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Black Hole Formation

The Collapse of Compact Stellar Objects to Black Holes

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Abstract

This paper attemps to prove the existence of black holes by combining observational evidence with theoretical findings. First, basic properties of black holes are explained. Then black hole formation is studied. The relativistic hydrostatic equations are derived. For white dwarfs the equation of state and the Chandrasekhar limit $M = 1.43 M_\odot$ are worked out. An upper bound of $M = 3.6 M_\odot$ for the mass of any compact object is determined. These results are compared with observational evidence to prove that black holes exist.

Contents

Chapter 1

Introduction

The idea of a object so heavy that even light cannot escape its gravitational well is very old. It was first considered by British amateur astronomer John Michell in 1783. Newtonian mechanics however can not accurately describe a black hole; so it was not until Einstein wrote down the theory of general relativity in 1916 that progress was made in the understanding of black holes. The term itself was coined in 1967 by John Wheeler[1] as an alternative to the scientifically accurate but cumbersome "gravitationally completely collapsed star".

The physics of black holes is a very rich field, of which this paper can only scratch the surface. The central question this paper attempts to answer, by combining observational evidence with theoretical findings, is whether black holes exist. To do this we look in detail at black hole formation. This is the subject of chapter 3. First the equations for relativistic hydrostatic equilibrium, known as the Tolman-Oppenheimer-Volkoff equation or also as the equations of structure, will be derived. Then we will treat the physics of white dwarfs in great detail and derive upper bounds on their mass. The same is done for neutron stars. In this way upper bounds on the mass of white dwarfs and neutron stars will be found. In chapter four these results will be linked with astronomical observations, to prove that black holes do indeed occur in nature.

This paper only treats non-rotating black holes. Rotating black holes display many interesting phenomena, but rotation does not significantly alter the upper limit on the mass of neutron stars[2], so for the purpose of this paper treating rotation is not really needed. Rotation blackholes are treated in the paper written for this theoretical physics colloquium by Stijn van Tongeren [1].

[1] According Kip Thorne, a former student and colleague of Wheeler, he spent a lot of time pondering the perfect name, and then just introduced the term during a 1967 conference, pretending they had always been called like this

[2] This is not true for white dwarfs. Fast rotating white dwarfs can be significantly heavier.

This paper will use relativistic notation throughout. Summation over repeated indices is implied, and the metric signature used is $(-1,+1,+1,+1)$. Tensors will be denoted in bold when they carry no indices, so $\boldsymbol{u}$ but u_μ. Ordinary vectors will be denoted with an arrow, for example $\vec{p}$. Finally natural units will be used throughout this paper, unless otherwise specified. So $c = G = 1$.

Chapter 2

Black Holes Basics

2.1 The Schwarzschild Metric

This paper will start with a brief summary of the basic properties of black holes. A complete treatment of black hole basics would be too involved for this paper. There are many excellent texts (For example Carroll [2]) that treat this topic in detail.

The central equations in General Relativity (GR) are the Einstein Field Equations. They read

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.1}$$

These equations link the curvature of spacetime on the left-hand side to the energy and momentum content of spacetime on the right-hand side. The simplest case possible is of course empty space, which means that $T_{\mu\nu}$ vanishes. In this case R will also vanish such that (2.1) reduces to

$$R_{\mu\nu} = 0. \tag{2.2}$$

A solution to this equation was found by Schwarzschild in 1915, within about a month of the publication of the theory of general relativity by Einstein. The solution Schwarzschild found is [2, 3]:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.3}$$

Here $d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$ is a shorthand for the metric on S^2. From now on we will put $c = 1$ and $G = 1$ everywhere in this paper, unless otherwise noted. This simplifies the Schwarzschild metric to

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.4}$$

It turns out that this solution is in fact the unique static, spherically symmetric and electrically neutral vacuum solution to the Einstein equations. This uniqueness was proven by Birkhoff in 1923 and is known as Birkhoff's theorem, but we shall not go into this.

The Schwarzschild solution is the unique solution for a non-rotating, electrically neutral black hole. Solutions for rotating and / or charged black holes also exist. Realistic black holes are not expected to have a net charge (because the universe as a whole does not seem to have a net charge), but they are expected to have at least some angular momentum. The relevant solution for rotating black holes is known as the Kerr metric. More information on this can be found in [3, 4] or in [1]

It should be stressed that the Schwarzschild metric is a vacuum solution. It does not hold inside regions where matter is present. Rather, it describes space outside a spherically symmetric, non-rotating mass distribution. Because of its uniqueness, this is true for any such mass distribution. So space around stellar objects such as stars, white dwarfs, neutron stars and black holes are all described by it. The metric contains the a priori unknown constant M, which can be identified with the total mass of the object, as can be seen by taking the Newtonian limit. In this limit we should have

$$\begin{aligned} g_{tt} &= \;-\left(1+2\Phi\right), \\ g_{rr} &= \;\left(1-2\Phi\right). \end{aligned} \tag{2.5}$$

This immidiately implies

$$\Phi = -\frac{GM}{r}, \tag{2.6}$$

which shows that in the weak field limit the parameter M can be associated with a mass. In fact M is measure of total energy content of the object, also called the 'total mass'. It contains not just the classical concept of mass, but also all other energy contributions such as binding energy, energy of electromagnetic fields, gravitational energy, etc. From now on we will refer to M as simply the mass of the object.

Apart from the weak field limit we can also take the limit of $r \to \infty$. In this limit the Schwarzschild metric reduces to the Minkowski metric, as one would expect.

2.2 Black Holes

One interesting feature of the Schwarzschild metric (2.4) is that it becomes singular for two different values of r. At $r = 2M$ the factor in front of dr^2 blows up. And at $r = 0$ both g_{tt} and g_{rr} diverge. As shall be shown in a moment, the singularity at $r = 2M$ is not a true singularity, but rather an artifact of the choice of coordinates. In a different coordinate system this singularity can disappear. The singularity at $r = 0$ however represents a true singularity, a point of infinite spacetime curvature.

In itself this is nothing to worry about. Space around a massive object is described by the Schwarzschild metric, but the object itself is not. Since any object has a finite size, we do not expect the Schwarzschild metric to be valid for arbitrarily small values of r. However there is a catch.

The most striking feature of the Schwarzschild metric is that g_{tt} and g_{rr} change sign for $r < 2M$. This has profound consequences. The roles of time and radius change, such that in a sense the time direction becomes *down* (the direction of decreasing r). But one can only go forward in time, so anything within the region $r < 2M$ can only go down. There are no timelike paths going in the direction of increasing r. This is illustrated in figure 2.1. Of course, since one needs to pass a coordinate singularity to get to this region, the Schwarzschild coordinates are not really a good way to describe it in the first place. We shall threat this in more detail later.

The surface $r = 2M$ is known as the event horizon, and it acts as a point of no return. An observer in a space ship falling towards a black hole can still escape as long as he is at $r > 2M$ (if the space ship has a good enough engine), but once he passes the event horizon he is utterly doomed. No amount of energy expenditure is going save him. Not only can he never escape, he must fall further down. In fact, since a geodesic maximizes proper time, any energy expenditure is only going to shorten his demise.

