Large Scale Structure Formation

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Outline

Introduction
- Static Toy Model
- Non-static Toy Model

Relativistic Perturbations

Numerical work

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Introduction

Overview

Hierarchial growth

- Quantum origin perturbations (Misha)
  small $\delta\rho/\rho_0 \ll 1$
- Inflation scales perturbations up
- Instability due to gravity (linear growth of structure)

$$\ddot{\delta\rho} \sim G\delta\rho.$$

- Contracting cold dark and ordinary matter
- Radiative cooling

Toy model in Minkowski space

Hydrodynamical matter

- The continuity equation

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + (v_i\partial_i)\rho = -\rho(\partial_i v_i).$$

- The Euler equation

$$\frac{dv_i}{dt} + \partial_i\phi = \frac{\partial v_i}{\partial t} + (v_j\partial_j)v_i + \partial_i\phi = -\rho^{-1}\partial_i p.$$

- The Poisson equation for Newtonian gravity

$$\nabla^2\phi = \partial_i\partial_i\phi = 4\pi G\rho.$$

- Entropy conservation

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + (v_i\partial_i)S = 0.$$

Linear Order Perturbations

Perturb solutions

$$\begin{aligned}\rho(\mathbf{x}, t) &= \rho_0 + \delta\rho(\mathbf{x}, t)\\ \mathbf{v}(\mathbf{x}, t) &= \delta\mathbf{v}(\mathbf{x}, t)\\ p(\mathbf{x}, t) &= p_0 + \delta p(\mathbf{x}, t)\\ \phi(\mathbf{x}, t) &= \phi_0 + \delta\phi(\mathbf{x}, t)\\ S(\mathbf{x}, t) &= S_0 + \delta S(\mathbf{x}, t).\end{aligned}$$

Equation of state

$$\delta p = c_s^2 \delta\rho + \sigma \delta S,$$

with

$$c_s^2 = \left(\frac{\delta p}{\delta \rho}\right)\bigg|_S \qquad \sigma = \left(\frac{\delta p}{\delta S}\right)\bigg|_\rho,$$

where c_s is identified with the speed of sound.

Governing equations

Inserting the perturbations in the hydrodynamical equations gives

- Continuity equation

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \partial_i \delta v_i = 0$$

- Euler equation

$$\frac{\partial \delta v_i}{\partial t} + \frac{1}{\rho_0}\partial_i \delta p + \partial_i \delta\phi = 0$$

$$\Longrightarrow \frac{\partial}{\partial t}\partial_i \delta v_i + \frac{1}{\rho_0}\partial_i\partial_i \delta p + \partial_i\partial_i\delta\phi = 0$$

- Poisson

$$\partial_i\partial_i\delta\phi = 4\pi G\delta\rho$$

- Equation of state

$$\delta p = c_S^2 \delta\rho + \sigma\delta S$$

Governing equations (continued)

So the hydrodynamical equations yield

$$\ddot{\delta\rho} - c_s^2 \partial_i \partial_i \delta\rho - 4\pi G \rho_0 \delta\rho = \sigma \partial_i \partial_i \delta S.$$

Furthermore one also has

$$\frac{dS}{dt} = 0.$$

Generalization to multicomponent fluid, where each component is distinguished by a index A

$$\ddot{\delta\rho}_A - c_{sA}^2 \nabla^2 \delta\rho_A - \sum_A 4\pi G \rho_0 \delta\rho_A = \sigma \nabla^2 \delta S_A.$$

Two types of perturbations are of particular interest

- So called adiabatic fluctuation, where the entropy fluctuations are set to zero. In a ‘realistic’ model multicomponent model, number of photons much greater then number of baryons, density perturbations of all baryons are determined by temperature fluctuations.
- Entropy fluctuations, where $\dot{\delta\rho} = 0$, but $\delta S \neq 0$.

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Jeans Length

We consider only adiabatic perturbations for the moment.
Fourier decomposition

$$\delta\rho(\mathbf{x},t) = \int e^{i\mathbf{k}\cdot\mathbf{x}}\delta\rho_k(t)\mathrm{d}\mathbf{k}$$

yields

$$\delta\ddot{\rho}_k(t) + c_s^2 k^2 \delta\rho_k - 4\pi G\rho_0 \delta\rho_k = 0.$$

