Seminar Theoretical Physics

Neutrino oscillations

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Chapter 1

Introduction

Before plunging into the many theoretical descriptions and experimental results in neutrino physics, we give a brief overview of the history of the neutrino.

1.1 Historical overview

In this section we will provide a brief overview of the discovery of the neutrino, which is based on the historical summary in [1]. For a more extended account see [2, Ch. 1] and references therein.

Without giving proper credit to Becquerel, Thompson, Rutherford and all those others who were involved in the discovery of and early attempts to explain radioactive phenomena, let us go back to the early 1900's and the first studies of β radiation. At the time, it was established that the particles making up the β radiation were actually the same ones as those coming from cathode ray tubes, namely electrons. The only other elementary particle was thought to be the proton, all the nuclei of elements heavier than hydrogen being built up from protons and electrons. The electrons emitted in β radiation were assumed to come from the nucleus. Now in spontaneous beta decay (that is, without any external stimulus such as light or electricity) some nucleus A with mass M_A turns into a nucleus A' of mass $M_{A'}$ while emitting an electron (and no more than that). Using conservation of momentum and energy, and the energy-mass equivalence from Einsteins new theory) then obviously the energy of the emitted electron should be proportional to $M_A - M_{A'}$. For a while, this result was supported by both theoretical and experimental evidence. However, both turned out to have serious problems. In 1914 detection techniques were sufficiently advanced that Chadwick could provide definite evidence that the energy E of the outgoing electron is not sharply peaked aroud $M_A - M_{A'}$ but continuously takes values up to $E \simeq M_A - M_{A'}$. It took another 15 years to show that this was in fact a "real" property, rather than one induced by interactions of the radiation with other nuclear particles or other radiation. A completely different problem also arose. Under the assumption of nuclei just containing electrons and protons, $^{7}_{14}\mathrm{N}$ should consist of $7 + 14 = 21$ fermions, although experiments had revealed that nitrogen was in fact a boson.

Rather than thinking the atomic model of nuclei built up from electrons and protons was wrong (which, of course, it is), other explanations were sought. Niels Bohr around 1930 went as far as to suspect that maybe the energy conservation principle would have to be abandoned. Clearly this would solve the problem of the "missing" energy in β decay. Based on the "strange" properties of electrons in nuclei, this idea was not immediately dismissed; in fact it was considered a very likely explanation. However, in 1930, Pauli wrote a hesitant letter [3] to the participants of a nuclear physics conference in Tübingen (Germany). He proposed that another, as yet unseen, elementary particle might exist which would have to be electrically neutral, very light and which he suspected to have spin $\frac{1}{2}$. He went on to demonstrate that this would solve both problems, as this new particle could carry away some energy from beta decays unseen, and adding an odd number of them to the nitrogen nucleus would make the number of fermions even. He immediately added that the idea was very daring and "may not seem very probable *a priori*", but "only the one who dares can win" and "from now on, every possible solution must be considered."

Some years later, in 1932, Chadwick did in fact discover a neutral nuclear constituent. Unfortunately it did not have the properties predicted by Pauli. In particular it was much heavier, with mass slightly greater than the proton. Soon after, Heisenberg, Majorana and Ivanenko all independently assumed that atomic nuclei were not built out of protons and electrons, but protons and neutrons. Thus they explained all the existing data. It was Enrico Fermi who, in a play of words, renamed Pauli's neutral particle to "neutrino", where –ino is an Italian diminuitive suffix (neutrino $\sim$ "small neutron"). Moreover, he wrote down a beta decay*

$$n \to p + e^- + \overline{\nu}_{\rm e}, \tag{1.1}$$

and worked out the quantum mechanical theory. In fact, this also allowed for the calculation of the cross-section of the inverse reaction,

$$\overline{\nu}_{\rm e} + p \to e^+ + n \tag{1.2}$$

which was confirmed by Reines and Cowan in 1956.

It took until well into the 1950's before definite direct experimental evidence for the neutrino with all the postulated properties was provided. We will come back to this in later sections where we will discuss the experimental "evolution" of the neutrino in more detail. Out of theoretical need, a second and even third neutrino type were later postulated. The second type, the muon neutrino ν_μ was found rather quickly, but the third one – the tau neutrino ν_τ – was discovered only much later. In fact, it was the last particle of the minimal Standard Model of elementary particles to be observed experimentally.

*in contemporary notation, with bars denoting anti-particles and being aware of the existence of two other neutrino flavours

Chapter 2

The Standard Model

In this chapter, we briefly review some important concepts from the minimal Standard Model of physics (SM). We recall the different particles and fields, the extremely important concept of gauging and we sketch how the Higgs mechanism works, which gives the particles their masses. We also consider the status of the neutrino in the SM and the shortcomings of this theory.

One can safely say, that the Standard Model of particle physics (SM) is one of the greatest successes of theoretical physics of the past decades. This theory incorporates all known elementary particles into a single framework, and it describes the electroweak and strong interactions. So far, any prediction that has been made by the Standard Model has been tested experimentally up to great accuracy. Though the Standard Model does not include the gravitational force it is a very useful theory. In fact there is not much physics which is *not* described by the SM. One of those exceptions are neutrinos, which are the topic of this paper. In particular the 1960's version of the Standard Model fails to accurately describe neutrino oscillations. Let us now first recapitulate how the (minimal) SM works by briefly summing up some of its building blocks.

2.1 Fermions

The basic ingredients of the SM are fermionic fields of spin $\frac{1}{2}$, divided into three generations*. Each generation consists of two quarks, a lepton and a neutrino, as indicated in table 2.1. Each of these fermions f also has an anti-particle $\overline{f}$. The two spin degrees of freedom of any (anti-)fermion are described in a *Weyl spinor*: a 2-component (complex) vector

$$\chi = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix}. \tag{2.1}$$

*This section is based on [4], with the conventions from [5]

	First generation	Second generation	Third generation
Quarks	Up (u) Down (d)	Charm (c) Strange (s)	Top (t) Bottom (b)
Lepton	Electron (e)	Muon (μ)	Tau (τ)
Neutrino	ν_e	ν_μ	ν_τ

Table 2.1: Fermions in the Standard Model

The constituents of the fermionic part of the Standard Model are then spin fields

$$\psi_\alpha(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} = \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix}, \tag{2.2}$$

where ϕ and χ are Weyl spinors and $\alpha = 1, 2, 3, 4$ is a spin label (it does not necessarily take d values in d dimensions, in general it runs through $2^{\lfloor d/2 \rfloor}$ values). We can consider ϕ as describing the fermion and χ the anti-fermion. In general, ϕ and χ may be independent, in which case we call ψ a *Dirac spinor*. If they are related by

$$\chi = -\sigma_2 \phi^*, \tag{2.3}$$

where σ_2 is a Pauli matrix and the star denotes complex conjugation, ψ is called a *Majorana spinor*.

The fermion fields satisfy the *Dirac equation*

$$(\mathrm{i}\partial\!\!\!/ - m)\psi = 0, \tag{2.4}$$

where m denotes the mass of the fermion. We have employed Feynman slash notation, $\partial\!\!\!/ \stackrel{\text{def}}{=} \gamma^\mu \partial_\mu$ with $\partial_\mu = \partial/\partial x^\mu$, where γ^μ ($\mu = 0, 1, 2, 3$) denote the Dirac matrices which satisfy

$$\{\gamma^\mu, \gamma^\nu\} \stackrel{\text{def}}{=} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}. \tag{2.5}$$

Whenever an explicit form is needed for the Minkowski metric $\eta^{\mu\nu}$, we will choose $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ (in accordance with [5]).

Defining the *Dirac conjugate spinor* by $\overline{\psi} \stackrel{\text{def}}{=} \psi^\dagger \gamma^0$ with $\psi^\dagger$ the Hermitian conjugate, the Dirac equation follows from the Euler-Lagrange formalism applied to

$$\mathcal{L} = \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi; \tag{2.6}$$

for example, it can be obtained immediately from the Euler-Lagrange equation $\partial\mathcal{L}/\partial\overline{\psi} = 0$.

Using the Dirac matrices, we can define

$$P_\mathrm{R} = \tfrac{1}{2}(1 + \gamma_5), \quad P_\mathrm{L} = \tfrac{1}{2}(1 - \gamma_5), \qquad \text{where} \qquad \gamma_5 = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3. \tag{2.7}$$

Using that $(\gamma_5)^2$ is the identity matrix, up to a sign (with our explicit choice of $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ it is even precisely the identity), it can be easily shown that these are orthogonal projection operators. The projection of the fermionic fields

$$\psi_{\mathrm{L,R}} = P_{\mathrm{L,R}}\psi \tag{2.8}$$

defines two *chiralities*, left-handed and right-handed respectively. We can decompose the Dirac Lagrangian (2.6) accordingly [4, § 6.9]:

$$\begin{aligned}\mathcal{L} &= \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi \\ &= \overline{\psi}(P_\mathrm{R} + P_\mathrm{L})(\mathrm{i}\partial\!\!\!/ - m)(P_\mathrm{R} + P_\mathrm{L})\psi \\ &= \mathrm{i}\overline{\psi}P_\mathrm{R}\partial\!\!\!/ P_\mathrm{L}\psi + \mathrm{i}\overline{\psi}P_\mathrm{L}\partial\!\!\!/ P_\mathrm{R}\psi - m\overline{\psi}P_\mathrm{L}P_\mathrm{L}\psi - m\overline{\psi}P_\mathrm{R}P_\mathrm{R}\psi \\ &= \mathrm{i}\overline{P_\mathrm{L}\psi}\partial\!\!\!/(P_\mathrm{L}\psi) + \mathrm{i}\overline{P_\mathrm{R}\psi}\partial\!\!\!/(P_\mathrm{R}\psi) - m\overline{P_\mathrm{R}\psi}(P_\mathrm{L}\psi) - m\overline{P_\mathrm{L}\psi}(P_\mathrm{R}\psi) \\ &= \mathrm{i}\overline{\psi_\mathrm{L}}\partial\!\!\!/\psi_\mathrm{L} + \mathrm{i}\overline{\psi_\mathrm{R}}\partial\!\!\!/\psi_\mathrm{R} - m\overline{\psi_\mathrm{L}}\psi_\mathrm{R} - m\overline{\psi_\mathrm{R}}\psi_\mathrm{L},\end{aligned} \tag{2.9}$$

so the kinetic term decomposes into two independent parts, whereas the mass term mixes the two chiralities. This concept is very important in the Standard Model.

In the Standard Model, the different chiralities play very different roles. In 1958, a rather ingenious experiment by Goldhaber, Grodzins and Sunyar had measured neutrino spins* and observed only left-handed neutrinos. Since no right-handed neutrinos were required to explain any phenomena at the time the right-handed neutrinos ν_R were, in the 1970s, not included in the Standard Model by choice. By the decomposition above, neutrinos automatically become massless, which also did not seem to contradict experiment.

2.2 Gauging and gauge bosons

So far we have only considered fermions with no interactions, except for the mixing between different chiralities. (Almost) all bosons in the SM arise as *gauge bosons*. They arise from local symmetries of the Lagrangian. Let us recall how this works.

Consider the Dirac Lagrangian (2.6). It is clear that under a global phase transformation $\psi(x) \to e^{\mathrm{i}q\xi}\psi(x)$, with ξ some real number, this Lagrangian is unchanged (because $\overline{\psi}(x) \to e^{-\mathrm{i}q\xi}\overline{\psi}(x)$). However, since the phase of ϕ is not measurable, we would like to have the freedom to choose the phase of ψ at every point in spacetime separately. Therefore we with to impose invariance of the Lagrangian under a *local* transformation $\psi(x) \to e^{\mathrm{i}q\xi(x)}\psi(x)$. It can be easily checked that now,

$$\partial_\mu\psi(x) \to \partial_\mu\left(e^{\mathrm{i}q\xi(x)}\psi(x)\right) = e^{\mathrm{i}q\xi(x)}\left(\partial_\mu\psi(x) + \mathrm{i}q(\partial_\mu\xi(x))\psi(x)\right). \tag{2.10}$$

To restore invariance, we introduce a covariant derivative ∇_μ with the property that $\nabla_\mu\psi(x) \to e^{\mathrm{i}q\xi(x)}\nabla_\mu\psi(x)$. The name covariant derivative has a rigorous meaning in differential geometry, we will ignore this point for now and will state only that the Lagrangian does become invariant when we replace the derivative ∂_μ by the covariant version ∇_μ defined by

$$\nabla_\mu\psi = (\partial_\mu - \mathrm{i}qA_\mu(x))\psi(x) \tag{2.11}$$

as long as the new field A_μ transforms as $A_\mu \to A_\mu + \partial_\mu\xi$. The new field A_μ is called a *gauge field*. After the above replacement, the Lagrangian (2.6) reads

$$\begin{aligned}\mathcal{L} &\to \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi + qA_\mu\overline{\psi}\gamma^\mu\psi \\ &\to \overline{\psi}(\mathrm{i}\nabla\!\!\!\!/ - m)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}.\end{aligned} \tag{2.12}$$

In the last term of the first line, we automatically get an interaction of the fermions with the gauge field. If we define $F_{\mu\nu} \overset{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$, we can add another invariant

*For the original paper see [6], a very accessible explanation is given in [7] term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, which turns out to be the kinetic term for A_μ when deriving the field equations for the gauge field; therefore we have also included this term on the second line.