This also means that there can be no stable matter configurations inside a black hole. All matter inside a black hole that is not already at $r = 0$ must fall towards it[1], which in turn means that the Schwarzschild metric will be valid even at arbitrary small values of r, because while the Schwarzschild solution breaks down outside a vacuum, in a black hole the vacuum is preserved even at abitrary small values of r. The center of a black hole must contain a singularity.

2.3 Eddington-Finkelstein Coordinates

To gain more insight in the nature of the horizon, one can rewrite the Schwarzschild metric in a different basis, the so called Eddington-Finkelstein Coordinates. In this basis the metric does not become singular at the horizon, $r = 2M$. We leave r, θ and ϕ untouched and replace t with ν in the following way:

$$t = \nu - \left(r + 2M \log \left| \frac{r}{2M} - 1 \right| \right). \tag{2.7}$$

With these coordinates the Schwarzschild metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\nu^2 + 2d\nu dr + r^2 d\Omega^2. \tag{2.8}$$

[1]and hit $r = 0$ in a finite amount of proper time, as shall be shown later.

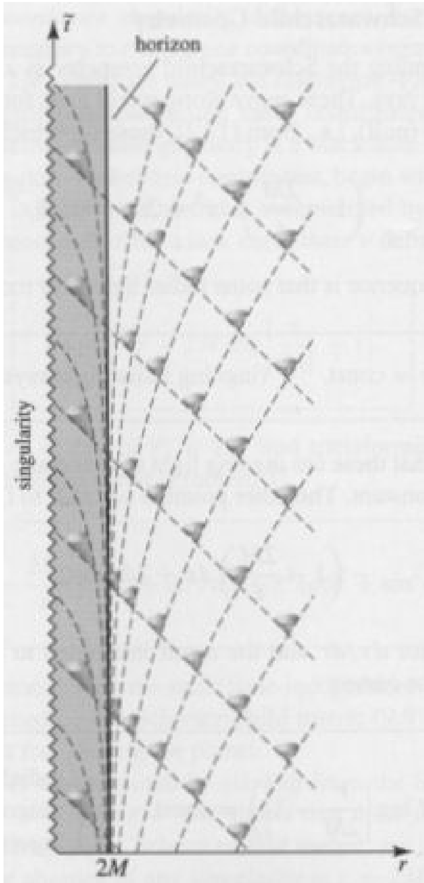

Figure 2.1: *The behavior of radial light rays near a black hole.* The ingoing light rays always move towards the black hole, but as one approaches the black hole the 'outgoing' light rays move slower and slower away from the black hole, until below the horizon they move inwards as well. Also shown in this figure are the light cones. Close to a black hole they will be tilted in the direction of decreasing radius, and past the event horizon the light cones will be tilted more than 45°. This means that there are no timelike paths or null paths out of the black hole. All paths lead further into the black hole. Image from Hartle [3], chapter 12.

This means that the metric is no longer singular at $r = 2M$, proving that the singularity found earlier is indeed an artifact of the chosen basis. The surface $r = 2M$ is special because it is the point of no return, the horizon. But space itself is normal at this surface, and an observer passing it will not notice anything special happening.

We now want to study radial light rays. For such a light ray one has

$ds^2 = 0$ and $d\theta = d\phi = 0$. This gives

$$-\left(1 - \frac{2M}{r}\right) d\nu^2 + 2d\nu dr = 0. \tag{2.9}$$

This has two solutions

$$d\nu = 0, \tag{2.10}$$

$$-\left(1 - \frac{2M}{r}\right) d\nu + 2dr = 0. \tag{2.11}$$

The first of these corresponds to incoming light rays. This can seen from (2.7). When t is increased r must decrease if ν is to be kept constant. The second equation can be solved for $d\nu/dr$ and integrated, which gives

$$\nu - 2\left(r + 2M \log\left|\frac{r}{2M} - 1\right|\right) = \text{const.} \tag{2.12}$$

Combining this with (2.7) we see that for $r > 2M$ this corresponds to outgoing light rays, but for $r < 2M$ the equation changes sign and the light rays are also incoming (see figure 2.1), which proves mathematically what we had already concluded: Beyond the event horizon nothing can escape a black hole.

2.4 Types of Black Holes

A typical star is made up of about 10^{60} different particles. To give a complete description of the star, in principe information about every single particle in needed. This is not the case for a black hole. Nothing can come out of a black hole, and that includes information about the nature and microscopic state of the matter inside it. A black hole is completely determined by one parameter, M, which is the total mass[2]. All black holes with the same mass are the same, regardless of what matter they are made up of. This is known as the 'no hair' theorem. Thus, in a way, black holes are the simplest objects imaginable.

Since black holes only differ in their total mass, it is natural to classify them by mass. Commonly astrophysicists distinguish between four types of black holes:

- **Stellar-mass black holes** have mass in the range of 1.5-3.0 to about 20-30 solar masses. They are remnants of stars and form directly from supernova explosions. The lower bound on the mass comes from the upper limit on the mass of neutron stars, which will be derived in chapter 3.6. Stars with initial mass significantly higher than the

[2]When rotation and charge are included, 4 extra parameters appear. The total angular momentum $\vec{J}$, and the total charge Q.

above upper bound may be formed, but these shed most of their mass during the initial stages of their evolutions, such that there is an effective upperbound on the mass of a black holes formed in supernova explosions.

- **Supermassive black holes** contain between 10^5 and 10^{10} solar masses. They are believed to lie in the centers of most, probably all, galaxies, and are thought to be responsible for Active Galactic Nuclei. They are most likely formed by coalescence of smaller black holes and accretion of stars and gas onto them. Another hypothesis is that they are formed directly from very large gas clouds. Such a cloud may form a single star a hundred thousand solar masses or more, which would be unstable and collapse directly into a black hole, without a supernova explosion. Yet another model says that the core of very dense stellar clusters may collapse into a black hole.

- **Intermediate-mass black holes** with masses around 10^3 solar masses are extremely rare. There is no known mechanism for them to form directly, so they must form from the merger of smaller black holes. This may happen in dense globular clusters. Their creation should produce intense bursts of gravitational waves which may be detected in the near future.

- Finally we have **Micro black holes**, with very low mass. Such black holes are so small that quantum effects start to play a role. One thus needs quantum gravity to describe them, and as a result they are very poorly understood. They are predicted by some inflationary models, and expected to have a very short lifetime. Smaller black holes have a higher temperature, and thus evaporate faster. A black hole of approximately the size of the moon will have temperature equal to the cosmic microwave background. So anything heavier will be stable, because it will absorb more energy from the cmb then it ratiates due to Hawking radiation, while anything smaller will evaporate. Hence the mass of the moon is often taken as an approximate upper limit for the size of micro black holes. This paper will not delve further into the subject of black hole temperature and evaporation. For more information see for example [4].

Chapter 3

Stellar Collapse and Black Hole Formation

3.1 Introduction

In the previous chapter it has been shown that black holes exist as valid solutions to the vacuum Einstein field equations. Whilst this is interesting, in and of itself it does not say anything about the actual existence of black holes. For that, we need a conceivable method of black hole formation. After all, if there would be no way of creating black holes, then they would not exist. This chapter aims at bridging this gap. As will be shown, stellar collapse can lead to the formation of black holes. In fact, if a star is sufficiently massive, a black hole *must* form. Even if our understanding of stellar collapse turns out to be wrong, and the final stages of stellar collapse are governed by new physical phenomena, then for sufficiently massive stars black holes must still form, assuming general relativity remains valid.

