Large Scale Structure Formation

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Contents

Preface

This report is written as part of the Student Seminar in Theoretical Physics. Its content concerns the formation of Large Scale Structure in the Universe. We follow the outline of [1] for the greater part, although we also rely on [2–7]. The study of Large Scale Structure formation concerns itself with how small fluctuations in, among others, the mass distribution grows as the universe evolves. The growth of these fluctuations explains the formation of galaxies and galaxy clusters. The theory of large scale structure formation allows us to understand why galaxies and hence stars come into being from some small fluctuations, which presumably arise from quantum fluctuations, but also links models for the very early Universe to observational data arising from measurements of the cosmic background radiation. We concern ourselves mainly with linear perturbation theory which may be treated analytically. Linear perturbation theory only provides information about how density fluctuations have grown but not the eventual collapse of matter into galaxies and stars. First we give a general overview of the process of structure formation. The second chapter is concerned with a simple Newtonian toy model which gives us some insights in the physical processes. In the third chapter we treat the results of the linearized General Relativistical treatments of the cosmological perturbations. In the fourth chapter we discuss a toy model for the quantum mechanical origins of the cosmological fluctuations. Finally we remark on some observational results, although these are greatly model dependent. These models are very complicated and cannot be treated analytically and are therefore not part of this report. Models for the final phases of the formation of structure that is how galaxies and stars form, are also left out although many computer models exist which give good predictions.

Chapter 1

Large Scale Structure Formation

The scenario of large scale structure formation is widely believed to be described by the following. Fluctuations in the density of matter (or rather energy) arise in the very early universe. These fluctuations are generally attributed to quantum fluctuations and were small in both length scale and density. That is $\delta\rho/\rho_0 \ll 1$, if $\delta\rho$ denotes the variation of the average density and the average density is denoted by ρ_0. Thanks to inflation these fluctuations blew up in size, the wavelengths could even have grown beyond the Hubble radius, although the fluctuations in density were still small. We note that during inflation the density ρ is caused by the inflaton. This implies that interestingly enough we need both Quantum mechanics and General Relativity to explain the formation of structure in the universe. After inflation the system relaxes and matter starts contracting.[1] Contraction is solely a consequence of the purely attractive nature of the gravitational force. In a Newtonian setting it is clear that, if one starts with a small patch of space with an excess $\delta\rho$ of matter, that this excess pulls surrounding matter towards the centre of the patch. The magnitude of this force is proportional to $\delta\rho$. So one has

$$\ddot{\delta\rho} \sim G\delta\rho. \tag{1.1}$$

From this simple equation we already see that instability is exponential in nature, if we ignore the expansion of the Universe and pressure effects. This initial phase of gravitational collapse is well described by linear perturbation theory and the process is referred to as linear growth of structure.

After some time the linear approximation breaks down and galaxies and stars start forming. These structures form in a two step process. First the cold dark matter and baryonic matter collapse together. This is referred to as the non-linear growth of structure. In a general relativistic setting this step and the previous era can better be viewed like this;

[1]At scales not of the size of the Hubble radius, at the Hubble radius the perturbations are frozen, see chapter 4 for a more extensive discussion.

regions of space with excess matter expanded more slowly until eventually they contract. In the second stage the baryonic matter loses its energy through radiative cooling so that the matter could condense into 'small' clouds of gas from which galaxies and stars could form. Dark matter particles however could not lose their energy via radiative cooling and therefore were unable to condense into very small clouds but remained as halos around the galaxies. This process is sometimes referred to as gastrophysical evolution.

In the following chapters we shall choose natural units unless specified otherwise. Moreover we shall sum over repeated indices, in particular in the non-relativistic treatment we will sum over repeated indices even if both indices are down.

Chapter 2

Newtonian Theory of Cosmological Perturbations

We shall start this chapter by describing the collapse of hydrodynamical matter in a Newtonian setting in a fixed background metric and discuss some critical length scale beyond which the collapse occurs. Then we move on to a semi-Newtonian treatment, where we take the background to be homogeneous, isotropic and expanding, meaning we take the background metric to be given by

$$ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2,$$

where t is viewed as a physical time, as usual, $d\mathbf{x}^2$ is the Euclidean metric and $a(t)$ denotes the scale factor. The matter distribution evolves Newtonian via the Poisson equation and we avoid a back reaction of the matter on the metric. The contents of this chapter is based on [1], whose notation we shall also use, but also relies on [4].

2.1 Perturbations in Minkowski Space

We start by discussing the evolution of matter in the form of an ideal fluid in a fixed non-expanding background. So we see matter as a perfect fluid and take gravity to be adequately described by the Newtonian gravitational potential ϕ. We denote the energy density by ρ, the pressure by p, the fluid velocity by $\mathbf{v}$ and the entropy density by S. We also know that the pressure is a function of the energy density and the entropy density. The dynamics of the fluid are governed by the following basic hydrodynamical equations:

- The continuity equation

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (v_i \partial_i)\rho = -\rho(\partial_i v_i), \tag{2.1}$$

which says that the change of energy density in a region of space must equal the energy flowing in minus the energy flowing out.

- The Euler equation

$$\frac{dv_i}{dt} + \partial_i\phi = \frac{\partial v_i}{\partial t} + (v_j\partial_j)v_i + \partial_i\phi = -\rho^{-1}\partial_i p. \tag{2.2}$$

This equation tells us that pressure in a small region of space is caused by fluid flowing in the region and by a driving force, in this case provided by gravity.

- The Poisson equation for Newtonian gravity

$$\nabla^2\phi = \partial_i\partial_i\phi = 4\pi G\rho. \tag{2.3}$$

- Entropy conservation, which is a consequence of the assumption that matter in this toy model behaves like an ideal fluid,

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + (v_i\partial_i)S = 0. \tag{2.4}$$

As mentioned above, we assume that we are working in a fixed static background and are thus faced with some background constants. The background energy density will be denoted by ρ_0, the background pressure by p_0, the background constant gravitational potential by ϕ_0 and the constant entropy density by S_0. The background velocity is set to zero. As the background gravitational potential is constant it is clear that

$$\nabla^2\phi_0 = \partial_i\partial_i\phi_0 = 0.$$

Although obviously

$$4\pi G\rho_0 \neq 0,$$

so that the background equations are not valid, if one only allows the 'usual' matter content (we shall discuss a solution to this problem below). Thus neither the background Einstein equations nor the background poisson equations are satisfied in this setting. Weinberg, page 562 of [8], puts this as follows 'The effects of gravitation were ignored in the unperturbed "solution" '. Jeans [9] (see in particular pages 49 and 50) has a different method based on taking some limit of a slowly rotating nebula, but he also notes that if certain boundary conditions at infinity are not met, 'the only solution is $\rho_0 = 0$'. This problem may be remedied by inserting a density distribution with negative energy density ρ_{vacuum} or matter with a negative pressure (if one allows for even more relativistical input), which does not exhibit same the hydrodynamical behavior as the other matter component. In a General Relativistical setting this ρ_{vacuum} would be associated with a cosmological constant Λ. This may be made more exact in the following manner: we know that, in the weak field approximation,[1] the following identification holds

$$R_{00} = \nabla^2 \phi \tag{2.5}$$

and furthermore that if $g_{\mu\nu} \simeq \eta_{\mu\nu}$ and the energy momentum tensor belongs to an ideal fluid

$$R_{00} = 4\pi G(\rho_m + 3p_m) - \Lambda. \tag{2.6}$$

If we associate an energy density to the cosmological constant

$$\rho_{\text{vacuum}} = \frac{1}{8\pi G}\Lambda,$$

whose equation of state reads

$$p_{\text{vacuum}} = -\rho_{\text{vacuum}},$$

we may recast (2.5) and (2.6) as

$$\begin{aligned}
\nabla^2 \phi = R_{00} &= 4\pi G\big[(\rho_m + 3p_m) - 2\rho_{\text{vacuum}}\big] \\
&= 4\pi G\big[(\rho_m + 3p_m) + (\rho_{\text{vacuum}} + p_{\text{vacuum}})\big] \\
&= 4\pi G\big[\rho_t + 3p_t\big],
\end{aligned}$$

where we used the index t to denote the sum of the matter contribution denoted by m and the vacuum contribution. We may now choose ρ_{vacuum} and thus Λ such that $(\rho_t + 3p_t) = 0$ and thus we satisfy the background equations. It must be noted that this description uses explicit relativistical input, for example there is no real Newtonian analog of a cosmological constant. We note that we may consider absorbing the pressure associated to the cosmological constant in the background pressure p_0 already mentioned. It is quite natural to need a cosmological constant to construct a static Universe as we see in classical example of the static Einstein Universe, although as we already mentioned the introduction of such an object will remain foreign in a semi-Newtonian setting.

