \\ N_j & q_{ij} \end{pmatrix}$$

Moving back to 3D

q_{ab} determines the intrinsic curvature ${}^{(3)}R$ and introduces a notion of a 3D covariant derivative D_a,

$$D_e T^h_{fg} = q^a_e q^b_f q^c_g q^h_d \nabla_a T^d_{bc},$$

and 3D extrinsic curvature

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which upon inserting definitions takes the form

$$K_{ij} = \frac{1}{2N}\left(\partial_t q_{ij} - D_i N_j - D_j N_i\right)$$

Now ${}^{(4)}R_{abcd}$ is determined by ${}^{(3)}R_{abcd}$ and K_{ab}.

Action Principle in General Relativity

A generic form of an action

$$S = k \int_{\mathcal{M}} d^n x \mathcal{L} = k \int_{\mathcal{M}} d^n x \left(p\dot{q} - \mathcal{H}(p,q) + \sum_i \lambda_i C_i(p,q) \right)$$

In GR we are dealing with the Einstein Hilbert action

$$S = S_{EH} + S_{\mathrm{matter}} = \frac{1}{\kappa} \int_{\mathcal{M}} d^4 x \sqrt{g}^{(4)} R + S_{\mathrm{matter}}$$

Einstein-Hilbert action in the ADM formalism

Introducing the ADM formalism the curvature scalar becomes

$$^{(4)}R = ^{(3)}R - K_{ab}K^{ab} + K^2 + 2\nabla_a\left(n^b\nabla_b n^a - n^a\nabla_b n^b\right)$$

and our action obtains the form.

$$S = \frac{1}{16\pi G_N}\int d^4x N\sqrt{q}\left(^{(3)}R - K_{ab}K^{ab} + K^2\right)$$

Finding the canonical momenta

$$\pi^{ab} = \frac{\partial\mathcal{L}}{\partial(\partial_t q_{ab})} = \frac{1}{16\pi G_N}\left(q^{ab}K - K^{ab}\right),$$

with Poisson bracket

$$\{q_{ij}(x), \pi^{kl}(x')\} = \delta^k_{(i}\delta^l_{j)}\tilde{\delta}\left(x - x'\right)$$

We can rewrite the action in the Hamiltonian formalism

$$S = \int d^4x \pi^{ab}\partial_t q_{ab} - \underbrace{N^a\mathcal{H}_a - N\mathcal{H}}_{\text{constraints}}$$

Constraints in the Action

The lapse function and the shift vector are the Lagrange multipliers introducing two constraints:

1. Momentum constraint

$$\mathcal{H}_a = -2D_b\pi_a^b$$

2. Hamiltonian constraint

$$\mathcal{H} = \frac{16\pi G_N}{\sqrt{q}}\left(\pi^{ab}\pi_{ab} - 1/2\pi^2\right) - \frac{\sqrt{q}}{16\pi G_N}{}^{(3)}R$$

In general in any constrained Hamiltonian system, the first class constraints generate gauge transformations. $\mathcal{H}_a$ generates surface deformations of Σ_t and $\mathcal{H}$ generates the time translation $\Sigma_t \longrightarrow \Sigma_{t+\xi_0}$

Dirac Quantization of ADM GR - the Birth of Quantum Geometrodynamics

Choosing the representation $\Psi_{kl} = -i\frac{\delta}{\delta q^{kl}}$, the Hamiltonian constraint upon quantization should be annihilating the physical states. This yields the Wheeler-DeWitt equation :

$$\left(16\pi G_N G_{ijkl}\frac{\delta}{\delta q_{ij}}\frac{\delta}{\delta q_{kl}} + \frac{\sqrt{q}}{16\pi G_n}\ {}^{(3)}R + \mathcal{H}_{\mathrm{matter}}\right)\Psi[q] = 0$$

where $G_{ijkl} = \frac{1}{2\sqrt{q}}\left(q_{ik}q_{jl} + q_{il}q_{jk} - q_{ij}q_{kl}\right)$ is the DeWitt supermetric. Problems:

- action on a functional of q_{ij} gives $\delta(0)$
- suffers from ordering ambiguities.
- unclear what BC should be used

Quantization as a Means to an End.

- Easily obtained Quantum Geometrodynamics, however difficult to solve and has shown to produce no new physics.
- Existence of gauge symmetries - perhaps one should follow the path taken by quantization of gauge theories.
- Need to reformulation of GR in terms of connections which brings them closer to gauge theories: similar kinematics, but different (and more difficult) dynamic framework.

Reformulation of the theory in terms of vielbeins e_a^j.

Vielbeins

Vielbein = a set of vectors e_μ^I that denote the frame, such that

$$e_\mu^I e_\nu^J \eta_{IJ} = g_{\mu\nu}$$

vielbeins obey the following Local Lorentz transformation rule: $e_\mu^I \rightarrow \Lambda^I{}_J e_\mu^J$.
When parallel transported the vielbein would also change, thus we need to introduce a connection that would define the way it changes.