3.2 Collapse of Dust

Imagine a giant ball of dust. Dust, in the context of relativity, is defined as pressureless matter. Particles that do not interact in any way, except gravitationally. The only force acting on these particles will be gravitational. So all particles will follow timelike geodesics. Intuitively one would say that all particles will just fall radially inward towards the center of mass of the matter distribution. The ball of dust will become smaller and smaller until its surface passes its own Schwarzschild radius, at which point a black hole will be born. Indeed, this intuition is precisely right. But we can be a bit more rigorous than this.

We start out with a properly normalized four-velocity:

$$\boldsymbol{u} \cdot \boldsymbol{u} = g_{\mu\nu} u^\mu u^\nu = -1.$$

Neglecting thermal motion, we can assume that the particles will follow a radial trajectory, because of spherical symmetry. So without loss of generality we set $d\theta = d\phi = 0$. Plugging this into the Schwarzschild metric (2.4) gives

$$-\left(1 - \frac{2M}{r}\right) \left(u^t\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1} \left(u^r\right)^2 = -1. \tag{3.1}$$

Since the Schwarzschild is time-independent there is a Killing vector $\boldsymbol{\xi}$, such that $\boldsymbol{\xi} \cdot \boldsymbol{u}$ is conserved. Call this conserved quantity (with a costumary minus sign) e. We have

$$e = -\boldsymbol{\xi} \cdot \boldsymbol{u} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}, \tag{3.2}$$

which implies

$$\left(1 - \frac{2M}{r}\right)^{-1} e = \frac{dt}{d\tau} = u^t. \tag{3.3}$$

Plugging this into (3.1) yields

$$-\left(1 - \frac{2M}{r}\right)^{-1} e^2 + \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 = -1. \tag{3.4}$$

What is the value of the constant e? A particle at rest in flat space time has $E = mu^t = m\,(dt/d\tau)$. So looking at (3.2) we see that a particle at rest at very large r willl have $e = 1$. In general e will depend on the initial velocity of the particles. So to not further complicate things we will assume the particles start at rest at infinity, which allows us to set $e = 1$. With this value (3.4) becomes

$$\left(\frac{dr}{d\tau}\right)^2 = -\left(1 - \frac{2M}{r}\right) + 1 = \frac{2M}{r}, \tag{3.5}$$

which in turns gives

$$r^{1/2} dr = (2M)^{1/2}\, d\tau. \tag{3.6}$$

Integrating both sides to solve this differentional equation we arrive at r as a function of τ. The overall sign is determined by the condition that the geodesic be inward. The result is

$$r(\tau) = (3/2)^{2/3} (2M)^{1/3} (\tau_* - \tau)^{2/3}. \tag{3.7}$$

Here τ_* is an arbitrary integration constant. It is equal to the proper time at $r = 0$. We now have r as a function of the proper time. We are also interested in knowning r as a function of the Schwarzschild time t. Using both (3.5) and (3.2) we find (keeping $e = 1$)

$$\frac{dt}{dr} = \frac{dt}{d\tau}\frac{d\tau}{dr} = \left(\frac{2M}{r}\right)^{-1/2}\left(1 - \frac{2M}{r}\right)^{-1}. \tag{3.8}$$

We can again integrate this to find

$$t = t_* + 2M\left[-\frac{2}{3}\left(\frac{r}{2M}\right)^{3/2} - 2\left(\frac{r}{2M}\right)^{1/2} + \log\left|\frac{(r/2M)^{1/2}+1}{(r/2M)^{1/2}-1}\right|\right], \tag{3.9}$$

where again τ_* is an arbitrary integration constant.

These equations describe radial plunge orbits. A particle falling radially into a black hole would be described by them. Recall however that the Schwarzschild metric does not just describe black holes. It describes the gravitational field outside any spherically symmetrical mass distribution. Since the particles in a ball of dust do not interact, they will all fall independently of each other. In particular the outer surface layer will follow this orbit, with M just the total mass of the dust sphere.

From these equations we can learn several important facts. A black hole will be formed as soon as the outer layer of the dust sphere falls past the $r = 2M$ point. Looking at (3.9) one might think that this will never happen, because t diverges as r comes closer to $2M$. So a black hole will never be formed in any finite time. However if we consider (3.7) we see that there is no divergence of the proper time. In the proper time of the infalling particles they will pass the $r = 2M$ point without any trouble. They will fall all the way to $r = 0$, creating a singularity. A black hole will be formed in a finite amount of proper time.

Imagine an experimental physicist standing on the surface of a collapsing star (see figure 3.1). From his point of view, he will fall inward faster and faster, and after some finite amount of time he will cross the event horizon of the newly formed black hole. Suppose he sends out light signals at regular intervals. An observer far away will see the falling physicist move slower and slower, as t (the proper time of the observer) diverges with respect to τ (the proper time of the falling physicist). The light signals he is sending out will become redshifted, and arrive at ever increasing intervals. At the event horizon they will become infinitely redshifted.

What does this mean? There is a popular scientific view that when you look at something falling into a black hole you will see it move slower and slower until it hangs suspended in time at the horizon. This is wrong. Something that is infinitely redshifted is invisible. In fact one could argue that it is no longer there. Infinitely redshifted signals have zero energy. What is correct in the above view is that you will never see an object pass the event horizon. Instead, you will see it become redder and redder, and fainter and fainter, until it disappears.

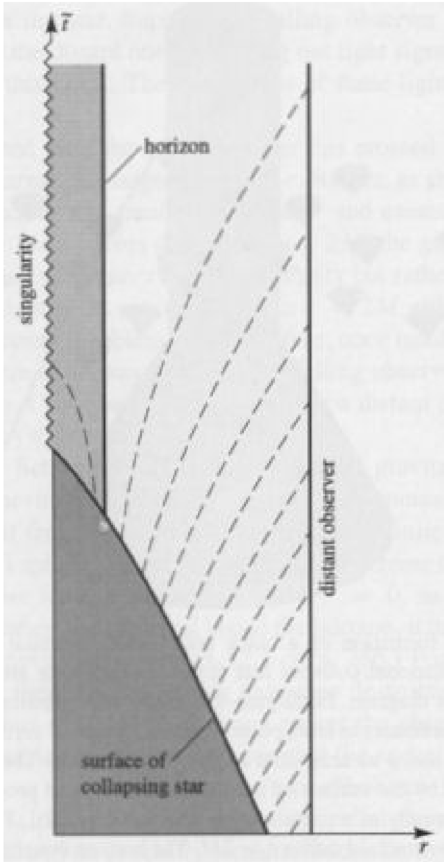

Figure 3.1: *The surface of a collapsing star.* Signals sent by a physicist sitting on the surface of a collapsing star will become progressively more redshifted, and arrive with progressively longer intervals in between. In his own proper time, the physicist sitting on the surface passes the event horizon in a finite time, but for an observer far away it takes infinitely long.
Image from Hartle [3], chapter 12.

A second important point we can learn from the collapse of a sphere of dust is very obvious, but important to mention. Gravity alone will always cause black holes to be formed. In the absence of other forces any mass distribution will collapse in on itself. A massive object will always be held together by a balance between gravitational forces pulling the matter inward, and a source of outward pressure. For example in the earth the gravitational force is countered by the outward pressure generated by the electromagnetic repulsion between the atoms making up the earth.