The idea from the above paragraphs can be generalised. Let G be some finitely generated Lie-group (called the *gauge group*) with generators $g_1, \ldots, g_N$. An n-dimensional representation ϕ of this group assigns to each generator g_i an invertible $n \times n$ matrix g_i^ϕ such that the image of the Lie bracket $[g_i, g_j]$ under ϕ coincides with the matrix commutator $[g_i^\phi, g_j^\phi] = g_i^\phi g_j^\phi - g_j^\phi g_i^\phi$. In a given representation ϕ, we can express the elements of the group as $n \times n$ matrices through

$$g(x) = \exp\left[\sum_{i=1}^{N} q\alpha_i(x) g_i^\phi\right], \tag{2.13}$$

where α_i is some set of real (or complex) smooth functions on spacetime and the exponent of a matrix is defined as usual by its power series. The constant q is called the *charge* of the field under the action of G. (If $q \equiv 0$ then the field does not change under the action of G and we say it is uncharged). Usually, the generators and commutation relations of the Lie algebra are themselves defined by matrices $g_i^{(\text{fund})}$, which we call the *fundamental representation* of G. Suppose we have some theory with a Lagrangian $\mathcal{L}$ which depends on a set of fields ϕ_i $(i = 1, \ldots, N)$. We may require that $\mathcal{L}$ be invariant under transformations from G, that is: for any $U \in G$ the Lagrangian does not change under

$$\phi(x) \to U\phi(x) \qquad \text{or, in matrix notation,} \qquad \phi_i \to \sum_{j=1}^{N} U_{ij}\phi_j. \tag{2.14}$$

To achieve this, we must again replace any derivatives ∂_μ in the Lagrangian by covariant ones ∇_μ, which act on the fields by $\nabla_\mu\psi \stackrel{\text{def}}{=} \partial_\mu\psi - W_\mu\psi$, with the gauge field $W_\mu = W_\mu^i g_i$ transforming as

$$W_\mu \to UW_\mu U^{-1} + (\partial_\mu U)U^{-1}. \tag{2.15}$$

We can also add to the Lagrangian a kinetic term for W_μ:

$$-\frac{1}{4}G_{\mu\nu}G^{\mu\nu}, \tag{2.16}$$

where

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu], \tag{2.17}$$

or, in some representation ϕ,

$$G^i_{\mu\nu}g_i^\phi = (\partial_\mu W_\nu(x)^i)g_i^\phi - (\partial_\nu W_\mu(x)^i)g_i^\phi - W_\mu(x)^i W_\nu(x)^j \left(g_i^\phi g_j^\phi - g_j^\phi g_i^\phi\right). \tag{2.18}$$

When imposing such a symmetry on the Lagrangian (2.6),

$$-\overline{\psi}_i(\partial\!\!\!/ + m)\psi_i \to -\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} - \overline{\psi}_i\left(\partial\!\!\!/ + m - g\gamma^\mu W_\mu\right)\psi_i. \tag{2.19}$$

For a one, two, three or n (> 3) dimensional representation ϕ of G, we say the the fields $\psi_1, \ldots, \psi_n$ transform as a singlet, doublet, triplet or n-tuplet respectively, under ϕ.

From the discussion above it should be clear that for every generator g_i of the gauge group G, a gauge field W_μ^i appears. These fields have one group index i and a spacetime vector index μ, meaning they correspond to spin-1 bosons, carrying interactions from one spacetime point to another. In the first case discussed in this section, we had $G = U(1)$ (the group of 1×1 unitary matrices) which has just one generator, Y. The associated charge q is the *electric charge*, the gauge field A_μ is the electromagnetic vector potential and the corresponding force mediating spin-1 particle is the photon.

The *gauge group of the Standard Model* is the group product

$$G_{\rm SM} = SU(3)_c \otimes SU(2)_{I_W} \otimes U(1)_Y. \tag{2.20}$$

We will focus on the last two factors. The subscripts will be explained below.

The $SU(2)$ group is the *weak isospin* group. Its fundamental representation is two-dimensional, the generators $t_i^{(\text{fund})}$ are one half times the Pauli matrices:

$$t_1^{(\text{fund})} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_2^{(\text{fund})} = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad t_3^{(\text{fund})} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.21}$$

The left handed fermions transform as doublets

$$\Psi^i_{\rm e,L} = \begin{pmatrix} \nu^i_{\rm L} \\ \ell^i_{\rm L} \end{pmatrix}, \qquad \Psi^i_{\rm q,L} = \begin{pmatrix} u^i_{\rm L} \\ d^i_{\rm L} \end{pmatrix} \tag{2.22}$$

under this representation, that is they have weak isospin charge $I_W = \frac{1}{2}$. In the above expression, $i = 1, 2, 3$ labels the three generations, ν^i, ℓ^i, u^i and d^i denote the corresponding neutrino, lepton and quarks, respectively. The right-handed fermions are singlets under $SU(2)$, in other words, their weak isospin charge is $I_W = 0$ and they do not transform under $SU(2)$ transformations). They are denoted $\psi^i_{\rm e,R}$, $\psi^i_{\rm u,R}$ and $\psi^i_{\rm d,R}$ for the right-handed electron (muon, tau), right-handed up (charm, top) quark and right-handed down (strange, bottom) quarks, respectively. Recall that in the minimal standard model, there are no right-handed neutrinos.

The $U(1)_Y$ group is not the $U(1)$ group from electromagnetism, the corresponding charge is not electric charge q but *hypercharge* Y. Each doublet or singlet has its own hypercharge, which in general is twice the (in the case of a doublet, average) electric charge.

The action then reads

$$\mathcal{L}_{\rm f} = \sum_{i=1}^{3} \mathrm{i}\left[\overline{\Psi^i_{\rm e,L}}\partial\!\!\!/\Psi^i_{\rm e,L}\overline{\Psi^i_{\rm q,L}}\partial\!\!\!/\Psi^i_{\rm q,L}\overline{\psi^i_{\rm e,R}}\partial\!\!\!/\psi^i_{\rm e,R}\overline{\psi^i_{\rm u,R}}\partial\!\!\!/\psi^i_{\rm u,R}\overline{\psi^i_{\rm d,R}}\partial\!\!\!/\psi^i_{\rm d,R}\right]. \tag{2.23}$$

If we replace the partial derivative by a covariant one, we must introduce four gauge fields: three fields $W_\mu^{1,2,3}$ for the $SU(2)_{I_W}$ group, and one field B_μ for $U(1)_Y$. The covariant derivative looks like

$$\nabla_\mu = \partial_\mu - \mathrm{i}g\sum_{a=1}^{3} W^a_\mu t^\phi_a - \mathrm{i}\frac{g'}{2}B_\mu Y. \tag{2.24}$$

The $SU(2)$ generators t^ϕ_a in representation ϕ will take their form according to the fermion representation they work on. For example, for left handed fermions they are the matrices from equation (2.21), while for right-handed fermions they are zero. Similarly, the representation of Y is in general proportional to an identity matrix, with the propoprtionality constant determined by the hypercharge of the fermion on which the derivative acts. The gauge fields from the $SU(3)_c$ *color* group, which we are neglecting at the moment, are the eight *gluon* fields G_μ. In the literature one usually encounters the charged and neutral W *bosons* $W^{\pm}$, W^{0}, the neutral Z *boson*, and the *photon* A. They are related to these gauge fields by

$$\begin{aligned} W^1_\mu &= \frac{1}{\sqrt{2}}\left(W^-_\mu + W^+_\mu\right), & W^2_\mu &= \frac{1}{\sqrt{2}\mathrm{i}}\left(W^-_\mu - W^+_\mu\right), \\ W^3_\mu &= \cos(\theta_W)Z_\mu + \sin(\theta_W)A_\mu, & B_\mu &= \cos(\theta_W)A_\mu - \sin(\theta_W)Z_\mu, \end{aligned} \tag{2.25}$$

where the *weak mixing angle* θ_W is defined by $\tan(\theta_W) = g'/g$.

Implementing the covariant derivative produces in addition to the terms in (2.23) with the replacement $\partial \to \nabla$, the following interaction terms [8]:

$$\mathcal{L}_{\mathrm{f,gauge}} = -\left(\frac{g}{\sqrt{2}} j^\mu_{\mathrm{c}} W^+_\mu + \mathrm{h.c.}\right) - \frac{g}{\cos\theta_{\mathrm{W}}} j^\mu_{\mathrm{n}} Z_\mu - e j^\mu_{\mathrm{em}} A_\mu. \tag{2.26}$$

In this expression we have defined the *charged* and *neutral vector current*

$$\begin{aligned} j^\mu_{\mathrm{c}} &= \sum_{i=1}^{3} \left[\overline{\nu_{i,\mathrm{L}}}\gamma^\mu e_{i,\mathrm{L}} + \overline{u_{i,\mathrm{L}}}\gamma^\mu d_{i,\mathrm{L}}\right]; \\ j^\mu_{\mathrm{n}} &= \sum_{f=\nu_i,e_i,u_i,d_i} \sum_{\sigma=\mathrm{L,R}} \overline{f_\sigma}(t^3 - \sin^2(\theta_{\mathrm{W}})Q) f_\sigma; \end{aligned} \tag{2.27}$$

respectively. Of course the nomenclature "charged" and "neutral" is related to the charge of the corresponding vector bosons, $W^{\pm}$ and Z^0. The neutral current contains the generator t^3 as well as the electric charge $Q = I_z + \frac{1}{2}Y$, while $j^\mu_{\mathrm{em}} \sim \overline{e_i}\gamma^\mu e_i$ is the standard electromagnetic current from QED.

2.3 Discrete symmetries and CP violation

Apart from the continuous symmetries discussed in the previous section, there are also discrete symmetries which are of importance in the Standard Model. Two of them are spacetime symmetries, acting on the spacetime labels, namely*: parity P $((x^0,\vec{x}) \to (x^0,-\vec{x}))$ and time reversal T $((x^0,\vec{x}) \to (-x^0,\vec{x}))$. The third one is charge conjugation C, which transforms a particle into its anti-particle. It acts by $C\phi = \phi^*, CW^\mu = -(W^*)^\mu$ (note that $(W^{\pm})^* = W^{\mp}$). The charge conjugate ψ^{c} of a fermionic field ψ is defined by

$$\psi^{\mathrm{c}} \stackrel{\mathrm{def}}{=} C\gamma^0\psi^*, \tag{2.28}$$

where C is the charge conjugation matrix, satisfying

$$(\gamma^\mu)^{\mathrm{T}} = -C^{-1}\gamma^\mu C, \qquad C^\dagger = C^{-1} \qquad \text{and } C^{\mathrm{T}} = -C. \tag{2.29}$$

In the Dirac representation of the gamma-matrices, we can take $C = \gamma^2\gamma^0$ and C takes the block form

$$C = \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \qquad \psi^{\mathrm{c}} = -\gamma^2\psi^*. \tag{2.30}$$

Using that $(\sigma_2)^{\mathrm{T}} = -\sigma_2$, $(\sigma_2)^* = -\sigma_2$ and $(\sigma_2)^{-1} = \sigma_2$, it is straightforward to verify that C satisfies the properties in equation (2.29).

*Note that in our notation C, P and T we do not discern between the operations themselves and the symmetry of the theory. For example: T is violated means that the Lagrangian is not invariant under application of T. Some authors prefer to make this distinction, and will write something like "$\mathcal{T}$ is violated, meaning $\hat{T}\mathcal{L} \neq \mathcal{L}$" instead.