Spin Connection

Let V^I be an element with a tetrad index, then when parallel transported it would behave as:

$$\nabla_\mu V^I = \partial_\mu V^I + \omega^I_{\mu J} V^J,$$

where $\omega^I_{\mu J}$ is the *tetrad compatible spin connection*. From $\nabla_\mu e^I_\nu = \partial_\mu e^I_\nu - \Gamma^\rho_{\nu\mu} e^I_\rho + \varepsilon^{IJK} \omega_{J\mu} e_{K\nu} = 0$ we can obtain its form. In this formulation now the EH action becomes

$$S = \frac{1}{16\pi G_N} \int d^4x |e| e^{\mu I} e^{\nu J} R_{\mu\nu IJ},$$

where $R_{\mu\nu IJ} = \partial_\mu \omega^I_{\nu J} + \omega^I_{\mu K} \omega^K_{\nu J} + (\mu \leftrightarrow \nu)$ or $R_{\mu\nu IJ} = R_{\mu\nu\rho\sigma} e^\rho_I e^\sigma_J$.
In the gauge $e^0_I = 0$, where $I = 1, 2, 3$ it is easy to check that

$$\Gamma^I_i = \frac{i}{2} \varepsilon^{0IJK} \omega_{iJK} \text{ and } \omega^{0I}_i = K^I_i = e^{jI} K_{ij}$$

where K_{ij} is the extrinsic curvature.
Note that in this gauge $SO(3,1) \rightarrow SO(3)$.

Action in terms of fields A and E

Then we can define the field

$$A^I_i(\gamma) = \Gamma^I_i + \gamma K^I_i,$$

where γ is the Immirzi parameter introduced by Immirzi and Barbero.The second field that will be relevant to us is:

$$E^i_I = \sqrt{q} e^i_I$$

which is just a densitised triad transforming under the vector representation of SU(2).
Now our action reads

$$S = \frac{1}{16\pi G_N} \int dt \int d^3x \frac{1}{\gamma} A^I_i \frac{\partial}{\partial t} E^i_I - iA_{0I} G^I + iN^i V_i + \frac{N}{2\sqrt{q}} H + h.c.$$

Constraints in the Gauge Theory Formalism

The Hamiltonian constraint

$$H = \varepsilon^{IJK} E^i_I E^j_J F_{ijK} - 2\frac{1+\gamma^2}{\gamma^2} E^i_{[J} E^j_{I]} \left(A^I_i(\gamma) - \Gamma^I_i\right)\left(A^J_j(\gamma) - \Gamma^J_j\right),$$

and the Gauss and the vector constraints:

$$\begin{aligned} G^I &= \partial_j E^{jI} + \varepsilon^I{}_{JK} A^J_j E^{jK} \equiv D_j E^{jI}, \\ V_i &= E^j_I F^I_{ij}, \end{aligned}$$

where

$$F^I_{ij} = \partial_i A^I_j - \partial_j A^I_i + \varepsilon^{IJK} A_{iJ} A_{jK}$$

This is a field strength tensor for an SO(3) (or SU(2)) field.

Theorem

For a phase space $\left(A^J_a, E^b_K\right)$ with the Poisson structure

$$\{E^a_J(x), E^b_K(y)\} = 0 = \{A^J_a(x), A^K_b(y)\} \text{and} \{E^a_J(x), A^b_K(y)\} = 8\pi G_n \gamma \delta^a_b \delta^K_J \delta(x-y)$$

and the above constraints, then this structure can be rewritten in the ADM phase space with the momentum and Hamiltonian constraints $\mathcal{H}^i$ and $\mathcal{H}$.

First steps into LQG

Overall the theory is invariant under local SO(3) (or SU(2)), the 3D diffeomorphism of the time slice and the coordinate time translation.
Start with a space of functionals $\Psi[A]$, and promote the Poisson bracket to a commutator, then

$$E^i_I = -8\pi\gamma G_N \frac{\delta}{\delta A^I_i} \tag{1}$$

The Gauss constraint generates SO(3) gauge transformations - quantum states need to be invariant under these. What are the states?

Parallel transport along the curve implies

For a gauge covariant derivative $D_\mu = \partial_\mu + A_\mu$, we can define the parallel transport equation for a curve $\alpha[0,1] \rightarrow \Sigma$

$$\frac{dx^\mu}{ds} D_\mu V_\nu = \frac{dV_\nu}{ds} + \underbrace{\frac{dx^\mu}{ds} A_\mu}_{\equiv A(s)} V_\nu = 0$$

Its a first order ODE. Solution $V_\nu\left(s\right) = U\left(s, 0\right) V_\nu\left(0\right)$ such that

$$v\left(s\right) = v(0) - \int\limits_0^s ds_1 A\left(s_1\right) v\left(s_1\right)$$

$$v\left(s\right) = v(0) - \int\limits_0^s ds_1 A\left(s_1\right) \left(v(0) - \int\limits_0^{s_1} ds_2 A\left(s_2\right) v\left(s_2\right) \right)$$