We now proceed to perturb the fluid variables around the background, which results in

$$\begin{aligned}
\rho(\mathbf{x}, t) &= \rho_0 + \delta\rho(\mathbf{x}, t) \\
\mathbf{v}(\mathbf{x}, t) &= \delta\mathbf{v}(\mathbf{x}, t) \\
p(\mathbf{x}, t) &= p_0 + \delta p(\mathbf{x}, t) \\
\phi(\mathbf{x}, t) &= \phi_0 + \delta\phi(\mathbf{x}, t) \\
S(\mathbf{x}, t) &= S_0 + \delta S(\mathbf{x}, t).
\end{aligned} \tag{2.7}$$

[1] See for example Prokopec [10].

Inserting formula (2.7) into the equations (2.1)-(2.4) and keeping only those terms linear in the perturbations, we see that

$$\frac{\partial(\rho_0+\delta\rho)}{\partial t}+(\delta v_i\partial_i)(\rho_0+\delta\rho)+(\rho_0+\delta\rho)(\partial_i\delta v_i)=0$$

yields

$$\frac{\partial\delta\rho}{\partial t}+\rho_0\partial_i\delta v_i=0. \tag{2.8}$$

The Euler equation

$$\frac{\partial\delta v_i}{\partial t}+(\delta v_j\partial_j)\delta v_i+\partial_i(\phi_0+\delta\phi)+(\rho_0+\delta\rho)^{-1}\partial_i(p_0+\delta p)=0,$$

yields

$$\frac{\partial\delta v_i}{\partial t}+\frac{1}{\rho_0}\partial_i\delta p+\partial_i\delta\phi=0. \tag{2.9}$$

Finally the Poisson equation and entropy conservation results in

$$\partial_i\partial_i\delta\phi=4\pi G\,\delta\rho$$

and

$$\frac{d\,\delta S}{dt}=0.$$

Letting ∂_i work on equation (2.9) and inserting (2.8) into the result we see that

$$\frac{\partial^2\delta\rho}{\partial t^2}-\nabla^2\delta p-\rho_0\nabla^2\delta\phi=0.$$

If we now use the Poisson equation and the equation of state

$$\delta p=c_s^2\delta\rho+\sigma\delta S, \tag{2.10}$$

with

$$c_s^2=\left(\frac{\delta p}{\delta\rho}\right)\Big|_S$$
$$\sigma=\left(\frac{\delta p}{\delta S}\right)\Big|_\rho,$$

where c_s is identified with the speed of sound, we may rewrite this equation as

$$\frac{\partial^2\delta\rho}{\partial t^2}-c_s^2\nabla^2\delta\rho-4\pi G\rho_0\delta\rho=\sigma\nabla^2\delta S.$$

Realizing that

$$\frac{\partial\delta\rho}{\partial t}=\frac{d\delta\rho}{dt}+\mathcal{O}(\delta^2),$$

we see that we may write

$$\ddot{\delta\rho} - c_s^2\nabla^2\delta\rho - 4\pi G\rho_0\delta\rho = \sigma\nabla^2\delta S, \tag{2.11}$$

as we work up to first order. We see that $4\pi G\rho_0\delta\rho$ gives us a purely attractive force which was already discussed from first principles, see equation (1.1). This attractive force causes instabilities. There is also a nice interpretation of $c_s^2\nabla^2\delta\rho$, this term is caused by pressure and tends to yield pressure waves. Together with equation (2.4), this gives us the evolution of the energy density fluctuations and the entropy perturbations.

Traditionally there are two types of fluctuations which are given special attention:

- Adiabatic fluctuation, where the entropy fluctuations are set to zero. In a 'realistic' multicomponent model, in the sense that the number of photons is much larger as the number of baryons, this means roughly speaking that the density perturbations of all baryons are determined by temperature fluctuations.[2]
- Entropy fluctuations, where $\dot{\delta\rho} = 0$, but $\delta S \neq 0$. Naturally we denote the total time derivative by an upper dot.

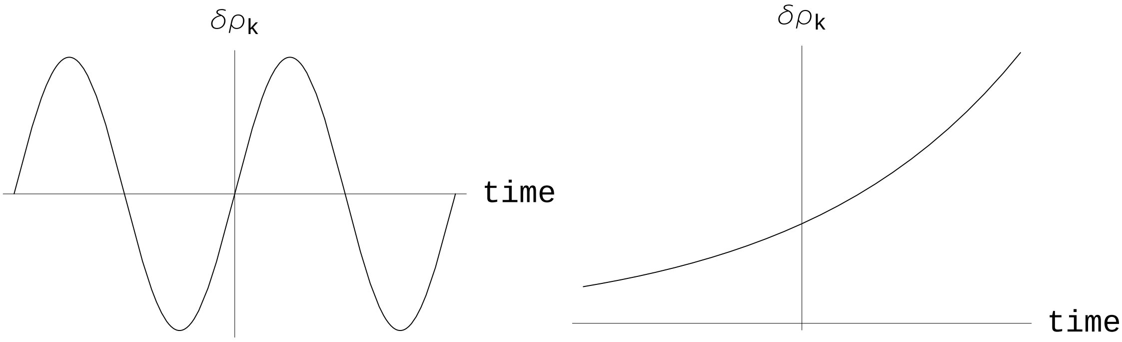

Figure 2.1: Typical oscillatory and exponentially growing solutions of the equations of motions for some mode of the density fluctuation are depicted.

From equation (2.11) and (2.4) we see that the entropy fluctuations do not grow in time and adiabatic fluctuations are time dependent. However, the density fluctuations do not necessarily grow. To exhibit this we shall focus on adiabatic fluctuations and fourier decompose the fluctuation field $\delta\rho(\mathbf{x}, t)$ as follows

$$\delta\rho(\mathbf{x}, t) = \int e^{i\mathbf{k}\cdot\mathbf{x}}\delta\rho_{\mathbf{k}}(t)\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}.$$

Inserting this into (2.11) we see that

$$\ddot{\delta\rho}_{\mathbf{k}}(t) + c_s^2\mathbf{k}^2\delta\rho_{\mathbf{k}} - 4\pi G\rho_0\delta\rho_{\mathbf{k}} = 0.$$

[2]For a full treatment see page 144 of [6]

It is now clear that a critical wavelength exists, the so called Jeans length k_J,

$$k_J = \left(\frac{4\pi G\rho_0}{c_s^2}\right)^{1/2},$$

such that if $|\mathbf{k}| = k < k_J$, $\delta\rho$ grows exponentially, but if $k > k_J$ the fluctuations $\delta\rho$ will oscillate. So more explicitly the solutions read

$$\delta\rho_{\mathbf{k}} = A_{\mathbf{k}} \sin\left(\sqrt{c_s^2\mathbf{k}^2 - 4\pi G\rho_0}t\right) + B_{\mathbf{k}} \cos\left(\sqrt{c_s^2\mathbf{k}^2 - 4\pi G\rho_0}t\right),$$

for $k > k_J$ and

$$\delta\rho_{\mathbf{k}} = A_{\mathbf{k}} e^{\sqrt{c_s^2\mathbf{k}^2 - 4\pi G\rho_0}t} + B_{\mathbf{k}} e^{-\sqrt{c_s^2\mathbf{k}^2 - 4\pi G\rho_0}t},$$

for k otherwise. We note that for sufficiently large t we may ignore the decaying exponential modes. The oscillatory and decaying solutions are depicted in figure 2.1.