Majorana spinors are now defined as those fermions which are their own anti-particle:

$$\psi = \psi^{c}, \qquad \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \implies \chi = -\sigma_2 \phi^*, \tag{2.31}$$

we have thus precisely recovered equation (2.3) for the Majorana condition.

Using the anticommutation properties of $\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$ we can show that applying projection operators $\frac{1}{2}(1 \pm \gamma^5)$ to the charge conjugate,

$$P_{\mathrm{L}}\psi_{\mathrm{L}}^{\mathrm{c}} = 0, \quad P_{\mathrm{R}}\psi_{\mathrm{L}}^{\mathrm{c}} = \psi_{\mathrm{L}}^{\mathrm{c}}, \quad P_{\mathrm{L}}\psi_{\mathrm{R}}^{\mathrm{c}} = \psi_{\mathrm{R}}^{\mathrm{c}}, \quad P_{\mathrm{R}}\psi_{\mathrm{R}}^{\mathrm{c}} = 0; \tag{2.32}$$

that is, charge conjugation inverts the chirality. In other words yet, the anti-particle of a left-handed fermion is right-handed and vice versa.

For Majorana fermions, we can decompose both the fermion ψ and its anti-fermion ψ^{c} in left-handed and right-handed fields, then the Majorana condition gives

$$\psi = \psi_{\mathrm{L}} + \psi_{\mathrm{R}} = \psi_{\mathrm{L}}^{\mathrm{c}} + \psi_{\mathrm{R}}^{\mathrm{c}}. \tag{2.33}$$

Acting with the left-handed (right-handed) projection operator we find that $\psi_{\mathrm{L,R}} = \psi_{\mathrm{R,L}}^{\mathrm{c}}$. Therefore, we can write $\psi = \psi_{\mathrm{L}} + \psi_{\mathrm{L}}^{\mathrm{c}}$ and the mass term reads (up to constant prefactors)

$$m_{\mathrm{Dirac}}\left(\overline{\psi_{\mathrm{R}}}\psi_{\mathrm{L}} + \overline{\psi_{\mathrm{L}}}\psi_{\mathrm{R}}\right) \quad \to \quad m_{\mathrm{Majorana}}\left(\overline{\psi_{\mathrm{L}}^{\mathrm{c}}}\psi_{\mathrm{L}} + \overline{\psi_{\mathrm{L}}}\psi_{\mathrm{L}}^{\mathrm{c}}\right). \tag{2.34}$$

The Standard Model violates not only C and P symmetry separately, also the combination CP is not preserved [9]. However, the famous CPT-theorem states that CPT *is* a symmetry of the standard model*. Thus, if CP is violated in the Standard Model, it immediately follows that T – time reversal symmetry – is also violated.

2.4 The Higgs mechanism

Experimentally it has been measured [11] that the vector bosons $W_\mu^\pm$ and Z_μ are rather massive, while the photon A_μ is of course massless. The first attempt to describe this in the Standard Model might be to insert a term like

$$\frac{1}{2}m_Z^2 Z_\mu Z^\mu + \frac{1}{2}m_W^2 (W_\mu^+ (W^-)^\mu + W_\mu^- (W^+)^\mu). \tag{2.35}$$

Apart from introducing new parameters into the Lagrangian by hand though, such a term breaks gauge invariance and renormalisability. It turns out that there is a very elegant way to make gauge bosons *and* fermions massive, which is the *Higgs mechanism*.

We add to the Standard Model a new (complex) scalar field $\Phi = \begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}^{\mathrm{T}}$, with Lagrangian

$$\mathcal{L}_\Phi = -(\partial_\mu \Phi^\dagger)(\partial^\mu \Phi) - V(\Phi), \qquad \text{where } V(\Phi) = \mu^2(\Phi^\dagger\Phi) + \lambda(\Phi^\dagger\Phi)^2, \tag{2.36}$$

which is the most general globally $SU(2)\otimes U(1)$ invariant renormalisable Lagrangian one can write down. In order to bound the energy spectrum from below, we must have $\lambda > 0$; however μ^2 may be either positive or negative. We now *choose* $\mu^2 < 0$. The Lagrangian has a $SU(2)\otimes U(1)$ symmetry, which we make into a local one:

$$\mathcal{L}_\Phi \to -(\nabla_\mu\Phi^\dagger)(\nabla^\mu\Phi) - \mu^2(\Phi^\dagger\Phi) - \lambda(\Phi^\dagger\Phi)^2 - \frac{1}{2}\,\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) - \frac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{2.37}$$

*For a "deaxiomatised" treatment of the CPT theorem see [10], which also contains references to the original proofs of Pauli-Lüders and Jost.

where

$$\begin{cases} \nabla_\mu = \partial_\mu - \mathrm{i}gW^i_\mu t_a^{(\mathrm{fund})} - \mathrm{i}\frac{g'}{2}B_\mu Y, \\ G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - \mathrm{i}g[W_\mu, W_\nu], \\ F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \end{cases} \tag{2.38}$$

As such a scalar doublet can be parametrised (in analogy to writing a complex number z as $re^{\mathrm{i}\phi}$) as

$$\Phi(x) = \overline{U}(x) \begin{pmatrix} 0 \\ \rho(x)/\sqrt{2} \end{pmatrix}, \tag{2.39}$$

where $\overline{U}(x)$ is an $SU(2)$ matrix; we can fix the $SU(2)$ gauge by demanding that $\overline{U}(x) \equiv \mathrm{I}_2$ (the 2×2 identity matrix) everywhere. Because we chose $\mu^2 < 0$, the minimum of the potential is not at $\rho(x) = 0$, but at $\rho(x) = v \overset{\mathrm{def}}{=} \sqrt{-\mu^2/\lambda}$, which is called the *vacuum expectation value (vev)*. If we expand Φ around its minimum Φ_0 we can write $\rho(x) = v + h(x)$, where $h(x)$ is the *Higgs field*. Working out the covariant derivative, the Lagrangian reads

$$\begin{aligned} \mathcal{L}_\Phi = &-\tfrac{1}{2}(\partial_\mu h)^2 - \tfrac{1}{2}(2v^2\lambda)h^2 + \mathcal{O}(h^3, h^4) \\ &- \tfrac{1}{8}\frac{g^2v^2}{\cos^2(\theta_W)}Z_\mu Z^\mu - \tfrac{1}{4}g^2v^2W^+_\mu(W^-)^\mu + \mathcal{O}(hWW, hZZ, h^2W^2, h^2Z^2). \end{aligned} \tag{2.40}$$

In the first line, we see the kinetic term and the mass term for the Higgs field ($m_h = 2v^2\lambda$) plus its interactions with itself. On the second line, we notice that the non-zero vev provides the mass for the Z and W bosons, whereas the photon stays massless.

Now instead of inserting the fermion masses "by hand" into the theory, we only consider the fermionic Lagrangian $\mathrm{i}\overline{\psi}\partial\!\!\!/\psi$ (cf. equation (2.23)) without the mass term and we see how the fermions couple to the Higgs field $h(x)$. From symmetry considerations (any term in the Lagrangian must be an $SU(2)_{I_W}$ singlet with zero total hypercharge) we can write down the most general term:

$$\mathcal{L}_{\mathrm{Yukawa}} = \sum_{i,j=1}^{3} \left[f^{ij}_{\mathrm{u}}\overline{\Psi^i_{\mathrm{q,L}}}\tilde{\Phi}\psi^j_{\mathrm{u,R}} + f^{ij}_{\mathrm{d}}\overline{\Psi^i_{\mathrm{q,L}}}\Phi\psi^j_{\mathrm{d,R}} + f^{ij}_{\mathrm{e}}\overline{\Psi^i_{\mathrm{e,L}}}\Phi\psi^j_{\mathrm{e,R}} + \mathrm{h.c.} \right], \tag{2.41}$$

where $\tilde{\Phi} \overset{\mathrm{def}}{=} \mathrm{i}\sigma_2\Phi$ serves to pick out the right component from the doublet. The fermions couple to the the Higgs doublet through so-called *Yukawa coupling*. After the symmetry has been spontaneously broken (that is, Φ has acquired a non-zero vev v), we can again expand in $v + h(x)$ and the Lagrangian reads

$$\mathcal{L}_{\mathrm{Yukawa}} = \sum_{i,j=1}^{3} \left[\overline{\tilde{\psi}^i_{\mathrm{u,L}}}m^{ij}_u\psi^j_{\mathrm{u,R}} + \overline{\psi^i_{\mathrm{d,L}}}m^{ij}_d\psi^j_{\mathrm{d,R}} + \overline{\psi^i_{\mathrm{e,L}}}m^{ij}_e\psi^j_{\mathrm{e,R}} + \mathrm{h.c.} \right], \tag{2.42}$$

or, in the less cumbersome notation of equation (2.22),

$$\mathcal{L}_{\mathrm{Yukawa}} = \sum_{i,j=1}^{3} \left[u^i_{\mathrm{L}}m^{ij}_u u^j_{\mathrm{R}} + d^i_{\mathrm{L}}m^{ij}_d d^j_{\mathrm{R}} + \ell^i_{\mathrm{L}}m^{ij}_\ell \ell^j_{\mathrm{R}} + \mathrm{h.c.} \right]. \tag{2.43}$$

The matrices $m^{ij}_\alpha \overset{\mathrm{def}}{=} \frac{1}{\sqrt{2}}vf^{ij}_\alpha$ are mass matrices. Unfortunately, in this formulation the mass matrices are not necessarily (and in general, will not be) diagonal which does not allow us to define *the* mass of the left-handed up-quark field, for instance. On the other hand, the up quark field is decoupled from, say, the strange quark field; the fields occuring in the above expression are therefore called *flavour eigenstates*. We can define new mass matrices

$$m'_\alpha = (S_{\alpha,\mathrm{L}})^\dagger \cdot m_\alpha \cdot S_{\alpha,\mathrm{R}} \tag{2.44}$$

for $\alpha = u, d, e$, where where $S_{\alpha,\chi}$ are unitary matrices for both chiralities $\chi = \mathrm{L}, \mathrm{R}$. It turns out that these matrices can be chosen, such that the new mass matrices m'_α are diagonal. If we then also define

$$\psi'^{i}_{\alpha,\chi} \stackrel{\mathrm{def}}{=} S_{\alpha,\chi} \cdot \psi_{\alpha,\chi}, \qquad \text{where } \alpha = u, d, e \text{ and } \chi = \mathrm{L}, \mathrm{R} \tag{2.45}$$

meaning

$$\ell'^{i}_{\mathrm{L}} = \sum_{j=1}^{3} S^{ij}_{\ell,\mathrm{L}} \ell^{j}_{\mathrm{L}}, \qquad u'^{i}_{\mathrm{L}} = \sum_{j=1}^{3} S^{ij}_{\mathrm{u,L}} u^{j}_{\mathrm{L}}, \qquad d'^{i}_{\mathrm{R}} = \sum_{j=1}^{3} S^{ij}_{\mathrm{d,R}} u^{j}_{\mathrm{R}}, \qquad \text{etc.} \tag{2.46}$$

we get "proper" mass terms, which only couple left-handed and right-handed components of the *same* fields:

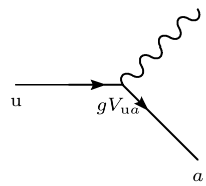
$$\mathcal{L}_{\mathrm{Yukawa}} = \sum_{i,j=1}^{3} \left[\overline{u'^{i}_{\mathrm{L}}} m'^{ij}_{u} u'^{j}_{\mathrm{R}} + \overline{d'^{i}_{\mathrm{L}}} m'^{ij}_{d} d'^{j}_{\mathrm{R}} + \overline{\ell'^{i}_{\mathrm{L}}} m'^{ij}_{\ell} \ell'^{j}_{\mathrm{R}} + \text{h.c.} \right]. \tag{2.47}$$

The primed fields are called the *mass eigenstates*.