So to know how black holes are formed, we need to know what forces inside stars and compact stellar objects create outward pressure, and under what conditions these forces will fail to contain gravity's all-consuming hunger.

3.3 Gravitational Balance

So what balances gravity? In a main sequence star this is the pressure generated by the thermonuclear reactions in the star's core. Hydrogen is fused into helium, which generates large amounts of heat. The result is a huge temperature gradient. The temperature in the core of a star is at least of order 10^7 Kelvin. This temperature gradient produces a pressure gradient, which in turn produces an outward force. This outward force exactly balances the gravitational forces, such that a star is in almost perfect hydrostatic equilibrium.

But, this equilibrium is not truly static. To maintain it, a star must burn hydrogen. As a result, this equilibrium can only ever last a finite time. And indeed we know that stars do not live forever. Discussing the final stages of the life of a star falls outside the scope of this paper, but the question of what happens *afterwards* interests us. Can whatever remains of a star, after all sources of energy have been exhausted, avoid collapse into a black hole?

To do this, a nonthermal source of pressure is needed. An object where this source of pressure is the Pauli exclusion principe acting between electrons is called a White Dwarf. Even in their ground state, at minimum energy, two electrons will still resist being pushed together. This simple principle generates enough pressure to keep a white dwarf from collapsing, as shall be shown in section 3.5. However it shall turn out that there is an upper limit to this. Above a certain mass, called the Chandrasekhar mass, the Pauli principe between electrons can no longer generate enough pressure.

Section 3.6 will investigate what happens in this case. But before we can study these processes in detail, we need to take a closer look at matter in hydrostatic equilibrium.

3.4 Equations of Structure

The goal of this section is deriving the equations that describe the relativistic hydrostatic equilibrium inside stars and compact stellar objects. These equations of structure for such objects are known as the Tolman-Oppenheimer-Volkoff equations [6]. Together with an equation of state (relating density to pressure) they completely determine the structure of a spherically symmetric body of isotropic material in equilibrium, as shall be shown.

We shall need to make several assumptions. First of all we take our system to be spherically symmetric and isotropic. This assumption is reasonable, at least in the zero temperature limit. Stars usually are not isotropic, as a look at the sunspots on the sun proves, but it is hard to see how a compact stellar object at zero temperature could not be isotropic. Like everwhere else in this paper we also assume that the system is non-rotating. Since we are interested in the equilibrium state of compact stars, we furthermore assume that the system is static. This means the metric is time-independent, and the four-velocity takes on the simplest possible form.

The goal is to solve the Einstein Field Equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}. \tag{3.10}$$

The left-hand side of this equation depends only on the metric. It is a well-known fact [7] that any spherically symmetric time-independent metric can be written in the form:

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.11}$$

Here $\nu(r)$ and $\lambda(r)$ are two as of yet unknown functions, that only depend on r. The Schwarzschild metric (2.4) for example has this form. Using this most general form of the metric the Christoffel symbols and the Riemann tensor can be calculated. There are 9 non-zero Christoffel symbols. They are

$$\begin{array}{llll} \Gamma^t_{rt} & = \frac{1}{2}\nu', & \Gamma^\theta_{\phi\phi} & = -\sin\theta\cos\theta, \\ \Gamma^r_{tt} & = \frac{1}{2}\nu' e^{\nu-\lambda}, & \Gamma^\theta_{r\theta} & = \frac{1}{r}, \\ \Gamma^r_{rr} & = \frac{1}{2}\lambda', & \Gamma^\phi_{r\phi} & = \frac{1}{r}, \\ \Gamma^r_{\theta\theta} & = -re^{-\lambda}, & \Gamma^\phi_{\theta\phi} & = \cot\theta, \\ \Gamma^r_{\phi\phi} & = -r\sin^2\theta e^{-\lambda}. & & \end{array} \tag{3.12}$$

Here the prime $'$ refers to differentiation with respect to r. Also for brevity we have dropped the argument r from λ and ν. From the Christoffel symbols the Ricci tensor can be calculated:

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}. \tag{3.13}$$

After having calculated the Ricci Tensor the Ricci scalar and the Einstein tensor can be calculated. The actual calculation is very straightforward, but also a lot of work. I will skip these details and just give the results. Only the diagonal elements survive. They are simplest in mixed form, where

$$G^t{}_t = -e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{3.14}$$

$$G^r{}_r = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{3.15}$$

$$G^\theta{}_\theta = G^\phi{}_\phi = e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right). \tag{3.16}$$

Having calculated the left-hand side of (3.10) we turn our attention to the right-hand side. In static equilibrium the interior of a star can be described as a perfect fluid. For a perfect fluid the stress energy tensor [8] is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + g_{\mu\nu} p. \tag{3.17}$$

The system is in static equilibrium. This means that $\boldsymbol{u}$ takes on a particularly simple form,

$$u^\mu = \left(e^{-\frac{1}{2}\nu(r)}, \vec{0}\right), \tag{3.18}$$

which can be immidiately seen by normalizing $\boldsymbol{u}$,

$$g_{\mu\nu}u^\mu u^\nu = g_{00}u^0u^0 = -e^\nu \left(e^{-\frac{1}{2}\nu}\right)^2 = -1. \tag{3.19}$$

Plugging this back into (3.17) gives us for the mixed form

$$T^\mu_{\ \nu} = \mathrm{diag}(-\rho, p, p, p). \tag{3.20}$$

Now that we have derived expressions for both $T^\mu_{\ \nu}$ and $G^\mu_{\ \nu}$, solving the Einstein field equations (3.10) becomes straightforward. As a first step we get [7]

$$8\pi\rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{3.21}$$

$$8\pi p = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{3.22}$$

$$8\pi p = e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right). \tag{3.23}$$

Now first differentiate (3.22):

$$8\pi\frac{dp}{dr} = \frac{2}{r^3} + e^{-\lambda}\left(-\frac{\lambda'\nu'}{r} - \frac{\lambda'}{r^2} + \frac{\nu''}{r} - \frac{\nu'}{r^2} - \frac{2}{r^3}\right). \tag{3.24}$$

Subtracting (3.23) from (3.22) and multiplying by $2/r$ we find

$$0 = -\frac{2}{r^3} + e^{-\lambda}\left(\frac{2\nu'}{r^2} + \frac{2}{r^3} - \frac{\nu''}{r} + \frac{\lambda'\nu'}{2r} - \frac{\nu'^2}{2r} - \frac{\nu' - \lambda'}{r^2}\right). \tag{3.25}$$

Adding these two equations gives

$$\begin{aligned} 8\pi\frac{dp}{dr} &= +e^{-\lambda}\left(-\frac{\lambda'\nu'}{2r} - \frac{\nu'^2}{2r}\right) \\ &= -\frac{\nu'}{2}\left[e^{-\lambda}\left(\frac{\lambda'}{r} + \frac{\nu'}{r}\right)\right]. \end{aligned} \tag{3.26}$$

In the term between the square brackets we recognize $8\pi(\rho + p)$ such that

$$\frac{dp}{dr} = -\left(\frac{\rho(r) + p(r)}{2}\right)\frac{d\nu}{dr}. \tag{3.27}$$