Jeans wavelength

$$k_J = \left(\frac{4\pi G\rho_0}{c_s^2}\right)^{1/2}.$$

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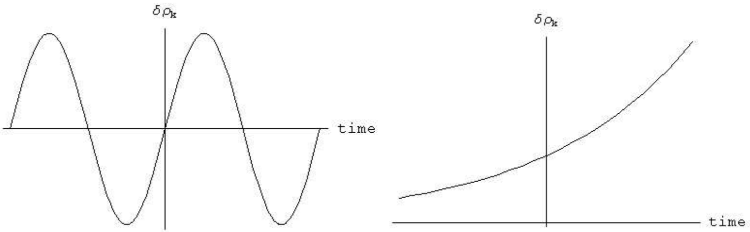

Interpretation:

$$\{\text{timescale for pressure readjustment}\} < \{\text{timescale for gravitational collapse}\}$$

Perturbations in expanding space

Background reads

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2(t), -a^2(t), -a^2(t)).$$

Perturbation

$$\rho(\mathbf{x}, t) = \rho_0(t)(1 + \delta_\epsilon(\mathbf{x}, t))$$
$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}_0(\mathbf{x}, t) + \delta\mathbf{v}(\mathbf{x}, t) = H(t)\mathbf{x} + \delta\mathbf{v}(\mathbf{x}, t)$$
$$p(\mathbf{x}, t) = p_0(t) + \delta p(\mathbf{x}, t)$$

Inserting in the hydrodynamical equations
From the zeroth order equation we get

$$\rho_0 \propto a^{-3},$$

consistent with Newtonian limit.

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Again perturbing to first order

- Continuity equation

$$d_t\partial_i\delta v_i + 2H\partial_i\delta v_i + \partial_i\partial_i\delta\phi + \rho_0^{-1}\partial_i\partial_i\delta p = 0$$

- Euler equation

$$d_t\delta_\epsilon + \partial_i\delta v_i = 0$$

- Poisson

$$\partial_i\partial_i\delta\phi = 4\pi G\rho_0\delta_\epsilon$$

- Equation of state

$$\delta p = c_S^2\delta\rho + \sigma\delta S$$

From the first order equations

$$d_t^2\delta_\epsilon + 2Hd_t\delta_\epsilon - c_s^2\nabla^2\delta_\epsilon - 4\pi G\rho_0\delta_\epsilon = \frac{\sigma}{\rho_0}\nabla^2\delta S,$$

with

$$d_t = \partial_t + Hx_j\partial_j = \partial_t + v_j\partial_j.$$

The one extra term $2Hd_t\delta_\epsilon$ identified with damping

Relativistic Cosmological Perturbation Theory

Scalar perturbations

Background flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe (first order perturbation theory).
We may decompose the perturbations of the metric

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}^{S} + \delta g_{\mu\nu}^{V} + \delta g_{\mu\nu}^{T}$$

(relies on the way the fields transform under the spacial coordinate transformations on a 3D surface of constant time)

$$\delta g_{\mu\nu}^{S} = a^2 \begin{pmatrix} 2\phi & -\partial_1 B & -\partial_2 B & -\partial_3 B \\ -\partial_1 B & 2(\psi - \partial_1\partial_1 E) & -2\partial_1\partial_2 E & -2\partial_1\partial_3 E \\ -\partial_2 B & -2\partial_2\partial_1 E & 2(\psi - \partial_2\partial_2 E) & -2\partial_2\partial_3 E \\ -\partial_3 B & -2\partial_3\partial_1 E & -2\partial_3\partial_2 E & 2(\psi - \partial_3\partial_3 E) \end{pmatrix}$$

with ϕ, ψ, B and E scalar fields.

Vector perturbations

$$\delta g^{V}_{\mu\nu} = a^2 \begin{pmatrix} 0 & -S_1 & -S_2 & -S_3 \\ -S_1 & 2\partial_1 F_1 & \partial_1 F_2 + \partial_2 F_1 & \partial_1 F_3 + \partial_3 F_1 \\ -S_2 & \partial_1 F_2 + \partial_2 F_1 & 2\partial_2 F_2 & \partial_2 F_3 + \partial_3 F_2 \\ -S_3 & \partial_1 F_3 + \partial_3 F_1 & \partial_2 F_3 + \partial_3 F_2 & 2\partial_3 F_3 \end{pmatrix}$$

where S_i and F_i are two three dimensional vectors, which also satisfy

$$\frac{\partial S_i}{\partial x^i} = \frac{\partial F_i}{\partial x^i} = 0.$$

From the equations of motions one can derive that these decay as a^{-2}.