The complete SM Lagrangian is then the sum of the Lagrangians in equations (2.26), (2.40) and (2.47). We have ensured diagonal mass matrices, at the expense of flavour mixing: in the former two expressions, terms like

$$\overline{\Psi_{\mathrm{u,L}}} \gamma^\mu \Psi_{\mathrm{d,L}} = \overline{\Psi'^{i}_{\mathrm{u,L}}} \gamma^\mu \left[(S_{\mathrm{u,L}})^\dagger S_{\mathrm{d,L}} \right]^{ij} \Psi'^{j}_{\mathrm{d,L}} \tag{2.48}$$

appear, leading to flavour changing interactions



(2.49)

where $a = \mathrm{d}$, s or b. The quark mixing matrix

$$V^{ij} \stackrel{\mathrm{def}}{=} \left[(S_{\mathrm{u,L}})^\dagger S_{\mathrm{d,L}} \right]^{ij} = \begin{pmatrix} V_{\mathrm{ud}} & V_{\mathrm{us}} & V_{\mathrm{ub}} \\ V_{\mathrm{cd}} & V_{\mathrm{cs}} & V_{\mathrm{cb}} \\ V_{\mathrm{td}} & V_{\mathrm{ts}} & V_{\mathrm{tb}} \end{pmatrix} \tag{2.50}$$

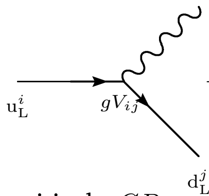
is known as the *Cabbibo-Kobayashi-Maskawa (CKM) matrix*. The latter two received the 2008 Nobel prize in physics "for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature" [12]. As an $n \times n$ matrix, it has in principle n^2 complex entries which means it has $2n^2$ real degrees of freedom. However the unitarity condition $\sum_j V^{ij} V^{jk} = 1$ imposes n^2 real constraints. From the previous discussion it is clear that the quark fields u^i and d^i are invariant under a global rephasing ($\psi \to e^{\mathrm{i}\delta}\psi$), and for n generations we have $2n$ of these. By simultaneously rephasing all of them nothing changes, so one of these is trivial (e.g. we can always make u_1 real by simultaneously rotating all the quark fields), so we can apply $2n-1$ phase transformations to get rid of complex phases in the CKM matrix. Of the remaining $n^2-(2n-1)=(n-1)^2$ real parameters, there are $\frac{1}{2}n(n-1)$ Euler angles which describe a rotation between the flavours, analogously to an ordinary $O(n)$ matrix, the last $(n-1)^2-\frac{1}{2}n(n-1)=\frac{1}{2}(n-1)(n-2)$ parameters are phases which cannot be removed and can lead to CP violation.

Applying the general counting argument to the Standard Model case $n=3$, we see that we need three real angles θ_1, θ_2, θ_3 and a complex phase δ to parametrise the CKM matrix. It is commonly written as

$$V^{ij}=\begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2s_3-s_2s_3e^{\mathrm{i}\delta} & c_1c_2s_3+s_2c_3e^{\mathrm{i}\delta} \\ s_1s_2 & c_1s_2c_3+c_2s_3e^{\mathrm{i}\delta} & c_1s_2s_3-c_2c_3e^{\mathrm{i}\delta} \end{pmatrix}, \tag{2.51}$$

where c_i and s_i are $\cos(\theta_i)$ and $\sin(\theta_i)$ respectively. These parameters have been measured to agree with the Standard Model predictions very well [13].

To show how the complex factor $e^{\mathrm{i}\delta}$ can lead to CP-violation, we look back at equation (2.48) and consider a typical flavour-changing interaction



$$+\ \text{h.c.} \propto V_{ij}\overline{u^i_\mathrm{L}}\gamma^\mu d^j_\mathrm{L}W^+_\mu + V^*_{ij}\overline{d^j_\mathrm{L}}\gamma^\mu u^i_\mathrm{L}W^-_\mu \stackrel{\text{def}}{=} \Gamma \tag{2.52}$$

Now intuitively, CP-conjugation means replacing all the particles by their respective antiparticles (C) and then flipping their chirality (P). However, nothing happens to the coupling constants, which are just numbers. Hence, the CP conjugated process of the above interaction reads

$$\Gamma' = V_{ij}\overline{d^j_\mathrm{L}}\gamma^\mu u^i_\mathrm{L}W^-_\mu + V^*_{ij}\overline{u^i_\mathrm{L}}\gamma^\mu d^j_\mathrm{L}W^+_\mu. \tag{2.53}$$

The fermion fields have changed positions with respect to the numerical prefactors, whence $\Gamma=\Gamma'$ only if $V_{ij}=V^*_{ij}$. If we take for example $i=2$, $j=3$ (charm – strange), the relevant CKM matrix element is $V_{23}=c_1c_2s_3-s_2s_3e^{\mathrm{i}\delta}$ which is obviously potentially non-real.

Chapter 3

Extending the Standard Model: Massive neutrinos

In the last chapter we have seen that in the Standard Model, neutrinos are assumed to be massless. However, several experiments have provided definite evidence of the existence of a neutrino mass. After an overview of these experiments, we will describe the possibile extensions of the Standard Model in which neutrinos become massive and go deeper into some of them in particular, namely the class of seesaw models.

3.1 Neutrino oscillations

3.1.1 Experiment

As explained in the previous chapter, at the time the Standard Model was constructed there was no hard evidence for the existence of a neutrino mass and all observed neutrinos were left-handed, so the Standard Model was constructed to have only left-handed, massless neutrinos. However, the predictions as made by the minimal Standard Model do not agree with experiment. As early (or late, considering the previous and first experiment was 10 years earlier) as 1968 astrophysicists Davis and Bahcall set up an experiment to measure the neutrino flux from the sun, the famous Homestake experiment [14]. This experiment was based on ideas worked out by Pontecorvo and Alvarez in the years 1946 – 1949 [15]. According to the *Standard Solar Model (SSM)*, neutrinos are produced in the fusion reaction

$$4\mathrm{p}^+ + 2\mathrm{e}^- \to {}^4\mathrm{He} + 2\nu_\mathrm{e}, \tag{3.1}$$

inside the sun. Because neutrinos hardly interact with matter, they escape the core and they can be measured by indicidental reactions that take place in a reactor on earth. The Homestake project made use of the reaction

$$\nu_\mathrm{e} + {}^{37}\mathrm{Cl} \to {}^{35}\mathrm{Ar} + \mathrm{e}^-, \tag{3.2}$$

by filling a tank with perchloroethylene solution and counting the number of argon atoms. However, Davis only measured about 1/3 of the expected flux [16]. Later experiments, which were also sensitive to muon and tau neutrinos, measured that the total flux did indeed agree with Bahcalls prediction. After this experiment, which has run for some five years, multiple experiments [17, 18, 19] have confirmed a "disappearance" of electron neutrinos, compensated by other flavours so that the total flux matches theoretical predictions.

A similar discrepancy occurred in measuring the number of neutrinos produced in atmospheric decay. They originate from decaying pions in the atmosphere, which are in turn reaction products of high energetic cosmic rays with nuclei in the atmosphere. These pions decay according to $\pi^+ \to \mu^+ + \nu_\mu$ and $\pi^- \to \mu^- + \overline{\nu_\mu}$, after which the muons decay by $\mu^- \to e^- + \nu^\mu + \overline{\nu_e}$ and $\mu^+ \to e^+ + \overline{\nu_\mu} + \nu_e$. Therefore, a measurement should show roughly twice as much ν_μ as ν_e. However, the observed ratio is only about 65% of the expected value [20]. Here the breakthrough only occurred in 1998, when the Super-Kamiokande collaboration [21] showed that there definitely is an asymmetry between upwards (taking a long way through the earth) and downwards (directly from the atmosphere) moving high-energy neutrinos, which contradicts an assumption that nothing happens to the neutrinos between their creation in the atmosphere and detection.

For a more complete overview of neutrino (mostly oscillation) experiments see [22, § 4], [23], [24], [25] and the rather extensive review [26] (more or less in chronological order). Also a very recent overview is given in [27].

3.1.2 Theory of three-flavour neutrino mixing

The best (and, as it seems [21], only) explanation for the observed flavour conversions is provided by neutrino oscillations. To explain this, we refer back to the discussion in section 2.4 on mass and flavour eigenstates. The flavour changing of for example quarks, is well established. When neutrinos are no longer considered to be massless, something similar happens: the mass eigenstates ν_i $(i = 1, 2, 3)$ differ from the flavour eigenstates ν_α $(\alpha = \mathrm{e}, \mu, \tau)$ (note that $\vec{x}$-dependence is implied and we are suppressing all other quantum numbers such as helicity and spin, which we assume fixed). Each flavour eigenstate is a superposition of mass eigenstates, but the precise composition of mass states is dynamic, because the propagation frequency depends on the mass. Experimentally only the neutrino flavour is measured, whence the amplitude to measure, say, an electron neutrino, depends on time. The mass and flavour fields are related through a 3×3 unitary matrix (similar to the CKM matrix from section 2.4):

$$\nu_\alpha = \sum_{i=1}^{3} U_{\alpha i} \nu_i. \tag{3.3}$$

The matrix U is called the *PMNS matrix* — after Pontecorvo, Maki, Nakagawa and Sakata — or sometimes by a slight abuse of language the neutrino mixing matrix. It can be parametrised in many different ways, one of the most customary is

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\phi_1} & 0 & 0 \\ 0 & e^{\mathrm{i}\phi_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.4a}$$

where c_{ij} and s_{ij} denote $\cos(\theta_{ij})$ and $\sin(\theta_{ij})$ respectively. There are three mixing angles θ_{12}, θ_{13} and θ_{23}, in the literature also denoted by θ_{sun}, θ_{atm} and θ_{CHOOZ} for reasons that will be explained later in section 3.3. There are three phase angles δ,

ϕ_1 and ϕ_2. The Dirac phase δ is connected to CP-violation, as we will show at the end of this section. The Majorana phases ϕ_i $(i = 1, 2)$ are connected to the question whether neutrinos are Dirac or Majorana particles; as only experiment can decide this we shall address this subject in section 3.3.2. The first factor is similar in form to the CKM matrix parametrisation (2.51), though it can also be expressed as a product of three rotation matrices (the second one of which is complex) [28]:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-\mathrm{i}\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{\mathrm{i}\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.4b}$$

Suppose now that at time $t = 0$ and position $x = 0$ a neutrino is created with flavour α. Our next goal is to derive the probability $P(\nu_\alpha \to \nu_\beta)$ of measuring a neutrino of flavour β at time t and distance x from the creation point, where we assume that the neutrino propagates essentially in one dimension. We will assume that the state describing a neutrino with flavour α is given by*

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} U^*_{\alpha i} |\nu_i\rangle , \tag{3.5}$$

where U is the same PMNS matrix. In fact, this corresponds to the approximation in which neutrinos are ultra-relativistic, that is, $m \ll E$. Using a loose upper-bound of $m \lesssim 1$ eV for the neutrino mass, and considering that all current and (near) future neutrino oscillation experiments are sensitive only to neutrino energies of $E \gg$ keV or even several MeV, this approximation is definitely justified.