It is customary to replace $\lambda(r)$ with $m(r)$ as follows:

$$e^{-\lambda(r)} \equiv 1 - \frac{2m(r)}{r}. \tag{3.28}$$

Doing this, (3.21) becomes

$$\begin{aligned}
8\pi\rho &= \left(1 - \frac{2m(r)}{r}\right)\left(\frac{d}{dr}\left[-\log\left(1-\frac{2m}{r}\right)\right]/r - \frac{1}{r^2}\right) + \frac{1}{r^2} \\
&= \frac{-1}{r^2}\left[\left(1-\frac{2m(r)}{r}\right)\left(\frac{1}{1-2m/r}\left(\frac{2m}{r} - \frac{2dm(r)}{dr}\right)+1\right)-1\right] \\
&= \frac{-1}{r^2}\left[1 - 2\frac{dm(r)}{dr} - 1\right] \\
&= 2\frac{1}{r^2}\frac{dm(r)}{dr}. && (3.29)
\end{aligned}$$

Next in (3.22) replace $e^{\lambda(r)}$ with its value from (3.28) and $\nu(r)$ with the value from (3.27). This gives

$$\begin{aligned}
8\pi p &= \left(1-\frac{2m(r)}{r}\right)\left(-\frac{1}{r}\frac{2}{\rho+p}\frac{dp}{dr} + \frac{1}{r^2}\right) - \frac{1}{r^2} \\
&= -\frac{2}{r}\frac{1}{\rho+p}\frac{dp}{dr} + \frac{1}{r^2} + \frac{4m(r)}{r^2}\frac{1}{\rho+p}\frac{dp}{dr} - \frac{2m(r)}{r^3} - \frac{1}{r^2} \\
&= \frac{2}{r^3}\left(-m(r) - r^2\left(1-\frac{2m(r)}{r}\right)\right)\frac{1}{\rho+p}\frac{dp}{dr}. && (3.30)
\end{aligned}$$

Rearranging (3.28), (3.30) and (3.27) we get the desired equations:

$$\frac{dm(r)}{dr} = 4\pi r^2\rho(r), \tag{3.31}$$

$$\frac{dp(r)}{dr} = -\left[\rho(r)+p(r)\right]\left(\frac{m(r)+4\pi r^3 p(r)}{r^2(1-2m(r)/r)}\right), \tag{3.32}$$

$$\frac{d\nu(r)}{dr} = -\frac{2}{\rho(r)+p(r)}\frac{dp(r)}{dr} = \frac{2m(r)+8\pi r^3 p(r)}{r^2(1-2m(r)/r)}. \tag{3.33}$$

These are the Tolman-Oppenheimer-Volkoff equations, also known as the equations of structure, describing matter in relativistic hydrostatic equilibrium. They describe the behavior of matter inside stars, but also insde white dwarfs and neutron stars, which is what they will be used for in this paper. It now also becomes clear why we replaced $\lambda(r)$ with $m(r)$. Looking at (3.31), we see that $m(r)$ gives the total mass up to radius r. These equations are three equations with four unknown variables ($m(r)$, $\nu(r)$, p and ρ). So together with an equation of state and suitable boundary conditions they completely determine the behavior of the system. The equation of state depends on the nature of the compact stellar object under investigation.

3.5 White Dwarfs

The first type of compact star we study is the white dwarf. As noted before, the inward force due to gravity is balanced by the electron degeneracy pressure. When it is first formed, a white dwarf is still very hot because of residual heat left over from the main sequence of the star. This is why white dwarfs are white, and visible. However in a white dwarf there are no more fusion reactions, and the left over thermal energy will slowly radiate away. Once all energy has been radiated away a white dwarf would become a black dwarf - cold invisible and inert. However the universe is not yet old enough for this to have happened in even the oldest white dwarfs[1]. So all white dwarfs in the universe have at least some residual thermal pressure working against gravity. However since this is usually very little, and we are interested primarily in the limiting case, this will be ignored. So we will only study the electron degeneracy pressure.

Two electrons cannot occupy the same quantum state. This means that if two electrons are forced together such that their positions overlap, they must be at a different energy level. As more and more electrons are pushed into the same space, the electrons will be pushed into higher and higher energy levels. This requires additional compression force. The result is an outward pressure balancing gravity, even at zero temperature.

To calculate how strong this pressure is, we start with a very simple quantum system. Put an electron in a box. A one-dimensional box of length L. The possible energy levels are quantized:

$$E_k = \frac{1}{2m}\left(\frac{k\pi\hbar}{L}\right)^2 = \frac{\vec{p}_k^2}{2m}, \qquad k = 1, 2, 3, \ldots \tag{3.34}$$

These are all the allowed energy levels. Since electrons are spin-$\frac{1}{2}$ particles, two electrons can be put in every energy level. This means that with N electrons the total energy will be

$$\mathcal{E} = \sum_{k=1}^{N/2}(2E_k). \tag{3.35}$$

The highest energy state the electrons are in, will of course be the Fermi energy E_F, and the accompanying momentum will be the Fermi momentum p_F. Instead of summing over all particles, one could sum over all momenta until p_F, where we have one state for every interval $\pi\hbar/L$ in p. For large N we can replace the sum with an integral

$$\mathcal{E} = \sum_{k=1}^{N/2}(2E_k) = 2\left(\frac{L}{2m}\right)\int_0^{p_F} dpE(p_k). \tag{3.36}$$

We can also express the total number of particles N as a function of the

[1] It is not known precisely how long it takes white dwarfs to cool, but estimates start at 10^{15} years.

Fermi level,

$$N = \sum_{k=1}^{N/2} 2 = 2\left(\frac{L}{2m}\right)\int_0^{p_F} dp. \tag{3.37}$$

So far this is all in one dimension. The universe however has three spatial dimensions. Upgrading to three dimensions however is easy. Let $\boldsymbol{p} = (p_x, p_y, p_z)$, and integrate over a sphere in momentum space with radius $< p_F$. Because only the octant in momentum space with positive p_x, p_y and p_z is relevant, an overall factor 1/8 should be added. For the total number of particles this gives

$$N = 2\left(\frac{L}{\pi\hbar}\right)^3 \frac{1}{8}\int_0^{p_F} 4\pi p^2 dp, \tag{3.38}$$

where the factor of 4π comes from integrating out the angles. Doing this integral gives

$$n \equiv \frac{N}{L^3} = \frac{p_F^3}{3\pi^2\hbar^3}. \tag{3.39}$$

Next we do the same for the total energy $\mathcal{E}$,

$$\mathcal{E} = 2\left(\frac{L}{\pi\hbar}\right)^3 \frac{1}{8}\int_0^{p_F} 4\pi p^2 E(p) dp. \tag{3.40}$$

Using the value of $E(p)$ from (3.34) this integration gives

$$\begin{aligned}
\rho \equiv \frac{\mathcal{E}}{L^3} &= \pi^{-2}\hbar^{-3}\int_0^{p_F} p^2 \frac{p^2}{2m} dp \\
&= \pi^{-2}\hbar^{-3}\frac{1}{5}\frac{p_F^5}{2m} \\
&= \frac{3}{10}\left(3\pi^2\right)^{2/3}\left(\frac{\hbar^2}{m}\right) n^{5/3},
\end{aligned} \tag{3.41}$$

where in the last line we used equation (3.39) to replace p_F in favor of n.