The Closed Solution to the Parallel Transport Equation

Let us define the path ordering operator

$$\mathcal{P}\left(\alpha\left(s_1\right)\beta\left(s_2\right)\right)=\left\{\begin{array}{ll}\alpha\left(s_1\right)\beta\left(s_2\right) & s_1>s_2\\ \beta\left(s_2\right)\alpha\left(s_1\right) & s_1<s_2\end{array}\right.$$

Using this we can write the iterating series as

$$U\left(s,0\right)=\sum_n\frac{\left(-1\right)^n}{n!}\mathcal{P}\left(\int\limits_0^s ds_1 A\left(s\right)\right)=\mathcal{P}\left(e^{-\int\limits_0^s ds_1 A(s)}\right)$$

which gives us the solution to the parallel transport equation.

Action of the gauge group on the $U(s, 0)$

Under the gauge group the matrix transforms as

$$U(s, s_1) \rightarrow g(s) U(s, s_1) g^{-1}(s_1)$$

For a loop the ends are the same, and we can then take the trace $\mathrm{Tr}[U_\alpha(0, 1)]$, a.k.a. holonomy, or the Wilson loop, which is a a gauge invariant function of the field A_μ.
Hence the name Loop Quantum Gravity.

Spin Networks

Spin Networks generalised Wilson loops to graphs.
Each spin network determines a gauge invariant functional $\Psi[A]$, using the algorithm:

- each edge labelled $s_1 =$ holonomy of A in rep. s_1
- Each vertex intertwiner that combines holonomies into an invariant.

Geometrical Operators

Anything that is gauge invariant would work e.g.

$$\begin{aligned}
\mathcal{T}[\alpha] &= -\mathrm{Tr}[U_\alpha(0,1)] \\
\mathcal{T}^i[\alpha](s) &= -\mathrm{Tr}[U_\alpha(s,s)E^i(s)] \\
\mathcal{T}^{i_1\ldots i_n}[\alpha](s_1,\ldots,s_n) &= -\mathrm{Tr}[U_\alpha(s_1,s_n)E^{i_n}(s_n)\ldots U_\alpha(s_2,s_1)E^{i_1}(s_1)].
\end{aligned}$$

Given a 2D surface choose a direction perpendicular to that surface and define a little patch of surface parametrized by σ_1 and σ_2. Then

$$E_I^3 = \varepsilon_{ijk}\frac{\partial x^i}{\partial \sigma_1}\frac{\partial x^j}{\partial \sigma_2}E_I^k,$$

then

$$\mathcal{T}^i[\alpha](s)$$

$$= -8\pi\gamma G_N \int_S d\sigma_1 d\sigma_2 \varepsilon_{ijk}\frac{\partial x^i}{\partial \sigma_1}\frac{\partial x^j}{\partial \sigma_2}\frac{\delta}{\delta A_k^I}\mathcal{P}\left(\exp\int_0^s ds_1 A^I(s)\,\tau_I\right)$$

Quantized Area

The area observable $\mathcal{A}_\Sigma$:

$$\mathcal{A}_\Sigma = \int_\Sigma dx^1 dx^2 \sqrt{\det h_{ij}} = \int_\Sigma dx^1 dx^2 \sqrt{E^3_I E^{3I}}$$

Thus for an operator

$$\mathrm{Area} = \int \sqrt{E^3_I E^{3I}}$$

we get the eigenvalue of $8\pi\gamma G_N \sqrt{j(j+1)}$ for a line with spin j intersecting the surface. Thus the spin networks are aeigenfunctions of the area operator.
The same can be done with the volume operator, but the calculations are much more elaborate.

Loop Quantum Cosmology

To get cosmological predictions just take the q_{ij} part of the FLRW metric

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2 \right) = -dt^2 + a(t)^2 q_{ij} dx^i dx^j.$$

Ashtekar, Bojowald and Lewandowski have also kept the matter Hamiltonian as a function of a scalar field i.e. $\mathcal{H}_{\mathrm{matter}} = \mathcal{H}_{\mathrm{matter}}(\phi)$.

LQC Predictions

Their procedure yielded the following results:

- the Big Bang singularity has been removed and replaced by a *Big Bounce*,
- at a large volume limit their equations reproduce the Wheeler-DeWitt equation, with a complicated choice of ordering of operators.

Problems:

- This is just a toy model - all but one degrees of freedom are suppressed.
- There are great problems with finding dynamics of this system.
- So far any proposed experimental tests or not feasible.

Conclusions

LQG ...

1. is a conservative canonical quantization of GR,
2. provides a background independent, nonperturbative treatment of gravity,
3. introduces geometric quanta with spectra bounded from below, thus it seems to be removing singularities that plague GR at small scales.

Questions?

Thank you for your attention.
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