Since from basic quantum mechanics

$$P(\nu_\alpha \to \nu_\beta) = |\langle \nu_\alpha \,|\, \nu_\beta(t, x)\rangle|^2 , \tag{3.6}$$

where $|\nu_\alpha\rangle \equiv |\nu_\alpha(t = x = 0)\rangle$, we have reduced our problem to calculating the time-evoluted state $|\nu_\beta(t, x)\rangle$. As is usually done in the literature, we consider a mass eigenstate $|\nu_k(t, x)\rangle$ with a well-defined momentum p_k and energy E_k which we can write as [2, § 8.1]

$$|\nu_k(t, x)\rangle = e^{-\mathrm{i}E_k t + \mathrm{i}p_k x} |\nu_k\rangle . \tag{3.7}$$

The ket $|\nu_k\rangle$ denotes the initial state; it is clear that this is a solution to both the Schrödinger evolution equation (the time dependence of the stationary states is simply given by $|\nu_k(p, t)\rangle = e^{-\mathrm{i}Et} |\nu_k(p, 0)\rangle$) and the momentum eigenvalue equation $\hat{p} |\nu_k(t, x)\rangle = p_k |\nu_k(t, x)\rangle$ (the neutrinos are emitted as plane waves $|\nu_k(x, 0)\rangle = e^{\mathrm{i}px} |\nu_k\rangle$) [29]. Writing the flavour eigenstate as a superposition of mass states, we find

$$|\nu_\alpha(t, x)\rangle = \sum_{k=1}^{3} U^*_{\alpha k} e^{-\mathrm{i}E_k t + \mathrm{i}p_k x} |\nu_k\rangle . \tag{3.8}$$

We can re-express this in terms of flavour states by inverting equation (3.5):

$$|\nu_\alpha(t, x)\rangle = \sum_{k=1}^{3} U^*_{\alpha k} e^{-\mathrm{i}E_k t + \mathrm{i}p_k x} \left(\sum_\beta U_{\beta k} |\nu_\beta\rangle \right) , \tag{3.9}$$

*Usually it is assumed that the neutrinos are produced through a process like $A \to B + \ell^+ + \nu_\alpha$, where ℓ is a lepton and α indicates the flavour, and detected through $\nu_\alpha + A \to B + \ell^-_\alpha$. Then we can write $|\nu_\alpha\rangle = \sum_{k=1}^{3} \mathscr{A}_{\alpha,k} |\nu_k, \ell^+, B\rangle$ with $\mathscr{A}_{\alpha,k}$ some amplitude. In [23, § 3] it is shown, that if the dependence of the neutrino interaction rate is insensitive to (differences of) the neutrino masses, then $\mathscr{A}_{\alpha,k}$ reduces to the PMNS-matrix U, up to some irrelevant phase factor. A similar argument holds for the detection process. Therefore, the flavour neutrino state of the production process and the detection process are both given by (3.5).

where β runs over the flavours (e, μ, τ). The probability we are interested in now follows by orthogonality of the flavour states ($\langle \nu_\alpha \,|\, \nu_\beta \rangle = \delta_{\alpha\beta}$):

$$P(\nu_\alpha \to \nu_\beta; t, x) = \left| \sum_{k=1}^{3} U_{\alpha k}^* e^{-\mathrm{i}E_k t + \mathrm{i}p_k x} U_{\beta k} \right|^2 . \tag{3.10}$$

Because we assume that the neutrinos are ultra-relativistic, we can expand the expression for the energy to the first contribution in the masses:

$$E_k = \sqrt{p_k^2 + m_k^2} \simeq p_k + \frac{m_k^2}{2p_k} + \mathcal{O}(m_k^4). \tag{3.11}$$

Since energy must be conserved in the neutrino creation process, and the neutrinos must satisfy $E_k^2 = p_k^2 + m_k^2$ it easily follows from dimensional considerations that we can write

$$E_k \simeq E + \xi \frac{m_k^2}{2E}, \qquad p_k \simeq E - (1-\xi)\frac{m_k^2}{2E}, \tag{3.12}$$

for some dimensionless parameter ξ. The numerical value of ξ depends on the details of the production process, for example: if the neutrino is created by pion decay $\pi^+ \to \mu^+ + \nu_\mu$, then $\xi = (1 + m_\mu^2/m_\pi^2)/2 \approx 0.8$. To simplify the calculations, we can assume that that all mass states carry the same momentum, $p_k \simeq E/c$, such that we can write $E_k \simeq E + \frac{m_k^2}{2E}$. This assumption is called the equal momentum assumption and corresponds to $\xi = 1$. A similar way of proceeding, is to assume that all mass states have the same energy $E_1 = E_2 = \cdots = E$, corresponding to $\xi = 0$. We will keep the parameter ξ unspecified for now. First we calculate

$$E_i - E_j = \xi \frac{\Delta m_{ij}^2}{2E}, \qquad p_i - p_j = -(1-\xi)\frac{\Delta m_{ij}^2}{2E}, \tag{3.13}$$

where we have introduced $\Delta m_{ij}^2 \overset{\text{def}}{=} m_i^2 - m_j^2$. Now it is possible to expand the modulus in equation (3.10):

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= \sum_{i=1}^{3}\sum_{j=1}^{3} U_{\alpha i}^* U_{\alpha j} U_{\beta i} U_{\beta j}^* e^{-\mathrm{i}(E_i - E_j)t + \mathrm{i}(p_i - p_j)x} \\ &= \sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2 + 2\,\mathrm{Re} \sum_{i>j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* e^{-\mathrm{i}\xi \Delta m_{ij}^2 t/2E - \mathrm{i}(1-\xi)\Delta m_{ij}^2 x/2E}. \end{aligned} \tag{3.14}$$

The last line follows from splitting the terms with $i = j$ and using that $z + z^* = 2\,\mathrm{Re}\, z$ for $z \in \mathbb{C}$. Note that there is still a dependence on the propagation time t, which is usually unknown in an experiment. Because the neutrinos are ultra-relativistic we can assume that they travel at the speed of light, which means that we will only measure it at points on (or negligibly close to) the light-cone, $t = x = L$, where L is the distance between the creation and the detection point*. Then the dependence of

* The approximation $t = x = L$ is actually a physical statement about a *particle*, whereas we are describing the neutrino as a plane wave. Of course a non-simplified derivation would have to describe the neutrino as a wave packet (superposition of such plane waves). For a rigorous derivation, implementing a proper time averaging of the relevant matrix density element and assuming the propagation velocity v to be the speed of light, see [30]. The same author has also shown in [31,32] that the effects of possible deviations from $v = c = 1$ are negligible.

the probability amplitude on the parameter ξ cancels from equation (3.14), and we obtain the literature expression

$$P(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2 + 2\,\mathrm{Re} \sum_{i>j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* e^{-\mathrm{i}\Delta m_{ij}^2 L/2E}. \tag{3.15}$$

A crucial observation is that the only mass dependence is through the mass difference Δm_{ij}^2 and not the m_i separately. The only other quantities entering the expression are the elements of the PMNS matrix and the experiment-dependent neutrino energy E and propagation distance L. Secondly, the probability amplitude for a neutrino created with flavour α to be detected with flavour β can be split into a constant part which *only* depends on the PMNS matrix entries, and some oscillating term. In cases where the oscillating term is not known well enough, the oscillations can be averaged out and only the constant first term can be measured.

We are now also in a position to understand why the Dirac phase δ in the PMNS matrix gives rise to CP-violating processes. In order to make this discussion completely explicit, let us consider the probability amplitude of $\mathcal{P} \stackrel{\text{def}}{=} P(\nu_\mathrm{e} \to \nu_\mu)$ of an electron flavour-neutrino going into a muon flavour-neutrino, and compare it to the probability $\overline{\mathcal{P}} \stackrel{\text{def}}{=} P(\overline{\nu_\mathrm{e}} \to \overline{\nu_\mu})$ of the charge conjugated process. As we have indicated in section 2.3, such probabilities are invariant under CPT-symmetry, so instead of looking at the charge-parity conjugated process we can also consider the time reversed process: $\overline{\mathcal{P}} = P(\nu_\mu \to \nu_\mathrm{e})$. In principle, we can now work out $\Delta\mathcal{P} = \mathcal{P} - \overline{\mathcal{P}}$ from equation (3.15). If CP is conserved, then $\Delta\mathcal{P}$ should vanish. After some extremely ugly although straightforward algebra, we find this expression to be equal to

$$\Delta\mathcal{P} = 16 \cos\theta_{12} \sin\theta_{12} \cos\theta_{23} \sin\theta_{23} \cos^2\theta_{13} \sin(\delta) \times \\ \sin\left(\frac{\Delta m_{12}^2 L}{4E}\right) \sin\left(\frac{\Delta m_{13}^2 L}{4E}\right) \sin\left(\frac{\Delta m_{23}^2 L}{4E}\right). \tag{3.16}$$

Due to the prominent $\sin(\delta)$ term, this expression will vanish if and only if $\delta = 0$ or $\delta = \pi$. The other two non-removable phases ϕ_i $(i = 1, 2)$ are absent in this expression, because they can be factored out in a diagonal (hence commuting) matrix, cf. equation (3.4a). Therefore the values of ϕ_1, ϕ_2 cannot be determined by any oscillatory experiment, as such an experiment necessarily measures the amplitudes from equation (3.15) which do not depend on these phases.

3.1.3 Two-flavour limit

To gain some intuition on the neutrino oscillation probabilities derived above, let us study the two-dimensional case in some detail. In this case, only one angle θ suffices to parametrise the mixing matrix,

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}; \tag{3.17}$$

any phases can be absorbed in redefinitions of the fields (this implies that in the two-flavour case, no CP-violation can take place). To find all the probability amplitudes, it suffices to calculate only one of them, for example, $P(\nu_\mathrm{e} \to \nu_\mathrm{e})$. The rest then follows, because $P(\nu_\mathrm{e} \to \nu_\mu) = 1 - P(\nu_\mathrm{e} \to \nu_\mathrm{e})$ (by complementarity: we only have two flavours), $P(\nu_\mu \to \nu_\mathrm{e}) = P(\nu_\mathrm{e} \to \nu_\mu)$ (by the CPT-theorem and the fact that two-flavour oscillations do not break CP-invariance, because of the absense of complex phases) and finally, $P(\nu_\mu \to \nu_\mu)$ (again, this follows from complementarity). The calculation can be easily done by hand, starting from the previous result (3.15):

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= |U_{e1}|^2|U_{\mu 1}|^2 + |U_{e2}|^2|U_{\mu 2}|^2 + 2\,\mathrm{Re}\left(U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}e^{-i\Delta m^2 L/2E}\right) \\ &= 2\sin^2\theta\cos^2\theta + 2\,\mathrm{Re}\left(-\sin^2\theta\cos^2\theta e^{-i\Delta m^2 L/2E}\right) \\ &= 2\sin^2\theta\cos^2\theta\left(1 - \cos\left(\Delta m^2 L/2E\right)\right) \\ &= \frac{1}{2}\sin^2(2\theta)\times 2\sin^2\left(\frac{1}{2}\Delta m^2 L/2E\right) = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2 L}{4E}\right), \end{aligned} \tag{3.18}$$

with $\Delta m^2 = |m_e^2 - m_\mu^2|$. On the last line the geometric identities $4\cos^2\theta\sin^2\theta = \sin^2(2\theta)$ and $1-\cos(2x) = 2\sin^2(x)$ have been used.

The result we then find is

$$\begin{aligned} P(\nu_e \to \nu_e) &= \mathcal{A}\mathcal{A}^* = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2 L}{4E}\right), \\ P(\nu_e \to \nu_\mu) &= 1 - P(\nu_e \to \nu_e). \end{aligned} \tag{3.19}$$

Apart from looking much simpler than the three-flavour expression (3.15), this expression has some historical value. Although the existence of the tau neutrino was suspected soon after the detection of the tau lepton in 1976, the first definite observation of ν_τ was not announced until 2000 by FermiLab [33]. Therefore, only two types of neutrinos have long been known and as such, the two flavour model has been the effective model of the theory in the three flavour case even before the more general three-flavour mixing was discovered. In fact it is still a good approximation for experiments in which one type of mixing is dominant, such as solar experiments (in which $\nu_e \leftrightarrow \nu_\tau$ hardly occurs because of the small mixing angle θ_{13}, see table 3.1 on page 28).

The probabilities for the two-flavour case have been plotted in figure 3.1. In the case of maximal mixing ($\theta = 45°$, solid black curve) we indeed observe that, if the neutrino starts out in the electron flavour eigenstate, some time (equivalently, distance) later, we have a 100% probability to detect it in the muon eigenstate, etc. For smaller mixing angles, the probability to find the original flavour will still hit 100% from time to time, however the mixing is clearly non-maximal (there is no definite chance of measuring another flavour).

3.1.4 Three neutrino mixing in matter

The theory presented in the previous section was derived for neutrinos propagating in a vacuum. Already in 1978 it was pointed out by Wolfenstein, that due to forward scattering of neutrinos off leptons, flavour changing effects – if present – could be enhanced or even only possible in matter. In the presence of such effects, even massless neutrinos could experience such an enhancement of the flavour changing probability. In 1984, Smirnov and Mikheev noticed, that for specific oscillation paramters and matter properties, a resonance could occur in which a small vacuum mixing angle becomes effectively very large in matter. This resonance effect has been named the *MSW effect* after these three people. Bethe [34] has shown in 1990 that these matter effects are mainly responsible for the discrepancy between Davis' result and Bahcalls prediction in the Homestake experiment.