So far we have neglected relativistic corrections. Equation (3.39) is a manifestly non-relativistic equation. Not only have we assumed no relativistic effects, we have also completely ignored the energy contribution due to the rest mass. We briefly reintroduce c because we want to compare the relativistic and non-relativistic case. This contribution then becomes $mc^2 n$, so that the total energy density in the non-relativistic case becomes

$$\rho = mc^2 n + \frac{3}{10}\left(3\pi^2\right)^{2/3}\left(\frac{\hbar^2}{m}\right) n^{5/3}. \tag{3.42}$$

In general the energy of an electron as a function of momentum is not given by (3.34) but by

$$E(p) = \sqrt{m^2c^4 + p^2c^2}. \tag{3.43}$$

In the ultra relativistic limit this of course becomes $E = pc$ and we can solve (3.40) as:

$$\begin{aligned}\rho &= \pi^{-2}\hbar^{-3}\int_0^{p_F} p^2(pc)dp \\ &= \pi^{-2}\hbar^{-3}\frac{1}{4}p_F^5 \\ &= \frac{3}{4}\left(3\pi^2\right)^{1/3}(\hbar c)\, n^{4/3}. \end{aligned} \tag{3.44}$$

This is the energy density inside a white dwarf as a function of the number density, in the ultra relativistic limit.

The next goal is to calculate the pressure. This is in fact rather easy. Recall the first law of thermodynamics:

$$d\mathcal{E} = -pdV, \tag{3.45}$$

where p means pressure, not momentum, and $V = L^3$ is the volume. We have $E = \rho V$ and $V = N/n$ which gives

$$p = n\left(\frac{d\rho}{dn}\right) - \rho. \tag{3.46}$$

Plugging this back into (3.42) and (3.44) immidiately gives

$$p = \frac{1}{5}\left(3\pi^2\right)^{2/3}\left(\frac{\hbar^2}{m}\right)n^{5/3} \qquad \text{(non-relativistic)}, \tag{3.47}$$

$$p = \frac{1}{4}\left(3\pi^2\right)^{1/3}(\hbar c)\, n^{4/3} \qquad \text{(relativistic)}. \tag{3.48}$$

Having expressions for both p and ρ as functions of n, one can elimite n to find $p(\rho)$. For the ultra relativistic case this expression is particularly simple:

$$p(\rho) = \frac{1}{3}\rho, \tag{3.49}$$

and that is the equation of state we were looking for. Together with the equations of structure derived in the previous section we now have all the tools to give a complete description of white dwarf stars.

The four equations that we now have form a system of first-order ordinary differential equations. They can be solved using standard numerical algorithms. The standard way to do this is to pick a certain energy density ρ_c at the center ($r = 0$) and integrate radially outwards. Boundary conditions are $m(0) = \nu(0) = 0$ and the value of $p(0)$ follows from ρ_c. At a specific radius R the pressure vanishes, which correponds to the surface of the star. The total mass M is then simply $m(R)$. An example of such a calculation is given in figure 3.2.

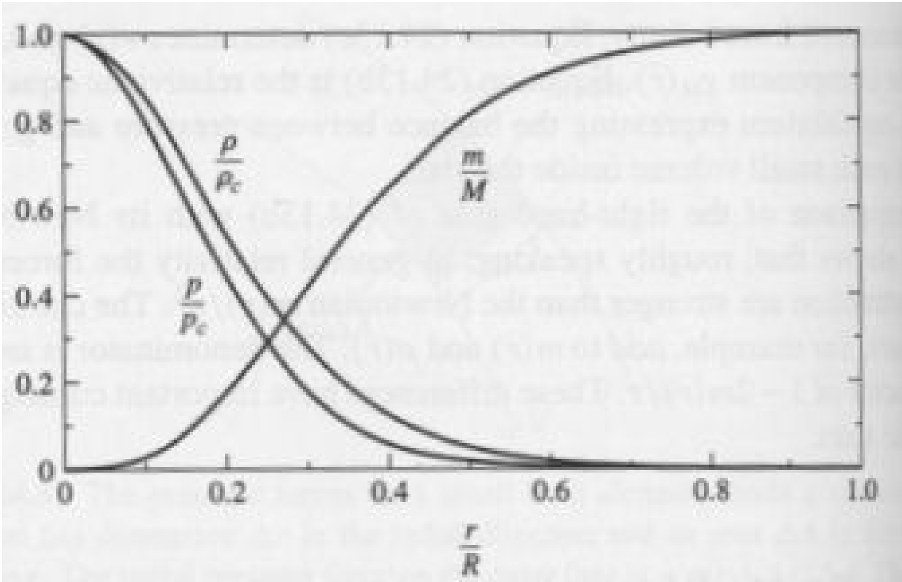

Figure 3.2: *Numerical solution of the relativistic equations of structure for a white dwarf.* The central density used is $\rho_c = 10^{10}$ g/cm^3. This gives $R = 1314$ km and $M = 1.42\ M_\odot$, which is near the maximum mass for a white dwarf.
Image from Hartle [3], chapter 24.

This can be done for different values of ρ_c. For each a total mass M and radius R will be found. These can be plotted against each other (see figure 3.3). The question arises whether the electrons should be treated relativistically or not. For white dwarfs of low mass, the electrons will be non-relativistic. However the average energy per electron increases with the total mass of the white dwarf. Since we are interested in an upper bound, we can therefor safely take the relativistic limit. It turns out that the radius as a function of total mass goes to zero at about $M \approx 1.43\ M_\odot$. This is known as the Chandrasekhar mass and it gives an upper bound on the mass of white dwarfs. As soon as the radius $R < 2M$ the white dwarf must collapse into a black hole. This will happen just below the Chandrasekhar limit.

White dwarfs are supported by the Pauli exclusion principle acting between electrons. We now have an upper bound on the strength of this effect. Above the Chandrasekhar mass a white dwarf must either collapse into a black hole, or new physics must come into play. The latter turns out to be the case.

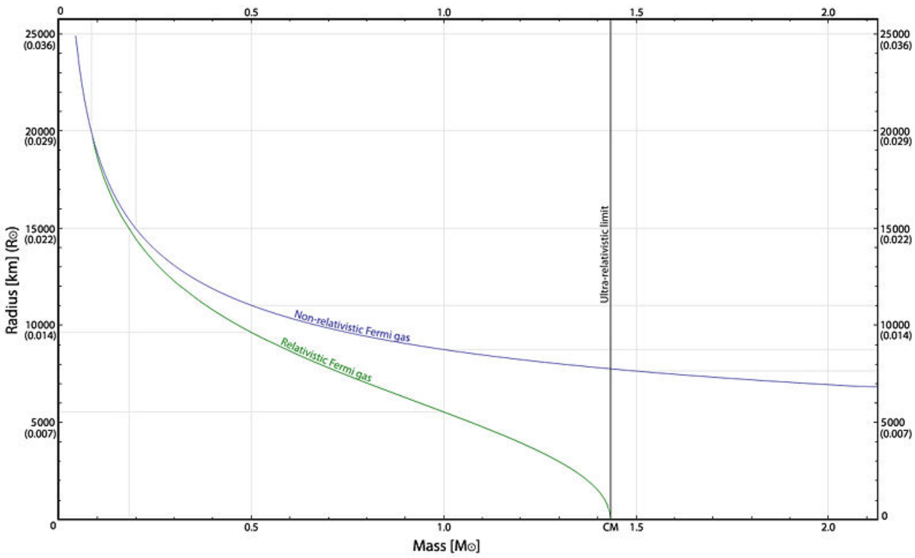

Figure 3.3: *The radius of a white dwarf as a function of its mass.* The radius of a white dwarf goes to 0 at the Chandrasekhar limit $M \approx 1.43\ M_\odot$. At this point the white dwarf will collapse. The non-relativistic solution does not exhibit the same behavior, but it is only valid for low mass.
Image from wikipedia *http://en.wikipedia.org/wiki/File:WhiteDwarf_mass-radius.jpg*.