2.2 Perturbations in Expanding Space

We shall now improve our derivations in the previous section by letting the background expand. Still we do not allow for a reaction of the matter fluctuations on the background metric. But in this setting the background equations are consistent, a significant improvement on the model in static Minkowski space which where we have seen that the background equations are very problematic in nature and even inconsistent without some relativistical input. To be precise we now define the metric as

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2(t), -a^2(t), -a^2(t)).$$

Furthermore we introduce the Hubble parameter $H(t)$

$$H(t) = \frac{\dot{a}(t)}{a(t)}.$$

The background cosmological model is described by a background energy density $\rho_0(t)$, note that we now allow for dependence on time, pressure $p_0(t)$ and the recessional velocity $\mathbf{v}_0 = H(t)\mathbf{x}$. We shall now make use of a slightly different notation for the perturbations, to shorten the formulae, and write

$$\begin{aligned}
\rho(\mathbf{x}, t) &= \rho_0(t)(1 + \delta_\epsilon(\mathbf{x}, t)) \\
\mathbf{v}(\mathbf{x}, t) &= \mathbf{v}_0(\mathbf{x}, t) + \delta\mathbf{v}(\mathbf{x}, t) = H(t)\mathbf{x} + \delta\mathbf{v}(\mathbf{x}, t) \\
p(\mathbf{x}, t) &= p_0(t) + \delta p(\mathbf{x}, t)
\end{aligned} \tag{2.12}$$

Note that δ_ϵ denotes the fractional density perturbation. Also using formula (2.10), we insert these expressions into equations (2.1)-(2.4). Then we see the following, where we drop terms beyond linear order in the perturbations whenever convenient:

$$\begin{aligned}\partial_t v_i + v_j\partial_j v_i + \partial_i\phi + \rho^{-1}\partial_i\delta p &= 0\\ \partial_t\partial_i v_i + (\partial_i v_j)(\partial_j v_i) + v_j\partial_j(\partial_i v_i) + \partial_i\partial_i\phi + \rho^{-1}\partial_i\delta p &= 0\\ \partial_t(3H + \partial_i\delta v_i) + (\delta_{ij}H + \partial_i\delta v_j)(\delta_{ji}H + \partial_j\delta v_i) + &\\ (Hx_j + \delta v_j)\partial_j(3H + \partial_i\delta_i) + 4\pi\rho_0(1+\delta_\epsilon) + \rho_0^{-1}\partial_i\partial_i\delta p &= 0\\ 3\partial_t H + 3H^2 + 4\pi G\rho_0 + \partial_t\partial_i\delta v_i + Hx_j\partial_j(\partial_i\delta v_i) + 2H\partial_i\delta v_i + 4\pi G\rho_0\delta_\epsilon + \rho_0^{-1}\partial_i\partial_i\delta p &= 0\\ 3\partial_t H + 3H^2 + 4\pi G\rho_0 + d_t\partial_i\delta v_i + 2H\partial_i\delta v_i + 4\pi G\rho_0\delta_\epsilon + \rho_0^{-1}\partial_i\partial_i\delta p &= 0,\end{aligned} \tag{2.13}$$

where we have defined

$$d_t = \partial_t + Hx_j\partial_j = \partial_t + v_j\partial_j,$$

From equation (2.1) and equation (2.2), we see that

$$\begin{aligned}\partial_t\rho + v_i\partial_i\rho + \rho\partial_i v_i &= 0\\ (\partial_t\rho_0)(1+\delta_\epsilon) + \rho_0\partial_t\delta_\epsilon + (Hx_i + \delta v_i)\partial_i(\rho_0(1+\delta_\epsilon)) + \rho_0(1+\delta_\epsilon)\partial_i v_i &= 0\\ \frac{\partial_t\rho_0}{\rho_0} + \partial_t\delta_\epsilon + Hx_i\partial_i\delta_\epsilon + \partial_i v_i &= 0\\ \frac{\partial_t\rho_0}{\rho_0} + 3H + d_t\delta_\epsilon + \partial_i\delta v_i &= 0.\end{aligned} \tag{2.14}$$

From the zeroth order of the above equation we deduce that

$$\frac{\partial_t\rho_0}{\rho_0} + 3H = 0,$$

so that

$$\rho_0 \propto a^{-3},$$

consistent with the Newtonian limit of the Friedmann universe (see for a more lengthy discussion pages 136 and 137 of [4]). The first order equation in the perturbations of (2.14) reads

$$d_t\delta_\epsilon + \partial_i\delta v_i = 0$$

and of (2.13)

$$d_t\partial_i\delta v_i + 2H\partial_i\delta v_i + 4\pi G\rho_0\delta_\epsilon + \rho_0^{-1}\partial_i\partial_i\delta p = 0.$$

Combining these equations we see

$$d_t^2\delta_\epsilon + 2Hd_t\delta_\epsilon - 4\pi G\rho_0\delta_\epsilon - \rho_0^{-1}\nabla^2\delta p = 0.$$

If we now use equation (2.10) which in this case reads

$$\delta p = c_s^2 \rho_0 \delta_\epsilon + \sigma \delta S,$$

we see that

$$d_t^2 \delta_\epsilon + 2H d_t \delta_\epsilon - c_s^2 \nabla^2 \delta_\epsilon - 4\pi G \rho_0 \delta_\epsilon = \frac{\sigma}{\rho_0} \nabla^2 \delta S, \tag{2.15}$$

where ∇^2 denotes the Laplacian as usual. Naturally we retain

$$\dot{\delta S} = 0,$$

as a consequence of equation (2.4). Sometimes the result is formulated in terms of comoving coordinates $\mathbf{q}$, "coordinates pinned on the expanding background"

$$\mathbf{x} = a(t)\mathbf{q}.$$

In this coordinate system the Laplacian, denoted by ∇_q^2, looks different

$$\nabla^2 = \frac{1}{a^2} \nabla_q^2,$$

so that (2.15) reads

$$d_t^2 \delta_\epsilon + 2H d_t \delta_\epsilon - \frac{c_s^2}{a^2} \nabla_q^2 \delta_\epsilon - 4\pi G \rho_0 \delta_\epsilon = \frac{\sigma}{\rho_0 a^2} \nabla_q^2 \delta S. \tag{2.16}$$

If we compare the equation (2.15) to the result of our derivation in a static background (2.11) we see that we only have one additional term:

$$2H d_t \delta_\epsilon,$$

which is interpreted as a damping term. However our discussion of the Jeans length is not dramatically influenced by this, because we have excluded perturbations of the metric. More explicitly, if we again set the entropy fluctuations to zero we have that

$$d_t^2 \delta_\epsilon + 2H d_t \delta_\epsilon - c_s^2 \nabla^2 \delta_\epsilon - 4\pi G \rho_0 \delta_\epsilon = 0,$$

which for convenience we shall write as

$$\ddot{\delta}_\epsilon + 2H \dot{\delta}_\epsilon - c_s^2 \nabla^2 \delta_\epsilon - 4\pi G \rho_0 \delta_\epsilon = 0.$$

Fourier transforming as we did in the discussion of perturbations in Minkowski space;

$$\delta_\epsilon(t) = \int e^{i\mathbf{k}\cdot\mathbf{x}} \delta_{\epsilon,\mathbf{k}}(t),$$

yields

$$\ddot{\delta_{\epsilon,\mathbf{k}}} + 2H \dot{\delta_{\epsilon,\mathbf{k}}} + c_s^2 \mathbf{k}^2 \delta_{\epsilon,\mathbf{k}} - 4\pi G \rho_0 \delta_{\epsilon,\mathbf{k}} = 0.$$

If we now introduce the variable

$$\ddot{F}_{\mathbf{k}} \equiv a\delta_{\epsilon,\mathbf{k}},$$

we see that we get, using $|\mathbf{k}| = k$,

$$\ddot{F}_{\mathbf{k}} = \left(\frac{\ddot{a}}{a} - c_s^2 k^2 + 4\pi G\rho_0\right) F_{\mathbf{k}}, \tag{2.17}$$

so that the Jeans wavelength is now given by

$$k_J = \frac{4\pi G\rho_0 + \frac{\ddot{a}}{a}}{c_s^2}.$$

If we furthermore assume that

$$a(t) = t^\alpha \qquad \text{and} \qquad \rho_0 = \frac{R_0}{t^2},$$

we see that equation (2.18) reads

$$\ddot{F}_{\mathbf{k}} = \left(\frac{\alpha(\alpha-1)}{t^2} + \frac{4\pi G R_0}{t^2} - c_s^2 k^2\right) F_{\mathbf{k}}. \tag{2.18}$$

We know that the solutions of this equation are the product of the square root of t and Bessel functions. The evolution of $\delta_{\epsilon,\mathbf{k}}$ for some, not general, initial conditions is depicted in figure 2.2.

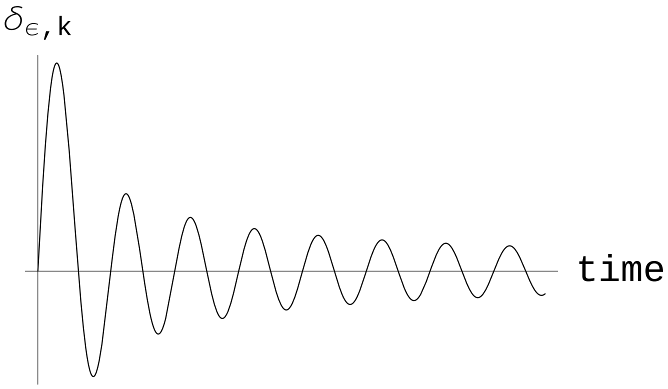

Figure 2.2: Some possible evolution of $\delta_{\epsilon,\mathbf{k}}$.