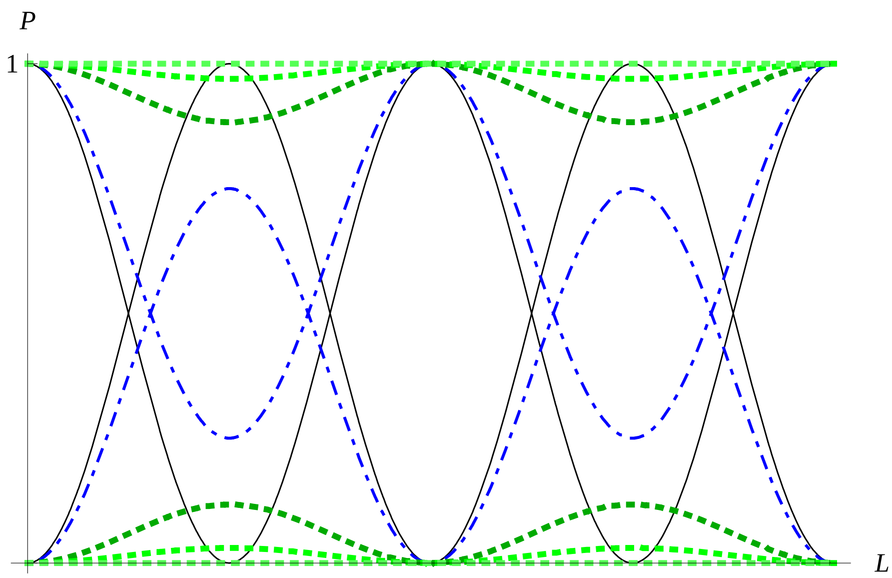


Figure 3.1: The probabilities $P(\nu_\mathrm{e} \to \nu_\mathrm{e})$ (top curve) and $P(\nu_\mathrm{e} \to \nu_\mu)$ (bottom curve) for a neutrino that starts out in the electron neutrino eigenstate, for mixing angles $\theta = 45°$ (solid black curve), $\theta = 30°$ (dash-dotted blue curve) and the expected limiting behaviour for $\theta = 10°$, $\theta = 5°$ and $\theta = 0$ in the darkest to lightest green dotted curves, respectively.

We attempt to point out the important steps only, a detailed calculation is given in [22, § 3.2].

As discussed section 2.2 on fermion-gauge interactions in the Standard Model, we explained the occurrence of so-called *charged current interactions*, in which a charged (W-)boson is exchanged, and *neutral current interactions*, where the exchanged boson is neutral (Z-boson). The interactions are shown in the diagrams of figure 3.2. Due to the electric charge conservation law, charged-current interactions are only possible when the neutrinos are of electron flavour. For simplicity, we will again consider a neutrino flavour state with a well-defined momentum

$$|\nu_\alpha(p)\rangle = \sum_{k=1}^{3} U^*_{\alpha,k} |\nu_k(p)\rangle \,, \tag{3.20}$$

a rigorous treatment requires a wave-packet description, as in [35]. We are making the same assumptions as in the previous section, in particular we assume the PMNS-matrix U is the matrix describing the mixing of flavour states. We write for the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_\mathrm{I}$, where $\mathcal{H}_0$ and $\mathcal{H}_\mathrm{I}$ are the free (vacuum) Hamiltonian which was implicit in the previous section, and a part containing the scattering, respectively. Of course, massive neutrino states with a well-defined momentum are eigenstates of the non-interacting part, $\mathcal{H}_0 |\nu_k(p,t)\rangle = E_k |\nu_k(p,t)\rangle$ where $E_k^2 = p^2 + m_k^2$. The interactions are felt by the flavour states though, $\mathcal{H}_\mathrm{I} |\nu_\alpha(p)\rangle = V_\alpha |\nu_\alpha(p)\rangle$. The potential can be written as $V_\alpha = V_\mathrm{c}\delta_{\alpha\mathrm{e}} + V_\mathrm{n}$, with $V_\mathrm{c} = \sqrt{2}G_\mathrm{F}N_\mathrm{e}$ the charged-current interaction potential (which is only felt by the electron flavours) and $V_\mathrm{n} = -(1/\sqrt{2})G_\mathrm{F}N_\mathrm{n}$ the neutral-current interaction potential. Here, G_F is the Fermi constant and $N_\mathrm{e,n}$ denotes the electron and neutron number densities of the medium. We do not wish to go into the details of this potential, they are given in [36]. Rather we just point out the sim-

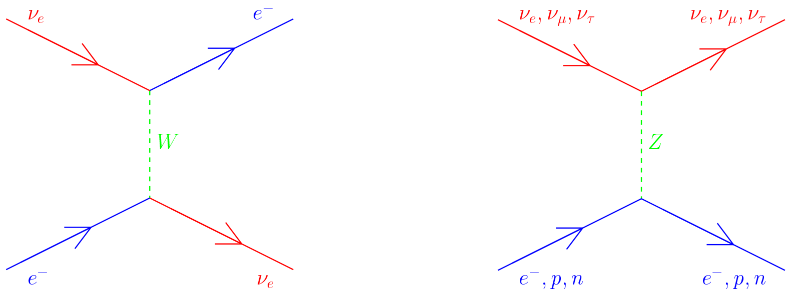


Figure 3.2: Feynman diagrams of the coherent forward elastic scattering processes that generate the charged current potential V_c through W exchange and the neutral current potential V_n through Z exchange; taken from [22].

ple form and the fact that the neutral current interaction has equal influence on all three flavours. Finally we mention that for anti-neutrinos all signs are reversed. The quantum-mechanical states $|\nu_\alpha(p)\rangle$ satisfy the Schrödinger equation, so if the state is initially purely α-flavoured,

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\,|\nu_\alpha(p,t)\rangle = \mathcal{H}\,|\nu_\alpha(p,t)\rangle\,, \qquad \text{where } |\nu_\alpha(p,0)\rangle = |\nu_\alpha(p)\rangle\,. \tag{3.21}$$

As in the previous section, we consider

$$\psi_{\alpha\beta}(p,t) \stackrel{\text{def}}{=} \langle \nu_\beta(p)\,|\,\nu_\alpha(p,t)\rangle\,, \tag{3.22}$$

with the initial condition $\psi_{\alpha\beta}(p,0) = \delta_{\alpha\beta}$. If we close the Schrödinger equation above on the left by an initial state ket, we can use the identities

$$|\nu_\alpha(p,t)\rangle = \sum_k U_{\alpha k}\,|\nu_k(p,t)\rangle\,, \qquad |\nu_k(p,t)\rangle = \sum_\alpha U^*_{\alpha k}\,|\nu_\alpha(p,t)\rangle \tag{3.23}$$

to derive a differential equation for the amplitude $\psi_{\alpha\beta}$. From the calculation

$$\begin{aligned}
\langle \nu_\alpha\,|\,\mathcal{H}\,|\,\nu_\beta(p,t)\rangle &= \langle \nu_\alpha|\,\mathcal{H}_0\left(\sum_k U_{\beta k}\,|\nu_k(p,t)\rangle\right) + \langle \nu_\alpha\,|\,\mathcal{H}_\mathrm{I}\,|\,\nu_\beta(p,t)\rangle \\
&= \sum_k \langle \nu_\alpha|\,U_{\beta k}E_k\left(\sum_\rho U^*_{\rho k}\,|\nu_\rho(p,t)\rangle\right) + \langle \nu_\alpha\,|\,V_\beta\,|\,\nu_\beta(p,t)\rangle \\
&= \sum_\rho \left\langle \nu_\alpha \,\middle|\, \sum_k U_{\beta k}E_k U^*_{\rho k} + V_\beta\delta_{\beta\rho} \,\middle|\, \nu_\rho(p,t)\right\rangle
\end{aligned} \tag{3.24}$$

we thus find

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi_{\alpha\beta}(p,t) = \sum_\rho\left(\sum_k U_{\beta k}E_k U^*_{\rho k} + \delta_{\beta\rho}V_\beta\right)\psi_{\alpha\rho}(p,t). \tag{3.25}$$

Applying the ultra-relativistic approximation ($t = x$) we find that $P(\nu_\alpha \to \nu_\beta) = |\psi_{\alpha\beta}(x)|^2$, where $\psi_{\alpha\beta}(x)$ satisfies

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\alpha\beta}(x) = \sum_\rho\left(\sum_k U_{\beta k}\frac{\Delta m^2_{k1}}{2E}U^*_{\rho k} + \delta_{\beta\mathrm{e}}\delta_{\rho\mathrm{e}}V_\mathrm{c}\right)\psi_{\alpha\rho}(x). \tag{3.26}$$

In the last step, we have redefined

$$\psi_{\alpha\beta}(x) \to \psi_{\alpha\beta(x)} \exp\left[-\mathrm{i}\left(p + \frac{m_1^2}{2E}\right)x - \mathrm{i}\int_0^x V_\mathrm{n}(x')\,\mathrm{d}x'\right] \tag{3.27}$$

to absorb a common phase factor. Since we are interested in the probability amplitudes $|\psi_{\alpha\beta}|^2$ only, the neutral-current interactions are therefore irrelevant for this theory.

Finally, we can rewrite the evolution equation to a matrix vector equation for $\Psi_\alpha(x) = (\psi_{\alpha\mathrm{e}}(x), \psi_{\alpha\mu}(x), \psi_{\alpha\tau}(x))$:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\Psi_\alpha(x) = \frac{1}{2E}(U\Delta U^\dagger + A)\Psi_\alpha, \tag{3.28}$$

where

$$\Delta = \mathrm{diag}(0, \Delta m_{21}^2, \Delta m_{31}^2), A = \mathrm{diag}(2\sqrt{2}G_\mathrm{F}N_\mathrm{e} = 2EV_\mathrm{c}, 0, 0). \tag{3.29}$$

The solution, from a basic theory of differential equations, can be formally written as

$$\Psi_\alpha(x) = \exp\left[-\frac{\mathrm{i}}{2E}(U\Delta U^\dagger + A)x\right], \tag{3.30}$$

where the exponent of a matrix M is defined through its power series, $e^M = \sum M^n/n!$.

For the rest of this section, we will only consider the two-dimensional case. This is a good approximation in special cases, for example when the vacuum mixing angle is small or the mass differences are well separated. In the three flavour case, calculations become less tractable, the theory is treated (partially analytically and then applied numerically to solar neutrino mixing) by Kuo and Pantaleone [37].

In our two-flavour approximation, the evolution equation can be written as

$$\mathrm{i}\frac{\mathrm{d}\Psi}{\mathrm{d}x} = \frac{1}{4E}M^2\Psi, \tag{3.31}$$

where the vector $\Psi = (\psi_\mathrm{ee}, \psi_{\mathrm{e}\mu})$ contains all the independent transition probabilities and

$$M = \begin{pmatrix} -\Delta m^2\cos(2\theta) + 4EV_\mathrm{c} & \Delta m^2\sin(2\theta) \\ \Delta m^2\sin(2\theta) & \Delta m^2\cos(2\theta) \end{pmatrix}, \tag{3.32}$$

contains a vacuum oscillation term and a term from the forward electron flavour scattering. This can be readily seen by writing it as

$$M = U\begin{pmatrix} -\Delta m^2 & 0 \\ 0 & \Delta m^2 \end{pmatrix}U^\dagger + 2E\begin{pmatrix} \sqrt{2}G_\mathrm{F}N_\mathrm{e} & 0 \\ 0 & -\sqrt{2}G_\mathrm{F}N_\mathrm{e} \end{pmatrix}. \tag{3.33}$$

where we have subtracted a multiple $V_\mathrm{c} = \sqrt{2}G_\mathrm{F}N_\mathrm{e}$ of the identity matrix to make the interaction part look more symmetric. Upon diagonalising M, we get the effective mass of the neutrinos in matter,

$$(\Delta m_\mathrm{m}^2)^2 = (\Delta m^2\cos(2\theta) - a)^2 + (\Delta m^2\sin(2\theta))^2, \tag{3.34}$$

with $a = 2\sqrt{2}EG_\mathrm{F}N_\mathrm{e}/\Delta m^2$ depends only on Δm^2 and properties of the material. The matrix used to diagonalise M has precisely the same form as the 2×2 PMNS matrix U, but with a different angle:

$$\begin{pmatrix} \cos\theta_\mathrm{m} & \sin\theta_\mathrm{m} \\ -\sin\theta_\mathrm{m} & \cos\theta_\mathrm{m} \end{pmatrix}, \qquad \text{where } \sin^2(2\theta_\mathrm{m}) = \frac{\sin^2(2\theta)}{(\cos(2\theta) - a)^2 + \sin^2(2\theta)}. \tag{3.35}$$

From this expression it is immediately clear, that if the vacuum mixing angle θ is very small, there is a resonance for $a = \cos(2\theta)$. Tracing back the definitions, this corresponds to an electron number density $\tilde{N}_\mathrm{e} = \frac{\Delta m^2 \cos(2\theta)}{2\sqrt{2}EG_\mathrm{F}}$.