3.6 Neutron Stars

As the mass of the white dwarf is increased, the average energy that each electron carries goes up. Once electrons carry more energy than

$$E_e > m_n c^2 - m_p c^2 = 1.3\text{MeV}, \tag{3.50}$$

the electrons can combine with protons to form neutrons, via inverse beta decay:

$$p + e^- \to n + \nu_e. \tag{3.51}$$

The result is known as a neutron star. Neutron stars are extremely dense and composed almost entirely of neutrons. Only at the outer layer, where the pressure is still relatively low, can ordinary atomic matter still survive. This matter will have more and more neutrons as we go deeper in. Deeper still neutrons will start leaking out of the nuclei and form a perfect neutron fluid. The remaining nuclei will become smaller and rarer with decreasing radius, until they disappear completely.

In principle the same method that was used in the previous section can be used to calculate the maximum mass of a degenerate gas of neutrons. This however is problematic because the results depend greatly on the equation of state. Above nuclear densities ($\rho = 2.8 \times 10^{14}$ g/cm^3) the equation of state is poorly understood, and there are several competing models. Depending on the equation of state chosen, estimates for the upper mass of a neutron star range from 1.5 $M_\odot$ to 2.7 $M_\odot$.

Another complication is that at sufficiently high densities, new physics may come into play. It is theorized that under sufficient pressure the individual neutrons in a neutron star break down into their constituent quarks (up and down). Some of these quarks might become strange quarks. The result is what is called a quark star or strange star. Another wildly speculative possibility is a preon star. Such new physics further complicate finding a reasonable equation of state.

Given this, it is desirable to derive an upper bound on the mass of any compact stellar object, be it neutron star or quark star or something else entirely, based on GR alone. This turns to be possible.

Assume that we know the equation of state up to some density ρ_0. Above this, unknown or poorly understood physics comes into play, but below this density we are reasonable sure about how matter behaves. A typical value for ρ_0 is the nuclear density which is about 2.9×10^{14} g/cm^3.

We know that we must have

$$\frac{dp}{d\rho} > 0, \tag{3.52}$$

which says nothing more than that pressure must increase with density. If this were not the case matter would not be stable.

Recall equation (3.32). Every term on the right hand side must be positive (the factor $1 - 2m(r)/r$ must be positive or the matter would be within its own Schwarzschild radius and form a black hole). This means that dp/dr is negative, so the pressure decreases with radius. By (3.52) the energy density must then also decrease with radius. This means we can separate the compact object into two region. An outer mantle with $\rho > \rho_0$ and a core with $\rho < \rho_0$. Let M_0 be the mass of the core, that is the value of $m(r)$ at the radius $\rho = \rho_0$. By (3.31) we have

$$M_0 = \int_0^{r_0} 4\pi r^2 \rho(r) \geq \int_0^{r_0} 4\pi r^2 \rho_0 = \frac{4}{3}\pi r_0^3 \rho_0. \tag{3.53}$$

However the core can not be inside its own Schwarzschild radius either, which gives:

$$M_0 \geq \frac{r_0}{2}. \tag{3.54}$$

From these two bounds we find

$$M_0 \leq \frac{1}{2}\left(\frac{3}{8\pi\rho_0}\right)^{1/2} \approx 8.0 M_\odot. \tag{3.55}$$

To this mass of the core the mass of the mantle should then be added. This mantle mass can be calculated by numerically solving the relativistic hydrodynamic equations combined with the known equation of state. It turns out that the mass of the mantle is negligible.

It is possible to derive a better upper bound, by enforcing the causality principle [9]. The gradient $\frac{dp}{d\rho}$ represents the speed of sound squared. So this must be smaller than the speed of light:

$$\frac{dp}{d\rho} \leq c^2 = 1. \tag{3.56}$$

This simple requirement is enough to significantly improve our previous estimate. The mass is in fact maximum when we have an equality in (3.56), such that we get

$$p = p_o + (\rho - \rho_0), \qquad \rho \geq \rho_0. \tag{3.57}$$

Using this equation of state and the Tolman-Oppenheimer-Volkof equations we can do a numerical integration, in the same way that it was done for white dwarfs. Including the mass of the mantle, the upper limit found is about 3.6 $M_\odot$. Assuming general relativity is correct, any object heavier than this must be a black hole.

Chapter 4

Astronomical Black Holes

In the second chapter the existence of black holes as valid solutions to the Einstein equations was shown. After that we showed that without sufficient pressure compact objects tend to collapse to black holes. Finally we showed that for a sufficiently massive object no source of pressure can be powerful enough. This means that *any* compact object heavier than eight solar masses must be a black hole.

The word 'compact' is important here. Main sequence stars can be much heavier than this. But their source of pressure is thermal, and thus finite. If we wait long enough, they will collapse.

So do black holes exist? Answering this question is now a simple matter of looking for compact objects - that is, objects that do not emit light - and measure their mass. If it is heavier than $3.6M_{\odot}$, it is a black hole. Physics can not get any simpler.

Of course, by their very nature black holes can not be observed directly. So they have to be observed indirectly, by the effect they have on other objects. For example, the movement of nearby stars will be affected by a black hole. Matter falling into a black hole will also be a strong source of x-rays[1]. We will distinguish between stellar mass black holes and supermassive black holes.

4.1 Stellar-Mass Black Holes

All known black hole candidates of stellar mass are X-ray binary systems. In such systems a compact object draws matter from its partner via accretion. This causes strong X-ray emissions which can be detected. Neutron stars and black holes radiate comparable amounts of energy, so distinguishing them by their emissions is very hard. However we can still look at the total mass. Since the binary stars rotate around a common center of gravity,

[1]A particle falling into a black hole can emit up to 42% of its rest mass as radiation. Black hole accretion is not treated in this paper, but see for example [1, 4, 5].

the mass of the invisible companion can be calculated from the orbit of the visible one.

Black Hole Candidates

Name	**Mass**
GRS 1915+105a	14 ± 4
V404 Cyg	12 ± 2
Cyg X-1	10 ± 3
LMC X-1	> 4
XTE J1819-254	7.1 ± 0.3
GRO J1655-40	6.3 ± 0.3
BW Cirb	> 7.8
LMC X-3	7.6 ± 1.3
XTE J1550-564	9.6 ± 1.2
4U 1543-475	9.4 ± 1.0
H1705-250	6 ± 2
GS 1124-684	7.0 ± 0.6
GS2000+250	7.5 ± 0.3
A0620-003	11 ± 2
GRS 1009-45	5.2 ± 0.6
GRO J0422+32	4 ± 1
XTE J1118+480	6.8 ± 0.4

Not a great many such systems are known. In 2004 about 40 X-ray binaries that contained candidates for black holes were known [10]. Tabel 4.1 gives a more recent list of stellar mass black holes that are considered to be confirmed [11]. We see that all have mass above the Tolman-Oppenheimer-Volkoff limit, and several have mass significantly above the maximum we derived for any compact object. This is thus direct evidence for the existence of solar mass black holes.