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Tensor perturbations

$$\delta g^T_{\mu\nu} = a^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & h_{13} \\ 0 & h_{12} & h_{22} & h_{23} \\ 0 & h_{13} & h_{23} & h_{33} \end{pmatrix}$$

where we have

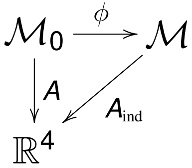
$$h_{ii} = 0 \text{ and } \frac{\partial h_{ij}}{\partial x^i} = 0.$$

The equations of motions yield that these perturbations decouple completely.

Active view

What about the gauge? We have chosen coordinates on the background or rather normal or conformal time.

Let $\mathcal{M}$ be our manifold and $\mathcal{M}_0$ the background and $A \in \mathcal{A}$ a chart in the atlas



Fixing a diffeomorphism ϕ induces a coordinate chart on $\mathcal{M}$.
In the following we will take time to be conformal.

Gauge issues

Since we work in linearized gravity we may assume that ϵ^μ is very small

$$x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$$

$$\delta g_{\mu\nu}(x) \to \delta g_{\mu\nu}(x) + \Delta g_{\mu\nu}(x).$$

Note that $\Delta g_{\mu\nu}$ is not independent of the background metric

$$\begin{aligned}
\Delta g_{\mu\nu}(x) &= \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) \\
&\simeq (g^{(0)}_{\rho\sigma}(x) + \delta g_{\rho\sigma}(x))(\delta^\rho_\mu - \partial_\mu \epsilon^\rho(x))(\delta^\sigma_\nu - \partial_\nu \epsilon^\sigma(x)) \\
&\quad -(\partial_\lambda \tilde{g}_{\mu\nu})\epsilon^\lambda - g^{(0)}_{\mu\nu}(x^\lambda) - \delta g_{\mu\nu}(x^\lambda) \\
&\simeq -g^{(0)}_{\lambda\mu}(x)\partial_\nu \epsilon^\lambda(x) - g^{(0)}_{\lambda\nu}(x)\partial_\mu \epsilon^\lambda(x) \\
&\quad -(\partial_\lambda g^{(0)}_{\mu\nu})(x)\epsilon^\lambda(x),
\end{aligned}$$

If we now rewrite

$$\epsilon_i = \partial_i \epsilon^S + \epsilon_i^V,$$

with

$$\partial_i \epsilon_i^V = 0,$$

which allows to separate the vector and scalar perturbations.
(Tensor perturbations don't transform)
The scalar fluctuations transform as

$$\tilde{\phi} = \phi - \frac{a'}{a}\epsilon^0 - (\epsilon^0)'$$
$$\tilde{B} = B + \epsilon^0 - (\epsilon^S)'$$
$$\tilde{E} = E - \epsilon^S$$
$$\tilde{\psi} = \psi + \frac{a'}{a}\epsilon^0$$

prime indicates the derivative with respect to conformal time η.

Solving gauge problems

1. Fix gauge
   - Longitudinal or Conformal Newtonian Gauge

$$B = E = 0$$

   - Synchronous Gauge

$$\phi = B = 0$$

2. Introduce Gauge Invariant Variables

$$\Phi = \phi + \frac{1}{a}[(B - E')a]'$$
$$\Psi = \psi - \frac{a'}{a}(B - E').$$

in the newtonian limit ϕ is identified with the Newtonian potential.
In the longitudinal gauge $\Phi = \phi$ and $\Psi = \psi$.

Generalizing

$$\delta G^{\nu}_{\mu} = 8\pi G\,\delta T^{\nu}_{\mu}.$$

Again write in a gauge invariant manner

$$\delta G_0^{0(\mathrm{gi})} \equiv \delta G_0^0 + ({}^{(0)}G_0^{\prime 0})(B - E')$$

$$\delta G_i^{0(\mathrm{gi})} \equiv \delta G_i^0 + \left({}^{(0)}G_i^0 - \frac{1}{3}{}^{(0)}G_k^k\right)\partial_i(B - E')$$

$$\delta G_j^{i(\mathrm{gi})} \equiv \delta G_j^i + ({}^{(0)}G_j^{\prime i})(B - E')$$

and

$$\delta T_0^{0(\mathrm{gi})} \equiv \delta T_0^0 + ({}^{(0)}T_{\prime 0}^{0})(B - E')$$

$$\delta T_i^{0(\mathrm{gi})} \equiv \delta T_i^0 + \left({}^{(0)}T_i^0 - \frac{1}{3}{}^{(0)}T_k^k\right)\partial_i(B - E')$$

$$\delta T_j^{i(\mathrm{gi})} \equiv \delta T_j^i + ({}^{(0)}T_j^{\prime i})(B - E'),$$