If the electron density of the matter is constant, then $\mathrm{d}\theta_\mathrm{m}/\mathrm{d}x = 0$. In this case it can be shown [22, § 3.2] that the effective massive neutrinos evolve independently and the transition probabilities are given by

$$P(\nu_\mathrm{e} \to \nu_\mu) = \sin^2(2\theta_\mathrm{m}) \sin^2\left(\frac{\Delta m_\mathrm{m}^2 x}{4E}\right) \tag{3.36}$$

(recall that this transition probability fixes all others). Note that this has exactly the same structure as the vacuum probability amplitude (3.19) where the parameters Δm^2 and θ have been replaced by their effective values in matter. For small propagation distances this is a good approximation, for example the density of the earth is approximately constant over a distance of 1 – 2 kilometers, which is the typical depth under the earth surface of an oscillation experiment. However, if the density is not constant, a numerical simulation must be performed. This is the case in the sun, for example, or when calculating neutrino fluxes at night, when the neutrinos approach from the other side of the earth and have to travel through $2R_\mathrm{earth} \approx 12\ 500$ kilometers of matter with strongly varying density. One way of computing results is approximating the density profile by patches of constant matter density and then patching solutions for Ψ together at the boundaries in a smooth way. This is usually done for the earth density profile. For propagation in the sun, one can use the approximation

$$N_\mathrm{e}(R) = N_\mathrm{e}(0) \exp(-R/R_0), \tag{3.37}$$

where $N_\mathrm{e}(0) = 245 N_\mathrm{A}/\mathrm{cm}^3$ and $R_0 = R_\odot/10.54$ for the electron density as a function of radial distance; for $0.1 \lesssim R/R_\odot \lesssim 0.9$ this is a rather accurate description of the actual Standard Solar Model density profile [22, § 3.2, Fig. 3]. Using such an approximation of the electron number density profile of the sun and a numerical approximation of the electron density of the earth it is then possible to derive various allowed combinations of Δm^2 and the vacuum mixing angle. This allows us to calculate the corrections to the vacuum oscillations rather well and at the same time put some restrictions on our theoretical parameters based on the available data.

As a final remark, note that in the three flavour case the only occurrence of the Majorana phases ϕ_1, ϕ_2 is again in the PMNS matrix U. This enters only in the combination $U \Delta U^\dagger$, hence in matter – just as in vacuum – these phases drop out. Not even a matter theory will give us a handle on measuring these elusive paramters, and we will definitely have to resort to non-oscillatory experiments to find them.

3.2 Adding neutrino mass terms to the Standard Model

For ease of notation and to exhibit the important ideas of this section more clearly, let us first neglect flavour and suppose that there is only one neutrino field ν of which neither left-handed nor right-handed chiral projection $\nu_\mathrm{L,R} = P_\mathrm{L,R}\nu$ vanishes *a priori*. We have seen in section 2.4 that mass terms can be generated by a Yukawa coupling of fields to the Higgs doublet Φ, leading to Dirac mass terms of the form

$$m_\mathrm{D} \left(\overline{\nu_\mathrm{R}} \nu_\mathrm{L} + \overline{\nu_\mathrm{L}} \nu_\mathrm{R}\right), \tag{3.38}$$

which couple left and right handed fields, as remarked below equation (2.9). The mass m_D is given by $m_\mathrm{D} = \frac{1}{2} y v$, where y is a Yukawa coupling constant and v is the vacuum expectation value (vev) of the Higgs field after electroweak symmetry breaking.

As we have shown in section 2.3, in the case of a Majorana fermion when $\psi = \psi^c$, this mass term reduces to

$$M\left(\overline{\psi_L^c}\psi_L + \overline{\psi_L}\psi_L^c\right). \tag{3.39}$$

In general, if we assume that the left-handed and right-handed chiral components are independent, we can add also a Majorana mass term for both of them, which — together with the Dirac term above — yields

$$\mathcal{L}_\nu = m_D\left(\overline{\nu_R}\nu_L + \overline{\nu_L}\nu_R\right) + M_L\left(\overline{\nu_L^c}\nu_L + \overline{\nu_L}\nu_L^c\right) + M_R\left(\overline{\nu_R^c}\nu_R + \overline{\nu_R}\nu_R^c\right). \tag{3.40}$$

We can write this in matrix form as

$$\mathcal{L}_\nu \propto \begin{pmatrix}\overline{\nu_L^c} & \overline{\nu_R}\end{pmatrix}\begin{pmatrix}M_L & m_D\\ m_D & M_R\end{pmatrix}\begin{pmatrix}\nu_L\\ \nu_R^c\end{pmatrix} + \text{h.c.} \tag{3.41}$$

or more compactly as

$$\mathcal{L}_\nu \propto \overline{N_L^c}MN_L + \text{h.c.} \qquad \text{where } M \stackrel{\text{def}}{=} \begin{pmatrix}M_L & m_D\\ m_D & M_R\end{pmatrix}, \quad N_L \stackrel{\text{def}}{=} \begin{pmatrix}\nu_L\\ \nu_R^c\end{pmatrix}. \tag{3.42}$$

The proportionality factor consists of minus signs and factors of $\frac{1}{2}$ which must be added to avoid overcounting when deriving the Euler-Lagrange equations, which we are not concerned with here. The doublet N_L has a left-handed index, because it only contains left-handed fields. In analogy with equation (2.45) we can write each of its components as a combination of left-handed mass eigenstates ν_L^i (for $i = 1, 2$) as

$$\nu_L = \sum_{i=1}^{2} U^{1,i}\nu_L^i, \quad \nu_R^c = \sum_{i=1}^{2} U^{2,i}\nu_L^i, \qquad \text{that is, } N_L = Un_L \tag{3.43}$$

for some unitary mixing matrix U and with n_L denoting the doublet of mass eigenstates. The matrix U is fixed by requiring that $U^t MU$ is a diagonal matrix, the diagonal elements m_1 and m_2 are then the masses of the mass eigenstates. After diagonalisation, the Lagrangian then reads

$$\mathcal{L}_\nu \propto \sum_{i=1}^{2} m_i\overline{(\nu_L^c)^i}\nu_L^i + \text{h.c.}, \tag{3.44}$$

which only contains Majorana mass terms for the massive states.

3.2.1 The see-saw mechanism

It is very interesting to consider the case $M_L = 0$, $|m_D| \ll M_R$. We will first explore this limit and only try to justify it afterwards. Upon diagonalizing the mass matrix in equation (3.42) for $m_L = 0$, we find entries equal to the eigenvalues

$$m_{1,2} = \frac{M_R \pm \sqrt{4m_D^2 + M_R^2}}{2} \simeq \frac{M_R \pm M_R(1 + 2m_D^2/M_R^2}{2} \tag{3.45}$$

where $\simeq$ indicates that the approximation $|m_D| \ll M_R$ has been used. Since the diagonal elements give the masses m_i of the neutrino mass states ν_L^i, we see that

$$m_1 \simeq -\frac{m_D^2}{M_R}, \qquad m_2 \simeq M_R + \cancel{\frac{m_D^2}{M_R}}. \tag{3.46}$$

The structure of this relations is somewhat similar to a seesaw: the heavier the right-handed mass state, the lighter the other one is. Therefore this construction is referred to as the *seesaw mechanism*. Though M_R can in principle take any value, let us assume that it is large. Because m_D was generated by the neutrino fields coupling to the Higgs field in precisely the same way as the other fermionic fields, one would naively expect m_D to be of the same order of magnitude as, for example, the electron mass. Then by our assumption on the Majorana mass M_R, one of the mass states will become very heavy while the mass of the other one will become very small compared to the electron mass.

If we assume that $M_\mathrm{L} \approx 0$ is not precisely zero, but very small compared to the other masses, the eigenvalues of the matrix are

$$
\begin{aligned}
m_{1,2} = {} & \frac{M_\mathrm{L} + M_\mathrm{R} \pm \sqrt{4m_\mathrm{D}^2 + (M_\mathrm{L} - M_\mathrm{R})^2}}{2} \\
& \simeq \frac{1}{2}\left(M_\mathrm{L} + M_\mathrm{R} \pm (M_\mathrm{L} - M_\mathrm{R})\left[1 + 2\frac{m_\mathrm{D}^2}{(M_\mathrm{L} - M_\mathrm{R})^2}\right]\right)
\end{aligned} \tag{3.47}
$$

which leads to the masses

$$
m_1 = M_\mathrm{L} + \frac{m_\mathrm{D}^2}{M_\mathrm{L} - M_\mathrm{R}} \simeq M_\mathrm{L} - \frac{m_\mathrm{D}^2}{M_\mathrm{R}}, \qquad m_2 = M_\mathrm{R} - \frac{m_\mathrm{D}^2}{M_\mathrm{L} - M_\mathrm{R}} \simeq M_\mathrm{R} \tag{3.48}
$$

for the massive neutrino states, upon diagonalisation. Indeed this approaches equation (3.46) as $M_\mathrm{L} \to 0$. We call this limiting case *type I seesaw*; we see, however, that also for small but non-zero left handed Majorana mass the seesaw effect occurs, where one mass is proportional to M_R (assumed to be large) and the other one inversely proportional. When M_R is in fact so large, that the M_L term dominates m_2, one speaks of *type II seesaw*.

Given that we can somehow justify the assumptions made above, this model therefore seems to offer a good explanation for the fact that the neutrino masses are so tiny compared to the other fermion masses. We shall now attempt to at least make these assumptions plausible.

A non-zero Majorana mass M_L for the left-handed chiralities is prohibited by the gauge symmetries of the Standard Model. Recall that the left-handed fields e_L and ν_L together form a weak isospin ($SU(2)_{I_W}$) doublet. However, it is clear that a term like (3.39) is not invariant under $SU(2)$ rotations. In particular, one can check that the term $\overline{\nu_\mathrm{L}^c}\nu_\mathrm{L}$ has a weak isospin eigenvalue of 1 (it is an $SU(2)$ triplet), whereas the lepton doublet has an eigenvalue of $1/2$. Therefore, to produce a $SU(2)$ singlet mass term it would have to be coupled to a Higgs triplet, which does not exist in the minimal Standard Model*. Since the right-handed neutrino field ν_R is a singlet under the Standard Model gauge group, there is no problem in adding a Majorana mass for this field.

Within the Standard Model, little can be said about the other assumption, $|m_\mathrm{D}| \ll M_\mathrm{R}$. As stated above, M_R is in principle just a parameter of the model, which can take any value. However, it is perfectly reasonable to expect that there is some more symmetric theory, which breaks down to the Standard Model at "low" energy scales. Such a theory is called a *Grand Unification Theory (GUT)*, the energy scale M_GUT at which its symmetry group $G_\mathrm{GUT} \supset G_\mathrm{SM}$ breaks to the Standard Model gauge group G_SM is generally expected to be of order 10^{14} GeV or higher. Standard GUT gauge

*Extensions of the Standard Model are possible, for example by a Higgs singlet (see [38]) or triplet (see [39]) scalar or by adding fields in the fermionic sector, which lead to the possibility of Majorana masses for left-handed neutrinos.

groups include products like $SU(3)_c \otimes SU(2)_\mathrm{L} \otimes SU(2)_\mathrm{R} \otimes U(1)$ or "easy" groups such as $SO(10)$ or E_8. It seems natural to expect, since we do not observe right-handed neutrinos in our low-energy experiment, that the right-handed neutrino mass is generated by some high-scale process such that $M_\mathrm{R} \simeq M_\mathrm{GUT}$. Such a theory could contain operators that are non-renormalisable in the Standard Model, such as [40]

$$\mathcal{L}_{5\mathrm{d}} \stackrel{\mathrm{def}}{=} \sum_{\alpha,\beta} \frac{g_{\alpha\beta}}{M_\mathrm{GUT}} \left(\overline{(\Psi^\alpha_{\mathrm{e,L}})^\mathrm{c}}\tilde{\Phi}\right)\left(\tilde{\Phi}^\dagger \Psi^\beta_{\mathrm{e,L}}\right), \tag{3.49}$$

where the sum in α and β runs over all flavours. After the Higgs fields Φ develops its non-zero vev, this leads to a Majorana mass

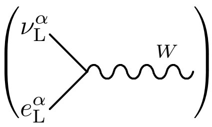
$$m_\mathrm{L} = \frac{gv^2}{M_\mathrm{GUT}}, \tag{3.50}$$

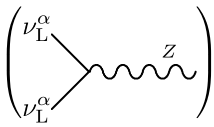
in the one-flavour case. For sufficiently high energy scales M_GUT, this is small even in comparison to $m_\mathrm{D} \propto Yv$, with Y some Yukawa coupling of the order of the electron coupling ($Y \simeq Y_\mathrm{e} \approx 0.5$ MeV$/v$).