4.2 Supermassive Black Holes

The centers of most, if not all, galaxies are believed to contain supermassive black holes. One example that has received a lot of attention recently is Sagittarius A* (see figure 4.1) , the black hole in the center of our own galaxy [12, 13]. Its mass has been estimated at 3.7 million solar masses. The current highest resolution measurements give its size as 37 micro arcsecond. At a distance of 26 thousand ly, this corresponds to a radius of 44 million kilometer. This is about 4 times the Schwarzschild radius for an object of that mass. It is hard to imagine anything that massive and small that is not a black hole.

The largest known black hole, as of november 2008, has a mass of 18

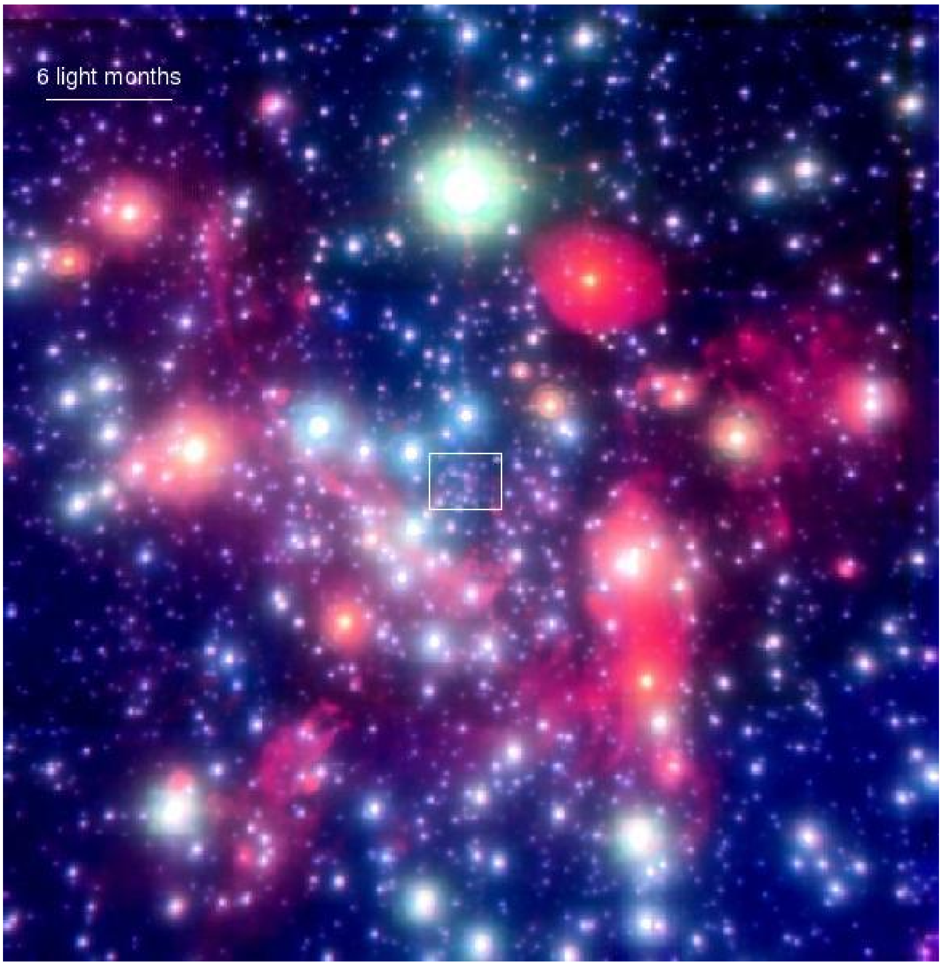

Figure 4.1: *Sagittarius A**, the bright and very compact astronomical radio source at the center of the Milky Way. It has been measured at 3.7 million solar masses with a radius of less than 44 million kilometers. These parameters imply strongly that it has to be a black hole.
Image from wikipedia *http://en.wikipedia.org/wiki/Sagittarius_A**.

billion solar masses. It is part of a pair of binary supermassive black holes in galaxy OJ 287.

An interesting fact about supermassive black holes is their low average density, which can be lower than that of air. Since the Schwarzschild radius goes up linearly with mass, the volume of a black hole will go cubic with mass. Supermassive black holes therefore do not have to be very dense. The gravitational forces on the surface are also a lot lower. One can safely pass the boundary of supermassive black hole without experiencing significant tidal forces.

Chapter 5

Discussion

5.1 Black hole alternatives

So far the evidence for black holes looks pretty convincing. But what alternatives are there? As noted before, the exact equation of state of a neutron star is not very well known. Different models give upper bounds on their masses between 1.5 $M_\odot$ to 2.7 $M_\odot$. There are several proposals for even denser objects that might have a higher upper limit. Quark stars and preon stars have already been mentioned.

Without any assumption about the equation of state of a compact object, so based on general relativity alone, an upper bound for the mass of a compact object of 3.6 $M_\odot$. This can be bumped up about 20% further by adding rotation. If the causality principle is dropped, such that signals are not bound by the speed of light, the upper mass goes up to 8.0 $M_\odot$, without rotation. Adding rotation might get you to 10 $M_\odot$ at most, but certainly not more. So if general relativity is correct black holes must exist.

Throwing out general relativity is an alternative. One alternative is the 'Gravastar' theory by Mazur and Mottola [14]. This is an extension of Bose-Einstein condensation to gravitational systems. The idea is that due to this Bose-Einstein condensation quantum effects could extend over the size of the entire black hole, such that GR would no longer be a good description of the system. Another alternative, from string theory, is the fuzzball.

Because the average density of supermassive black holes is so low, an alternative to supermassive black holes might be a swarm of smaller, stellar sized black holes. However it is not clear how such a swarm of black holes could exist without undergoing gravitational collapse.

5.2 Conclusion

The overall conclusion that can be drawn from this paper is that black holes do indeed most likely exist. They are not merely theoretical curiosities, but actually occur in nature.

Without a pressure to counter its effects, gravity will cause objects to collapse to a black hole. In main sequence stars the nuclear reactions generate enough thermal pressure to balance gravity. Once the star runs out of fuel another source of pressure must be found. White dwarfs survive due to the Pauli exclusion principe acting between electrons, but this can only work up to around 1.43 solar masses. Neutron stars can be a bit heavier, but not much. Even hypothetical and speculative objects such as strange stars or preon stars can not be heavier than 8 solar masses, an upper bound derived on principles from GR alone, without any assumption about the nature of the matter making up the compact star.

Astronomical observations provide strong evidence for believing such heavy objects exists. Several X-ray binaries have been found with masses above the upper limit. Additionally the centers of galaxies and some globular clusters are believed to contain black holes.

The existence of black holes is relevant because they provide potentional explanations for some observed phenomena, such as Active Galactic Nuclei, Ultra-luminous X-Rays or Gamma Ray Bursts.

Further study might include attempts to calculate the effects of rotation in the upper bound on the mass of compact objects. This upper bound could also be refined by a better understanding of matter at densities above the nuclear density.

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