Equation (2.16) may be easily extended to a system where matter consists of several components. Let us distinguish the fluid variables of each component with an under index A, with

$$A \in \{1, 2, \ldots, k\}.$$

Now (2.16) generalizes to

$$\ddot{\delta}_{\epsilon\, A} + 2H\dot{\delta}_{\epsilon\, A} - \frac{c_{s\,A}^2}{a^2}\nabla_q^2 \delta_{\epsilon\, A} - \sum_{A=1}^{k} 4\pi G\rho_{0\,A}\delta_{\epsilon\, A} = \frac{\sigma_A}{\rho_{0\,A} a^2}\nabla_q^2 \delta S_A.$$

This equation gives some ideas about the treatment of cold dark matter in combination with less exotic forms of matter, however a good description of relativistic gasses is not really possible, because we have taken Newtonian limits.

From our investigations of the Newtonian treatment we already get a basic understanding of the first steps of structure formation. We have seen quite explicitly in what way matter contracts.

Chapter 3

Relativistic Cosmological Perturbation Theory

In this chapter we will treat all the details of the calculations as we have done in the previous chapter, because they can be found in the excellent book by Weinberg [6]. Note that we shall often use the same notation and convention of the sign of the metric as Brandenberger [1], which differs in both notation and convention of the sign of the metric from Weinberg [6]. In this chapter we shall, like Weinberg, always assume that the background is a flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe.

The need to use General Relativity to describe cosmological fluctuations arises when considering scales larger than the Hubble scale . This is the point at which the metric will start to have real influence on the dynamics. This is made more explicit in the next paragraphs.

We begin by expanding the metric about the FLRW background metric, which we shall denote by $g^{(0)}_{\mu\nu}$

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$$

We note that the metric fluctuations $\delta g_{\mu\nu}$ depend on space and time and since it is a symmetric tensor there are at first glace 10 degrees of freedom. We shall see that not all degrees of freedom are physical but some depend on the coordinate system we use. The freedom to choose our coordinate system is referred to as gauge freedom in this context and the nonphysical degrees of freedom as gauge artifacts. Traditionally the fluctuations of the metric are subdivided into fluctuations which correspond to scalar, vector and tensor fluctuations. To be precise we write

$$\delta g_{\mu\nu} = \delta g^S_{\mu\nu} + \delta g^V_{\mu\nu} + \delta g^T_{\mu\nu},$$

where we may take $\delta g_{\mu\nu}$ to be ether a function of time t or a function of conformal time η, the latter will be of importance later. The subdivision refers to the manner in which the fields describing the perturbations in the metric transform under a coordinate transformation of a three dimensional constant time hypersurface. We may derive that there are four degrees of freedom which can be written in terms of scalars. We may verify that the scalar fluctuations can always be written as[1]

$$\delta g^S_{\mu\nu} = a^2 \begin{pmatrix} 2\phi & -\partial_1 B & -\partial_2 B & -\partial_3 B \\ -\partial_1 B & 2(\psi - \partial_1\partial_1 E) & -2\partial_1\partial_2 E & -2\partial_1\partial_3 E \\ -\partial_2 B & -2\partial_2\partial_1 E & 2(\psi - \partial_2\partial_2 E) & -2\partial_2\partial_3 E \\ -\partial_3 B & -2\partial_3\partial_1 E & -2\partial_3\partial_2 E & 2(\psi - \partial_3\partial_3 E) \end{pmatrix}$$

with ϕ, ψ, B and E scalar fields as announced earlier. We know that traditionally the δg_{00} is identified in the Newtonian limit with (two times) the Newtonian potential in a flat static background, see for example page 147 of [11]. However we have not resolved all gauge issues for these scalar fluctuations.

The vector fluctuations make up for four degrees of freedom and we may always write the vector fluctuations as follows

$$\delta g^V_{\mu\nu} = a^2 \begin{pmatrix} 0 & -S_1 & -S_2 & -S_3 \\ -S_1 & 2\partial_1 F_1 & \partial_1 F_2 + \partial_2 F_1 & \partial_1 F_3 + \partial_3 F_1 \\ -S_2 & \partial_1 F_2 + \partial_2 F_1 & 2\partial_2 F_2 & \partial_2 F_3 + \partial_3 F_2 \\ -S_3 & \partial_1 F_3 + \partial_3 F_1 & \partial_2 F_3 + \partial_3 F_2 & 2\partial_3 F_3 \end{pmatrix}$$

where S_i and F_i are two three dimensional vectors, which also satisfy

$$\frac{\partial S_i}{\partial x^i} = \frac{\partial F_i}{\partial x^i} = 0.$$

The tensor fluctuations correspond to the two polarizations of gravitational waves and can be written as

$$\delta g^T_{\mu\nu} = a^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & h_{13} \\ 0 & h_{12} & h_{22} & h_{23} \\ 0 & h_{13} & h_{23} & h_{33} \end{pmatrix}$$

where we have

$$h_{ii} = 0 \text{ and } \frac{\partial h_{ij}}{\partial x^i} = 0.$$

Having written $\delta g_{\mu\nu}$ like this we can apply the linearized Einstein equations to each of fluctuations individually, because in linear approximation the interaction between the terms vanishes. From applying the linearized Einstein equations to the vector fluctuations we may derive that the vector fluctuations decay in a expanding background. So generally they are considered uninteresting. For a more extensive discussion see Weinberg (pages 224-227 of [6]). We also may derive that the gravitational waves decouple, up to linear

[1]See page 224 of Weinberg[6] for the explicit argument.

order, from matter or energy density fluctuations and are therefore not of interest to us, again for a more extensive discussion see Weinberg (227-228 [6]). This means that we will concentrate solely on scalar perturbations.

We shall now deal with the gauge freedom of the system. Here we must note that the whole idea of gauge fixing might seem a bit strange as we already have chosen a coordinate system on the background. According to the active view promoted by Mukhanov, Feldman and Brandenberger [7] we can consider two manifolds the background isotropic homogenous universe $\mathcal{M}_0$ on which we have fixed our coordinate system by choosing either conformal or cosmological time and the universe with fluctuations $\mathcal{M}$. In this context a choice of gauge or coordinates is a diffeomorphism between $\mathcal{M}_0$ and $\mathcal{M}$, since we have fixed coordinates on $\mathcal{M}$. To confront the issue of gauge freedom we first consider a spacetime coordinate transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$$

As we are working in a linearized setting we require that ϵ^μ is small. We now wish to investigate $\Delta g_{\mu\nu}$ which we define as

$$\delta g_{\mu\nu}(x) \to \delta g_{\mu\nu}(x) + \Delta g_{\mu\nu}(x).$$

Note that $\Delta g_{\mu\nu}$ is not independent of the background metric

$$\begin{aligned}
\Delta g_{\mu\nu}(x) &= \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) \\
&\simeq (g^{(0)}_{\rho\sigma}(x) + \delta g_{\rho\sigma}(x))(\delta^\rho_\mu - \partial_\mu \epsilon^\rho(x))(\delta^\sigma_\nu - \partial_\nu \epsilon^\sigma(x)) - (\partial_\lambda \tilde{g}_{\mu\nu})\epsilon^\lambda - g^{(0)}_{\mu\nu}(x^\lambda) - \delta g_{\mu\nu}(x^\lambda) \\
&\simeq -g^{(0)}_{\lambda\mu}(x)\partial_\nu \epsilon^\lambda(x) - g^{(0)}_{\lambda\nu}(x)\partial_\mu \epsilon^\lambda(x) - (\partial_\lambda g^{(0)}_{\mu\nu})(x)\epsilon^\lambda(x),
\end{aligned}$$

where we have used that the background does not transform. We would like to concentrate on the scalar fluctuations so we are only interested in the transformation of ϕ, B, E and ψ under coordinate transformations. To find nice expressions we split up the spacial part of the infinitesimal transformation vector ϵ^μ into a gradient of a scalar plus a divergence-less vector:

$$\epsilon_i = \partial_i \epsilon^S + \epsilon_i^V,$$

with

$$\partial_i \epsilon_i^V = 0.$$

Using this notation and conformal time we find the transformation rules of ϕ, B, E and

ψ to be[2]

$$\begin{aligned}
\tilde{\phi} &= \phi - \frac{a'}{a}\epsilon^0 - (\epsilon^0)' \\
\tilde{B} &= B + \epsilon^0 - (\epsilon^S)' \\
\tilde{E} &= E - \epsilon^S \\
\tilde{\psi} &= \psi + \frac{a'}{a}\epsilon^0
\end{aligned}$$

where the prime indicates the derivative with respect to conformal time η.