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The linearized Einstein equations yield

$$-3H(H\Phi + \Psi') + \nabla^2\Psi = 4\pi G a^2 T_0^{0(\mathrm{gi})}$$
$$\partial_i(H\Phi + \Psi') = 4\pi G a^2 \delta T_i^{0(\mathrm{gi})}$$
$$[(2H' + H^2)\Phi + H\Phi' + \Psi'' + 2H\Psi']\delta_j^i$$
$$+\frac{1}{2}\nabla^2 D\delta_j^i - \frac{1}{2}\gamma^{ik}\partial_i\partial_k D = -4\pi G a^2 \delta T_j^{i(\mathrm{gi})},$$

where γ^{ij} denotes the spacial part of the background metric, $H = a'/a$ is the hubble parameter and

$$D \equiv \Phi - \Psi.$$

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Perfect fluid content

Energy momentum tensor

$$T_\mu^\nu = (\rho + p)u^\nu u_\mu - p\delta_\mu^\nu,$$

Equation of state

$$\delta p = c_s^2 \delta\rho + \sigma\delta S.$$

The perturbation of T_μ^ν is given by

$$\delta T_0^0 = \delta\rho \qquad \delta T_i^0 = \frac{1}{a}(\rho_0 + p_0)\delta u_i \qquad \delta T_j^i = -\delta p\, \delta_j^i.$$

One may derive that

$$\nabla^2\Phi - 3H\Phi' - 3H^2\Phi = 4\pi G a^2 \delta\rho.$$

This generalizes the Poisson equation, Φ is referred to as the relativistic (Newtonian) potential.

The governing equation now reads

$$\Phi'' + 3H(1 + c_s^2)\Phi' - c_s^2\nabla^2\Phi + [2H' + (1 + 3c_s^2)H^2]\Phi = 4\pi G a^2 \sigma\delta S.$$

To quantum perturbations

Matter content described by scalar field. We start with action

$$
\begin{aligned}
S_t &= S_{EH} + S_m \\
S_m &= \int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right],
\end{aligned}
$$

In the longitudinal gauge

$$\varphi(\mathbf{x},\eta) = \varphi_0(\eta) + \delta\varphi(\mathbf{x},\eta).$$

$$\nabla^2\phi - 3H\phi' - (H' + 2H^2)\phi = 4\pi G\left(\varphi_0'\delta\varphi' + \frac{dV}{d\varphi}a^2\delta\varphi\right)$$

$$H\phi + \phi' = 4\pi G\varphi_0'\delta\varphi$$

$$\phi'' + 3H\phi' + (H' + 2H^2)\phi = 4\pi G\left(\varphi_0'\delta\varphi' - \frac{dV}{d\varphi}a^2\delta\varphi\right).$$

Combing yields

$$\phi'' + 2\left(H - \frac{\varphi_0''}{\varphi_0'}\right)\phi' - \nabla^2\phi + 2\left(H' - H\frac{\varphi_0''}{\varphi_0'}\right)\phi = 0.$$

Numerical work

Overview

Hierarchial growth

- Quantum origin perturbations (Misha)
  small $\delta\rho/\rho_0 \ll 1$
- Inflation scales perturbations up
- Instability due to gravity (linear growth of structure)

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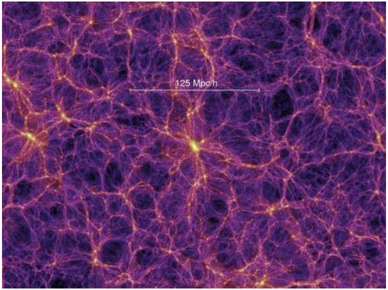
Millennium Simulation

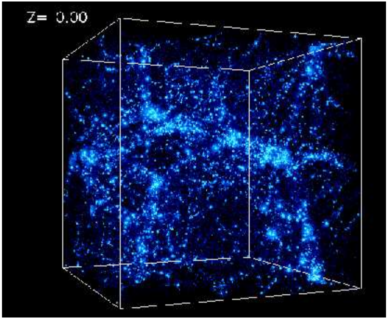
Spectrum of perturbations: nearly-scale invariant Gaussian random field (Harrison-Zel'dovich spectrum)

- Numerical work on non-linear collapse of matter.
- N-body Simulation with $N = 2160^3 \simeq 10^{10}$.
- From redshift $z = 127$ to present
- We see the present 'universe'



National center for Supercomputer Simulations

- From $z = 30$ to $z = 0$
- Size is only 43 *Mpsc*
- Time evolution is exhibited



Questions?