There are two way of dealing with the gauge freedom the first being the obvious: choosing a gauge or defining gauge independent variables.

The Longitudinal or Conformal Newtonian gauge will be very convenient

$$B = E = 0.$$

The synchronous gauge is also quite popular and sets

$$\phi = B = 0.$$

The latter is quite interesting since ϕ is often identified in a flat static background with the Newtonian potential, obviously such a one to one correspondence is no longer viable. We shall come back to the Newtonian Potential at a later stage.

For gauge independent variables we introduce the following variables due to Bardeen[12]

$$\begin{aligned}
\Phi &= \phi + \frac{1}{a}[(B - E')a]' \\
\Psi &= \psi - \frac{a'}{a}(B - E').
\end{aligned}$$

In the Longitudinal or Conformal Newtonian gauge we see that the gauge invariant variables become $\Phi = \phi$ and $\Psi = \psi$. To see what the role of the field Φ is we must first determine the equations of motion in a linearized setting.

We remember the Einstein equations of motion

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

with $G_{\mu\nu}$ the Einstein tensor as usual, $g_{\mu\nu}$ the metric and $T_{\mu\nu}$ the energy momentum tensor. We prefer to work in conformal time from this point onwards. By expanding the Einstein equation up to linear order in the perturbations of the energy momentum tensor and the Einstein tensor we naturally see that

$$\delta G^{\nu}_{\mu} = 8\pi G\, \delta T^{\nu}_{\mu}.$$

[2]One can find the calculations fully spelt out in Weinberg [6], however one must be really careful because not only the notation differs, there are discrepancies in the definitions for example the 'E' in Weinberg equals $-2a^2\phi$ not 2ϕ as one might think at first glace.

The components of both tensors are not formulated in a gauge invariant manner but we know thanks to Mukhanov, Feldman and Brandenberger [7] (page 218 onwards) that one can redefine the components in a gauge invariant manner

$$\begin{aligned}
\delta G_0^{0(\mathrm{gi})} &\equiv \delta G_0^0 + ({}^{(0)}G_0'^{0})(B - E') \\
\delta G_i^{0(\mathrm{gi})} &\equiv \delta G_i^0 + \Big({}^{(0)}G_i^0 - \frac{1}{3}{}^{(0)}G_k^k\Big)\partial_i(B - E') \\
\delta G_j^{i(\mathrm{gi})} &\equiv \delta G_j^i + ({}^{(0)}G_j'^{i})(B - E')
\end{aligned}$$

and

$$\begin{aligned}
\delta T_0^{0(\mathrm{gi})} &\equiv \delta T_0^0 + ({}^{(0)}T_0'^{0})(B - E') \\
\delta T_i^{0(\mathrm{gi})} &\equiv \delta T_i^0 + \Big({}^{(0)}T_i^0 - \frac{1}{3}{}^{(0)}T_k^k\Big)\partial_i(B - E') \\
\delta T_j^{i(\mathrm{gi})} &\equiv \delta T_j^i + ({}^{(0)}T_j'^{i})(B - E'),
\end{aligned}$$

where (gi) denotes the gauge invariance. We may now write the Einstein equations as

$$\delta G_\nu^{\mu(\mathrm{gi})} = 8\pi G\, \delta T_\nu^{\mu(\mathrm{gi})}.$$

Using this we rewrite the equations of motions in a gauge independent manner

$$-3H(H\Phi + \Psi') + \nabla^2\Psi = 4\pi G a^2 \delta T_0^{0(\mathrm{gi})} \tag{3.1a}$$

$$\partial_i(H\Phi + \Psi') = 4\pi G a^2 \delta T_i^{0(\mathrm{gi})} \tag{3.1b}$$

$$\begin{aligned}
&[(2H' + H^2)\Phi + H\Phi' + \Psi'' + 2H\Psi']\delta_j^i \\
&\quad + \frac{1}{2}\nabla^2 D\delta_j^i - \frac{1}{2}\gamma^{ik}\partial_i\partial_k D = -4\pi G a^2 \delta T_j^{i(\mathrm{gi})},
\end{aligned} \tag{3.1c}$$

where γ^{ij} denotes the spacial part of the background metric, $H = a'/a$ is the hubble parameter and

$$D \equiv \Phi - \Psi.$$

These equations may also be derived, relatively easily, in the longitudinal gauge where $\Phi = \phi$, $\Psi = \psi$ and $\delta T_\mu^{\nu(\mathrm{gi})} = \delta T_\mu^\nu$.

We may now investigate the role of Φ further and relate to the content of the first chapters in a perfect fluid setting, in this we follow [2]. To do so we consider the energy momentum tensor given by

$$T_\mu^\nu = (\rho + p)u^\nu u_\mu - p\delta_\mu^\nu,$$

where again (2.10) holds

$$\delta p = c_s^2 \delta\rho + \sigma\delta S.$$

The perturbation of T_μ^ν is given by

$$\begin{aligned}\delta T_0^0 &= \delta\rho\\ \delta T_i^0 &= \frac{1}{a}(\rho_0 + p_0)\delta u_i\\ \delta T_j^i &= -\delta p\ \delta_j^i.\end{aligned}$$

Since δT_j^i is diagonal we easily see from equation (3.1c) that $\Phi = \Psi$ and one may obtain[3]

$$\nabla^2\Phi - 3H\Phi' - 3H^2\Phi = 4\pi G a^2 \delta\rho.$$

This is the generalized Poisson equation in this general relativistical setting. The field Φ is called the relativistic (Newtonian) potential. Note that we have seen that Φ is gauge invariant. From the equation (3.1) we may further derive that

$$\Phi'' + 3H(1+c_s^2)\Phi' - c_s^2\nabla^2\Phi + [2H' + (1+3c_s^2)H^2]\Phi = 4\pi G a^2 \sigma\delta S.$$

Although this certainly reminds us of some of the results of the first chapters we are not quite finished. To achieve a greater similarity with the results of our discussion of perturbations in expanding space, we focus our attention on matter described by a single scalar field φ with the action[4]

$$S_m = \int d^4x\sqrt{-g}\Big[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\Big] \tag{3.2}$$

and we expand φ as follows

$$\varphi(\mathbf{x},\eta) = \varphi_0(\eta) + \delta\varphi(\mathbf{x},\eta).$$

We derive from this that the fluctuations in the energy momentum tensor are diagonal. In this case (3.1) reads[5] in the longitudinal gauge

$$\nabla^2\phi - 3H\phi' - (H' + 2H^2)\phi = 4\pi G\Big(\varphi_0'\delta\varphi' + \frac{dV}{d\varphi}a^2\delta\varphi\Big) \tag{3.3a}$$

$$H\phi + \phi' = 4\pi G\varphi_0'\delta\varphi \tag{3.3b}$$

$$\phi'' + 3H\phi' + (H' + 2H^2)\phi = 4\pi G\Big(\varphi_0'\delta\varphi' - \frac{dV}{d\varphi}a^2\delta\varphi\Big). \tag{3.3c}$$

Combining these one finds

$$\phi'' + 2\Big(H - \frac{\varphi_0''}{\varphi_0'}\Big)\phi' - \nabla^2\phi + 2\Big(H' - H\frac{\varphi_0''}{\varphi_0'}\Big)\phi = 0.$$

[3]See page 10 of [2] and [7] for more details.

[4]During inflation this description is actually a not unreasonable approximation as the matter contents of the universe will be dominated by the inflaton field.

[5]See [7] for all details.

One may also derive[6] that the equations of motion for the perturbations of the scalar field $\delta\varphi$ are

$$\delta\varphi'' + 2H\delta\varphi' - \nabla^2\delta\varphi + \frac{\partial^2 V}{\partial\varphi^2}a^2\delta\varphi - 4\varphi_0'\phi' + 2\frac{\partial V}{\partial\varphi}a^2\phi = 0.$$

If we now compare this equation to formula (2.15) with the entropy fluctuations set to zero

$$\ddot{\delta}_\epsilon + 2H\dot{\delta}_\epsilon - c_s^2\nabla^2\delta_\epsilon - 4\pi G\rho_0\delta_\epsilon = 0,$$

we see that the first terms are identical, since $c = c_s = 1$. We again see that there exists an attractive force, a damping term coming from the Hubble parameter called the Hubble friction term and a pressure term creating 'pressure waves'. We also see the appearance of a critical length below which we get oscillatory solutions.

This concludes our discussion of the general relativistical approach to cosmic fluctuations.

[6] See section 6 of the review by Mukhanov, Feldman and Brandenberger [7].

Chapter 4

Quantum Cosmological Perturbations

In this chapter we shall shortly discuss the quantum origin of cosmological fluctuations for a more or less identical simplified model which we discussed in the latter part of the previous chapter. The discussion will rely on quantum field theory in curved spacetime. To introduce this subject we have inserted a sketchy discussion of some of the important issues in quantum field theory on curved spacetime. This section will follow Wald [13], but mostly use Posthuma [14], at first in discussing why the remarkable results of quantum field theory on curved spacetime are unique to this field and do not arise in ordinary quantum mechanics and then it will follow Ford [15] in introducing some of these results. The discussion of the quantum mechanical origins of the cosmological fluctuation will follow the notes by Brandenberger [1], the article by Brandenberger, Feldman and Mukhanov [2] and the review by Mukhanov, Feldman and Brandenberger [7] closely and integrate these.

As we already mentioned in the introduction we need both an understanding of quantum mechanics and General Relativity to grasp the origin of the fluctuations, while General Relativity suffices to understand how these fluctuations were firstly scaled up during inflation and consecuently amplified by gravitational collapse. The generation of these cosmological fluctuations is assumed to have taken place during inflation by some models. We shall concentrate on this scenario. The fact that we need both general relativity and quantum mechanics to understand the generation of quantum fluctuation appears to prevent us from dealing with this issue. However the fluctuations from the average today are very small and thus the fluctuations were even smaller for the earlier universe and we may analyze the fluctuations linearly. This allows for a unified treatment of both the metric and matter fields and a very straightforward quantization. Since we will concentrate on a greatly simplified model, as in the previous chapter we will be able to reduce the theory to a theory of a single scalar field. We shall see that the non-static background will yield a time dependent “mass” of this scalar field. The time dependence of a mass term is generally identified with particle creation which is in this case also identified with generation of cosmological fluctuations.

4.1 Quantum Field Theory in Curved Spacetime and Particle Creation

In ordinary quantum mechanics one generally starts with some Lagrangian or symplectic manifold and "quantizes" this. We will only consider the quantisation of a symplectic vector space, in particular the coordinates may be assumed to be

$$q_1, \dots, q_n, p_1, \dots, p_n,$$

with the interpretation of position and momentum, and the symplectic form can be given as

$$\omega_{2n}\big((q,p),(\tilde{q},\tilde{p})\big) = \sum_{i=1}^{n} q_i\tilde{p}_i - \tilde{q}_i p_i.$$

Although there are some issues with the old way of quantizing, that is "replacing Poisson brackets with commutator brackets", see in particular van Hoves Theorem,[1] one has at least the Stone-von Neumann theorem. To understand the statement made in the Stone-von Neumann theorem we shall first introduce the Heisenberg group. Let (V, ω) be a symplectic vector space. We define the Heisenberg group to be

$$\tilde{V} = V \times \mathbb{T},$$

we shall denote its elements by (v, z), and the multiplication by

$$(v_1, z_1) \cdot (v_2, z_2) = \big(v_1 + v_2, e^{i\pi\omega(v_1,v_2)} z_1 z_2\big).$$

The definition of this group was inspired by the remark by Hermann Weil that for

$$(V, \omega) = (\mathbb{R}^2, \omega_2)$$

and

$$U(x) = \exp(ix\hat{q}) \qquad V(y) = \exp(iy\hat{p}),$$

where $\hat{q}$ and $\hat{p}$ are the usual quantum mechanical position and momenta operators, one has that

$$U(x)V(y) = e^{i\hbar\omega_2(x,y)} V(y)U(x).$$

We shall now give the Stone-von Neumann theorem:
The Heisenberg group $\tilde{V}$ has, up to isomorphism, a unique irreducible representation, which is faithful on its centre.

[1]For a pedagogical review see Ali and Englis[16].

Basically the Stone-von Neumann theorem says, see Ali and Englis[16], that the only way to realize the commutation relations

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{4.1}$$

on the Hilbert space $L^2(\mathbb{R}^n, dx))$ is by choosing,using the notation of Ali and Englis[16] and Bates and Weinstein[17]

$$q_i\psi(x) = x_i\psi(x)$$

and

$$p_i\psi(x) = -i\hbar\frac{\partial}{\partial x_i}\psi(x)$$

This theorem clearly establishes the nature of states but leaves us with the issue of normal ordering, as is most apparent in the already mentioned van Hoves theorem.

In quantum field theory there does not exist an equivalent theorem. Therefore the representation of states and observables is very difficult. In quantum field theory in Minkowski space these problems, to some extend, can be remedied by making use of Poincaré symmetry. However, thanks to ambiguities in the states we now see possibilities for 'particle' creation.[2]

We now become a bit more specific about quantum field theory on curved spacetime[3] in particular we give a somewhat explicit example of quantum field theory in a Minkowski space and in a non-flat space, and discuss the possibilities of 'particle' creation.

Consider a real massive scalar field with the Lagrangian

$$L = \frac{1}{2}(\partial_\alpha\phi\partial^\alpha\phi - m^2\phi^2 - \xi R\phi^2)$$

where ξ is a coupling constant. One easily sees that the equation of motion reads

$$\Box\phi + m^2\phi + \xi R\phi = 0$$

Let f_1 and f_2 denote two solutions of the said wave equation. We now define $\mathrm{d}\Sigma^\mu = \mathrm{d}\Sigma n^\mu$, where $\mathrm{d}\Sigma$ is a volume element in a given spacelike hypersurface and n^μ the unit normal to the surface Σ. We now define the inner product to be

$$(f_1, f_2) = i\int(f_2^*\partial_\mu f_1 - f_1\partial_\mu f_2^*)\mathrm{d}\Sigma^\mu.$$

[2] As mentioned the concept of particle is not well defined in quantum field theory on curved spacetime, for a full discussion see for example [13].

[3] In this we shall follow Ford[15] as his discussion is a somewhat simplified example of the discussion by in section 11 of Mukhanov, Feldman and Brandenberger[7], which discusses the generation of quantum fluctuations of the model based on one scalar field very extensively.

This definition is independent of the choice of hypersurface Σ. We may now define $\dot{\phi} = n^\mu \partial_\mu \phi$ and the canonical momentum as

$$\pi = \frac{\delta L}{\delta \dot{\phi}},$$

so that we can carry out the quantization by imposing canonical commutation relations. Furthermore let $\{f_i\}$ be a complete basis of positive norm solutions of the wave equation, hence $\{f_i^*\}$ comprises the negative norm solutions of the wave equation, such that

$$\begin{aligned}(f_j, f_{j'}) &= \delta_{j\,j'} \\ (f_j^*, f_{j'}^*) &= \delta_{j\,j'} \\ (f_j, f_{j'}^*) &= 0.\end{aligned}$$

Note that $\{f_j, f_j^*\}$ forms a complete basis of solutions. We now expand the field ϕ in terms of annihilation and creation operators as follows

$$\phi = \sum_j a_j f_j + a_j^\dagger f_j^*,$$

with as usual $[a_j, a_{j'}^\dagger] = \delta_{j\,j'}$. Because one has annihilation operators one also has a vacuum. We will now consider an asymptotically flat spacetime in the past and in the future. Let $\{f_j\}$ be a base of positive frequency solutions in the past and $\{F_j\}$ in the future. We may expand the f_j's in terms of the F_j's as

$$f_j = \sum_k (\alpha_{jk} F_k + \beta_{jk} F_k^*)$$

and the field ϕ in terms of the f_js and F_js as

$$\phi = \sum_j a_j f_j + a_j^\dagger f_j^* = \sum_j b_j F_j + b_j^\dagger F_j^*,$$

with a the annihilation operators in the asymptotic past and b in the future. These annihilation operators define a vacua $|0\rangle_{\text{past}}$ and $|0\rangle_{\text{future}}$. With a Bogoliubov transformation, one may rewrite the a_js in terms of the b_js and vice versa

$$\begin{aligned}a_j &= \sum_k \alpha_{jk}^* b_k - \beta_{jk} b_k^\dagger \\ b_k &= \sum_j \alpha_{jk} a_k - \beta_{jk}^* a_j^\dagger.\end{aligned}$$

It is now easily seen that

$$\langle N_k^{\text{future}} \rangle_{\text{past}} = {}_{\text{past}}\langle 0 | b_k^\dagger b_k | 0 \rangle_{\text{past}} = \sum_j |\beta_{jk}|^2,$$

where $N_k^{\text{future}} = b_k^\dagger b_k$ is the number operator of mode k in the asymptotic future. This implies particle creation if $\beta_{jk} \neq 0$ for some j.

4.2 Generation of Fluctuations

We will firstly give a brief overview of the scenario of the generation of quantum cosmological perturbations.[4] At some initial time we set each fourier mode in their vacuum state. For as long the wavelength is smaller than the Hubble radius, the state undergoes quantum fluctuations. The accelerated expansion of the background increases the length scale beyond the Hubble radius. The fluctuations freeze out when the length scale us equal to the Hubble radius. Beyond the Hubble radius the fluctuations grow as the scale factor, in classical general relativity this effect is due to self gravity.

We shall begin with the action in this case the sum of the Einstein-Hilbert action and matter action given by (3.2)

$$S = \int d^4x\sqrt{-g}\Big[-\frac{1}{16\pi G}R + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\Big]$$

We follow [1] and continue in the longitudunal gauge so that one has

$$\begin{aligned} ds^2 &= a^2(\eta)[(1+2\phi(\eta,\mathbf{x}))d\eta^2 - (1-2\psi(\eta,\mathbf{x}))d\mathbf{x}^2] \\ \varphi(\eta,\mathbf{x}) &= \varphi_0(\eta) + \delta\varphi(\eta,\mathbf{x}). \end{aligned}$$

Again we have that $\psi = \phi$. We now wish to expand the action up to quadratic order and write[5]

$$S \simeq S_0 + \delta_2 S,$$

where $\delta_2 S$ is quadratic in in the perturbations. One may derive the following expression due to Mukhanov for $\delta_2 S$

$$\delta_2 S = \frac{1}{2}\int d^4x[v'^2 - \delta^{ij}\partial_i v\partial_j v + \frac{z''}{z}v^2], \tag{4.2}$$

where v is the so called Mukhanov variable, a gauge invariant combination of matter and metric perturbations

$$v = a\Big[\delta\varphi + \frac{\varphi_0'}{H}\phi\Big],$$

with again $H = a'/a$ and

$$z = \frac{a\varphi_0'}{H}.$$

A full treatment of this calculation may be found in the review by Mukhanov, Feldman and Brandenberger[7], in particular the calculation of the variation of the purely gravitational contribution, which relies heavily on the ADM formalism, may be found in section

[4] In this we again follow [1].

[5] Following the notation of [2]

10.1, the matter part is spelled out in section 10.3. The part of the action quadratic in the fluctuations $\delta_2 S$ has the same form as a scalar field with a time dependent mass square $-z''/z$. Furthermore we note that in slow role inflation H and φ_0' are proportional, so that

$$z \propto a.$$

From this we see that

$$k_H^2 \equiv \frac{z''}{z} \simeq H^2,$$

where we have defined k_H^2 to make its role in the equations of motion more transparent. From the action (4.2) we get that

$$v'' - \nabla^2 v - \frac{z''}{z} v = 0$$

which reads in momentum space

$$v_{\mathbf{k}}'' + \mathbf{k}^2 v_{\mathbf{k}} - k_H^2 v_{\mathbf{k}} = 0. \tag{4.3}$$

Again we see a behavior similar to that in the classical setting as we have discussed when treating the Jeans length, namely a critical wavelength where the sign in front of the term with no derivatives switches.

From the action (4.2) we may easily proceed to quantize the theory. First we impose canonical commutation relations

$$[\hat{v}(\eta, \mathbf{x}), \hat{v}(\eta, \mathbf{x}')] = [\hat{\pi}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] = 0, \qquad [\hat{v}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] = i\delta(\mathbf{x} - \mathbf{x}'),$$

with the momentum

$$\pi(\eta, \mathbf{x}) = \frac{\partial L}{\partial v'}$$

and the delta function in normalized with respect to the metric on a time slice. According to [2] it is convenient to expand the operator $\hat{v}$ in terms creation and annihilation operators a_k^+ and a_k^-

$$\hat{v} = \frac{1}{2} \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} \Big[e^{i\mathbf{k}\cdot\mathbf{x}} v_{\mathbf{k}}^*(\eta) a_{\mathbf{k}}^- + e^{-i\mathbf{k}\cdot\mathbf{x}} v_{\mathbf{k}}(\eta) a_{\mathbf{k}}^+ \Big],$$

where again $v_{\mathbf{k}}$ satisfies (4.3). As we have already noticed the effective “mass” of the field is time dependent, this leads to the production of particles.[6] More generally one might say that we expect particle creation anyway as this is quantum field theory on a curved spacetime. This must be interpreted as follows: if $|\psi_0\rangle$ is a vacuum state at some time

[6] For a more extensive discussion of particle production in this setting see for example Birrell and Davies[18].

t_0 and $N(t)$ denotes the number operator then in general one has, as we have discussed in section 4.1 in an expanding universe,

$$\langle\psi_0|N(t)|\psi_0\rangle \neq 0.$$

It is important to note the existence of the critical wave length k_H beyond which quantum fluctuations become less relevant.

To connect more to the final part of chapter 3 we will now focus on the gauge invariant potential Φ, which equals ϕ in the longitudinal gauge. The field Φ may be quantized in the same manner as the field v, so one writes

$$\hat{\Phi}(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\frac{\varphi_0'}{a}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}[u^*_{\mathbf{k}}(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}a^-_k + u_{\mathbf{k}}(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}a^+_k],$$

where $u_{\mathbf{k}}(\eta)$ is related to $v_{\mathbf{k}}(\eta)$ by[7]

$$u_k(\eta) = -4\pi G\frac{z}{k^2}\left(\frac{v_k}{z}\right)'.$$

We now follow Mukhanov, Feldman and Brandenberger [7] in defining the power spectrum of metric perturbations by means of the two-point function of $\hat{\Phi}$

$$\langle\psi_0|\hat{\Phi}(\mathbf{x},\eta)\hat{\Phi}(\mathbf{x}+\mathbf{r},\eta)|\psi_0\rangle = \int_0^\infty \frac{\mathrm{d}k}{k}\frac{\sin(kr)}{kr}|\delta_k|^2,$$

where again $|\psi_0\rangle$ denotes the vacuum at some time t_0. Traditionally in realistic models, the vacuum before inflation is chosen. One may derive that[8]

$$|\delta_k(\eta)|^2 = \frac{1}{4\pi^2}\frac{\varphi_0'^2}{a^2}|u_k(\eta)|^2k^3$$

Here $|\delta_k(\eta)|^2$ characterizes the relative mass perturbations inside a sphere of radius k^{-1} squared [2]

$$\left(\frac{\delta M}{M}\right)^2 \sim |\delta_k|^2$$

This provides us with some ideas about the generation of cosmological perturbations. A 'generalization'of this discussion to hydrodynamical matter may be found in part II of Mukhanov, Feldman and Brandenberger [7].

We summarize the scenario as follows. One starts at some initial time t_0 in the vacuum state for that moment. Due to particle creation in curved spacetime we see the generation of cosmological fluctuations at scales smaller then the Hubble radius. Through inflation these fluctuations are blown up in size beyond the Hubble radius. Finally we simply apply General Relativity to see how these fluctuations are amplified, as we have discussed in chapter 3.

[7] See formula (13.8) of Mukhanov, Feldman and Brandenberger [7] or formula (40) of Brandenberger, Feldman and Mukhanov [2].

[8] See section 13 of Mukhanov, Feldman and Brandenberger[7]. In this section spectrum of the fluctuations are discussed, from which one may derive that there is a small deviation from scale invariance in the spectrum.

Chapter 5

Cosmological Perturbations and Cosmic Microwave Background Radiation

In this chapter we shall give an overview of the relation between the anisotropies in the cosmic microwave background radiation and cosmological perturbations. This discussion will be a short overview of the extensive treatment by Weinberg in his book [6], since the a full discussion will take too long.

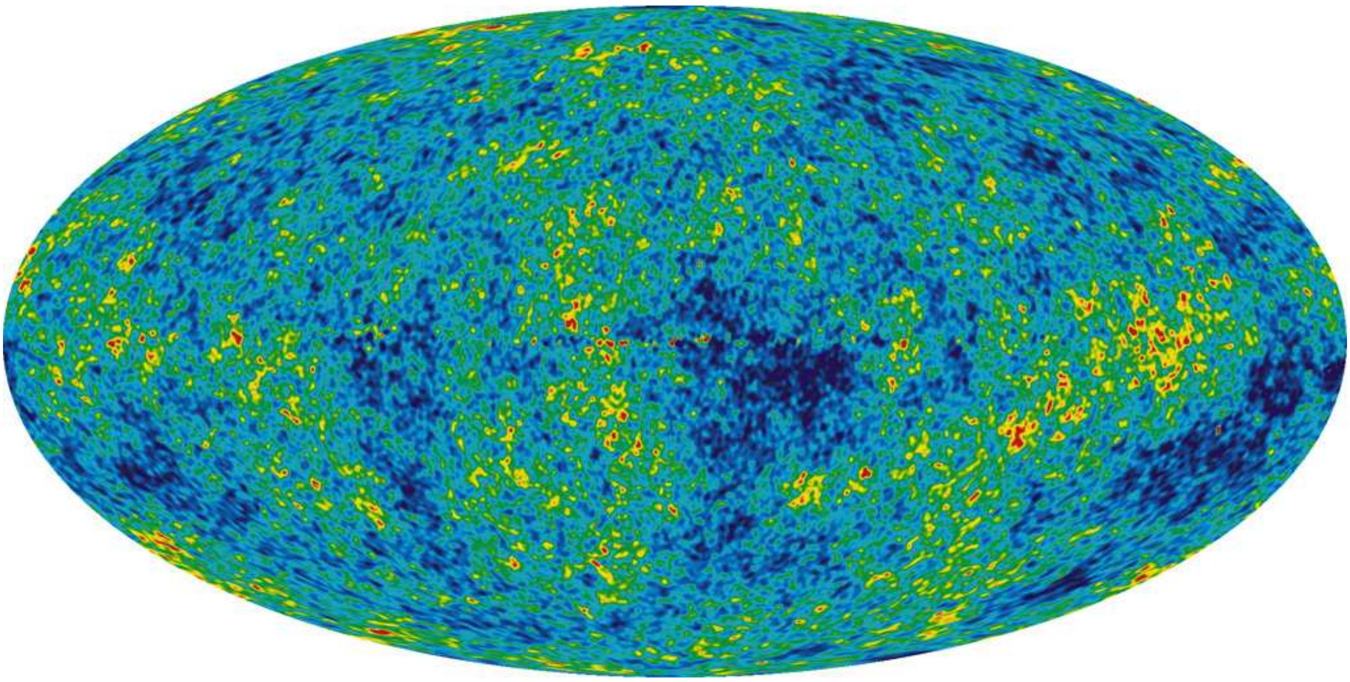

Figure 5.1: The Cosmic microwave background as observed by WMAP.

We first note that for sufficiently high temperature the constituents of matter in the universe are in thermal equilibrium. Furthermore the proper energy density of black body radiation is proportional to the fourth power of the temperature, the fractional perturbation in the temperature of the radiation coming from a direction $\hat{n}$ is one-fourth of the fractional perturbation in the proper energy density of photons traveling in the direction $-\hat{n}$. This proportionality gives us a manner in which we can translate the energy density of photons in a multi-component matter theory to temperatures observed in the cosmic microwave background.

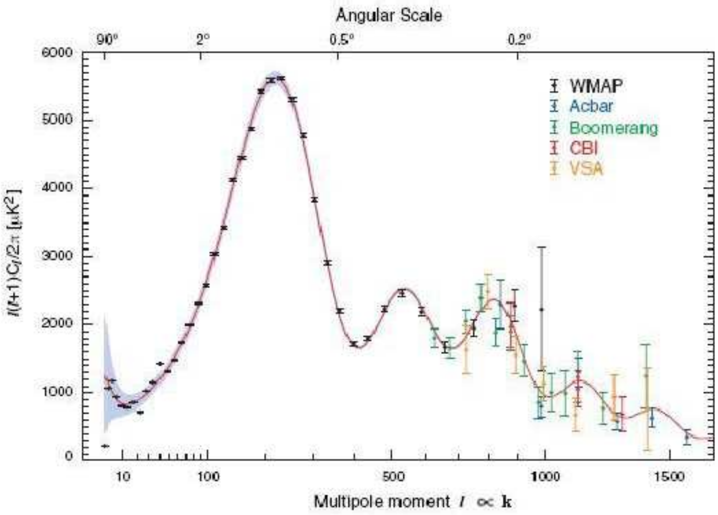

Figure 5.2: The multipole coefficients for a simple model, the Photon fluid approximation, where the fluid is taken to consist of only photons and matter effects are ignored. Picture courtesy of "universe-review.ca". A full explanation may be found in [6].

According to Weinberg there exists a division between the different causes for the anisotropies in the cosmic microwave background radiation. There are so-called recent effects such as the motion of the earth relative to the average direction of the photons of the cosmic microwave background radiation and the scattering of light by intergalactic electrons in clusters of galaxies. Furthermore, there are primary anisotropies whose origins lie in the early universe. These primary anisotropies are subdivided into:

- Intrinsic temperature fluctuations in the electron-nucleon-photon plasma at the time of last scattering. These temperature fluctuations are, as we have seen in the discussion above,determined by the energy density fluctuations which been have discussed throughout this paper.

- The Doppler effect due to the velocity fluctuations in the plasma at last scattering. Velocity fluctuations in hydrodynamical fluids have been discussed in the first chapters.

- The so-called Sacks-Wolfe effect, which describes the gravitational redshift or blueshift due to the gravitational potential at the moment of last scattering.
- Blue- or redshift due to gravitational effects after the moment of last scattering. This is known as the integrated Sachs-Wolfe effect.

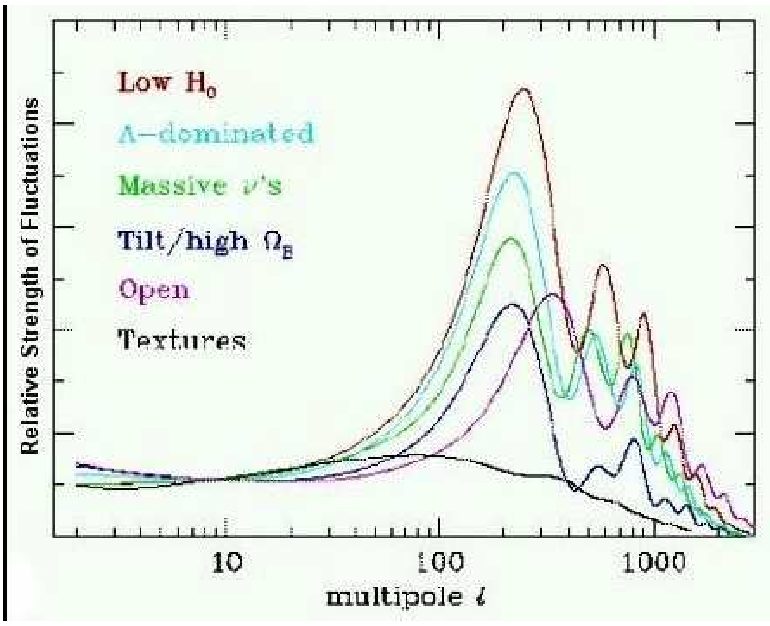

Figure 5.3: Figure to illustrate the great number of models available. Picture courtesy of "universe-review.ca".

We define T_0 to be the present mean value of the presently observed microwave radiation temperature. Denote by $T(\hat{n})$ the temperature in the direction of the unit vector $\hat{n}$ and

$$\Delta T(\hat{n}) = T(\hat{n}) - T_0.$$

We are now interested in the value of

$$\Delta T(\hat{n})\Delta T(\hat{n}'),$$

which has been measured. To be more precise the coefficients, denoted by C_l of the decomposition into spherical harmonics of $\Delta T(\hat{n})\Delta T(\hat{n}')$

$$C_l = \frac{1}{4\pi}\int d^2\hat{n}d^2\hat{n}' P_l(\hat{n}\cdot\hat{n}')\Delta T(\hat{n})\Delta T(\hat{n}'),$$

where P_l denotes the Legendre polynomial, is the data which is distilled from the measurement of the cosmic microwave background radiation. These so-called multipole coefficients may also be derived from various models for the contents of the universe. These models nearly always require a vast amount of computation and mostly require (super-) computers. We stress that there exists a great number of different models giving different values for the C_ls. One may find a figure which compares the results of a popular model, the so called photon fluid approximation, with observed values. To illustrate the great number of models available we have also inserted figure 5.3 depicting the results of a number of models.

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