In conclusion, from the viewpoint of a higher energy unifying theory, the assumptions for the seesaw mechanism can be more or less justified. The question whether the flavour neutrinos are indeed Majorana particles or not will be addressed in section 3.3.2, due to the experimental nature of this issue.

3.2.2 Three flavours

Let us now generalise the simplification in the preceding section to the actual case of three flavours. Recall that in the Standard Model fermion-gauge interactions (2.26) there are three left-handed neutrinos (corresponding to the three lepton types e, μ and τ) which take part in the weak charged and neutral current interactions, to wit:

$\sum_{\alpha=\mathrm{e},\mu,\tau}$


and $\sum_{\alpha=\mathrm{e},\mu,\tau}$


(3.51)

(with e^i_L denoting the left-handed leptons), respectively. Note that there is no flavour changing. The charge of the W boson depends on the direction of the charge flow, which has been omitted in the diagrams above. Because these neutrinos interact (although only weakly, by which they are hard to detect) they are also called *active neutrinos*. It has been experimentally verified that the electron, mu and tau neutrinos are the only active "light" neutrinos (that is, with mass less than approximately 46 GeV) [41]. The right-handed neutrino fields do not interact, and are therefore also called *sterile neutrinos*. Precisely because of this sterility, we have no experimental bounds on their number, but we will assume that there are only three, ν^i_R. We can again write down Majorana masses for the left- and right-handed chiralities separately, and a Dirac mass which couples the both:

$$\mathcal{L}_\nu = \overline{\nu_\mathrm{R}} M^\mathrm{D} \nu_\mathrm{L} - \frac{1}{2}\overline{\nu^\mathrm{c}_\mathrm{L}} M^\mathrm{L} \nu_\mathrm{L} - \frac{1}{2}\overline{\nu^\mathrm{c}_\mathrm{R}} M^\mathrm{R} \nu_\mathrm{R} + \text{ h.c.} \tag{3.52}$$

where the factors $-\frac{1}{2}$ were again added to avoid overcounting when applying the Euler-Lagrange formalism. The masses are no longer scalar numbers, but 3×3 matrices. The explicit form of the Lagrangian is

$$\mathcal{L}_\nu = \sum_\alpha \sum_{i=1}^3 \overline{\nu^i_\mathrm{R}} M^\mathrm{D}_{i\alpha} \nu^\alpha_\mathrm{L} - \frac{1}{2}\sum_{\alpha,\beta} \overline{(\nu^\mathrm{c}_\mathrm{L})^\alpha} M^\mathrm{L}_{\alpha\beta} \nu^\beta_\mathrm{L} - \frac{1}{2}\sum_{i,j=1}^3 \overline{(\nu^\mathrm{c}_\mathrm{R})^i} M^\mathrm{R}_{ij} \nu^j_\mathrm{R} + \text{h.c.} \tag{3.53}$$

where Greek indices such α, β run over the left-handed neutrino flavours e, μ, τ and roman indices i, j run over the massive states $1, 2, 3$.

Analogous to the one-flavour case, we can define a "doublet"

$$N_L = \begin{pmatrix} \nu_{\mathrm{L}} \\ \nu_{\mathrm{R}}^{\mathrm{c}} \end{pmatrix} \qquad \text{where} \qquad \nu_{\mathrm{L}} = \begin{pmatrix} \nu_{\mathrm{L}}^{\mathrm{e}} \\ \nu_{\mathrm{L}}^{\mu} \\ \nu_{\mathrm{L}}^{\tau} \end{pmatrix}, \text{ and } \nu_{\mathrm{R}}^{\mathrm{c}} = \begin{pmatrix} (\nu_{\mathrm{R}}^{1})^{\mathrm{c}} \\ (\nu_{\mathrm{R}}^{2})^{\mathrm{c}} \\ (\nu_{\mathrm{R}}^{3})^{\mathrm{c}} \end{pmatrix}. \tag{3.54}$$

Then we can write the Lagrangian as

$$\mathcal{L}_{\nu} = \frac{1}{2} \overline{N_{\mathrm{L}}^{\mathrm{c}}} M N_{\mathrm{L}} \qquad \text{with } M = \begin{pmatrix} M^{\mathrm{L}} & (M^{\mathrm{D}})^{\mathrm{t}} \\ M^{\mathrm{D}} & M^{\mathrm{R}} \end{pmatrix} \tag{3.55}$$

the 6×6 mass matrix. When we diagonalise this we find 6 mass eigenstates ν_p and the mass terms in the previous Lagrangian reduce to

$$\mathcal{L}_{\nu} = -\frac{1}{2} \sum_{p=1}^{6} m_p \overline{\nu_p^{\mathrm{c}}} \nu_p + \text{h.c.}, \tag{3.56}$$

whence we see that the mass states are pure Majorana particles. The six mass states are mixed by some mixing matrix V, such that

$$\nu_{\mathrm{L}}^{\alpha} = \sum_{p=1}^{6} V_{\alpha p} \nu_p, \qquad (\nu_{\mathrm{R}}^{i})^{\mathrm{c}} = \sum_{p=1}^{6} V_{ik} \nu_p, \tag{3.57}$$

from which we conclude first of all that the mass states are left-handed and secondly, that this description in no way prohibits oscillations between active and sterile neutrinos.

If we again assume that $M^{\mathrm{L}} = 0$, using a similar plausibility argument as before (based on the SM gauge group and the absence of Higgs singlets or triplets) and that the eigenvalues of M^{R} are much larger than those of M^{D}, we can (block) diagonalise M:

$$M \to \begin{pmatrix} M_{\text{light}} & 0 \\ 0 & M_{\text{heavy}} \end{pmatrix} \qquad \text{where } M_{\text{light}} \simeq -(M^{\mathrm{D}})^{\mathrm{t}} (M^{\mathrm{R}})^{-1} M^{\mathrm{D}}, \quad M_{\text{heavy}} \simeq M^{\mathrm{R}} \tag{3.58}$$

up to order $(M^{\mathrm{R}})^{-1} M^{\mathrm{D}}$. This provides a separation between the heavy (sterile) and light (active) sectors, where we can neglect the former at low-energy scales ($E \ll M_{\text{heavy}} \simeq M_{\mathrm{GUT}}$, where M_{GUT} is again some unifying energy scale at which the right handed neutrino masses are generated). We can then write for the active neutrinos

$$\nu_{\mathrm{L}}^{\alpha} = \sum_{p=1}^{3} U_{\alpha p} \nu_k, \tag{3.59}$$

because three of the mass eigenstates do not contribute after the diagonalisation. The matrix U is the unitary matrix that diagonalises M_{light} and is precisely the PMNS-matrix from equations (3.4).

3.3 Experimental evidence

However beautiful a physical theory, it is useless if it cannot be related to experiments, does not agree with the outcomes of such experiments, and cannot eventually make predictions before experiments are actually done. Let us inventarise which parameters are present in our theory and what experiments have told us (or might be able to tell us in the future).

3.3.1 Oscillation experiments

The important physical parameters which definitely would have to be determined to complete our theory are the neutrino masses m_1, m_2 and m_3 and the mixing angles $\theta_{\alpha\beta}$ ($\theta_{12} = \theta_{e\mu}$, $\theta_{13} = \theta_{e\tau}$ and $\theta_{23} = \theta_{\mu\tau}$). Oscillation experiments might in principle be able to give us the latter, as the oscillation amplitudes (3.15) depend on them through the entries of the PMNS matrix. However, as we have already remarked, oscillation experiments do not allow for the determination of absolute mass scales, just mass differences, $\Delta m^2_{ij} = m^2_j - m^2_i$. In fact there are only two independent mass differences; one convention is to use Δm^2_{12} and Δm^2_{23}, such that

$$\begin{aligned} &\Delta m^2_{13} = m^2_3 - m^2_1 = m^2_3 - m^2_2 + m^2_2 - m^2_1 = \Delta m^2_{12} + \Delta m^2_{23}, \\ &\Delta m^2_{21} = -\Delta m^2_{12}, \\ &\qquad \text{etc.} \end{aligned} \tag{3.60}$$

Obviously, taking Δm^2_{12} and Δm^2_{13} as independent is an equally valid choice. Other conventions do exist however, in which different mass squared differences and a labelling of the massive states are chosen which are more convenient for the purposes of a specific paper*. Most oscillation experiments are focused on a specific oscillation between two neutrino flavours and therefore measure $\theta \overset{\text{def}}{=} \theta_{ij}$ and $\Delta m^2 \overset{\text{def}}{=} \Delta m^2_{ij}$ for some specific i and j. The oscillation experiments can be crudely divided into two categories, which are characterised by the source - detector distance L and the typical neutrino energy E. Looking back at equation (3.15), or the effective two-dimensional theory (3.19), we see that the oscillation probability depends on $\Delta m^2 L/(4E)$. For the oscillations to be measurable this quantity cannot be too small and must be greater than order 0.1 – 1 [22]. Therefore, the parameters L and E also give a bound on the mass difference Δm^2 to which the experiment is sensitive. We then discern short baseline (SBL) and long baseline (LBL) experiments (some authors also speak of very long baseline (VBL) or even more categories). For short baseline experiments, $L/E \lesssim 1$ eV^{-2} yielding a mass sensitivity of $\Delta m^2 \gtrsim 0.1$ eV^2. Such experiments use for example nuclear reactors or muon accelerators as their source of neutrinos, whence the source – detector distance is relatively short. Long baseline (LBL) typically rely on atmospheric decays and the solar fusion processes as neutrino sources. For atmospheric experiments, $L \sim 1$ – 1000 km with a sensitivity of $\Delta m^2 \sim 10^{-2}$ – 10^{-3} eV^{-2}. Solar neutrino experiments, for which $L \sim 10^8$ km, can go as far as $\Delta m^2 \gtrsim 10^{-12}$ eV^2.

Rather recently, several global fits have been published, combining the data of many oscillation experiments. The data from this publications agree rather well. As an example, we quote the results from Fogli [42] in table 3.1, for a more complete overview see [27, § 7] and the references therein.

Insufficient data are currently available to determine the mixing angle θ_{13}, or, equivalently, $|U_{e,3}|$. The lower bound given in the tabel is a formal one (corresponding to the trivial requirement $\sin^2(\theta_{13}) \geq 0$); the current experimental upper limit is $\sin^2(2\theta_{13}) < 0.17$ and comes from the CHOOZ reactor experiment [43]. Currently preparations are taking place for the experiment, scheduled to begin in 2009, to measure up to $\sin^2(\theta_{13}) > 0.03$ and to provide either a value or a new (much lower) upper limit [44]. Whereas the 'first neutrino event' is expected by summer 2009, the full sensitivity of $\sin^2(\theta_{13}) \sim 0.03$ will only be reached after approximately 5 years of operation (see figure 1b from [44]).

*When discussing mass hierarchies below, different sources will set up things such that in both hierarchies $m_1 < m_2 < m_3$, or that changing from one hierarchy to the other amounts to changing the sign of one of the mass differences squared [42].

$$
\begin{aligned}
\sin^2\theta_{13} &= 0.9^{+2.3}_{-0.9} \times 10^{-2} \\
\Delta m^2_{12} &= 7.92^{+0.09}_{-0.09} \times 10^{-5} \;\; \mathrm{eV}^2 \\
\sin^2\theta_{12} &= 0.314^{+0.18}_{-0.15} \\
\Delta m^2_{23} &= 2.4^{+0.21}_{-0.26} \times 10^{-3} \;\; \mathrm{eV}^2 \\
\sin^2\theta_{23} &= 0.44^{+0.41}_{-0.22}
\end{aligned}
\qquad (3.61)
$$

Table 3.1: Current standings in determination of the parameters in the PMNS matrix (taken from [42])

An other important missing piece of information is the precise mass hierarchy. From the experimental data in table 3.1 it is clear that $|\Delta m^2_{12}| \ll |\Delta m^2_{23}|$ and therefore $|\Delta m^2_{13}| \sim |\Delta m_{23}|^2$. By convention, we label the two mass states with the smallest mass-difference squared ν_1 and ν_2, such that $m_1 < m_2$. That is, by a relabelling we can always get $\Delta m^2_{12} > 0$; however, the sign of Δm^2_{23} has physical importance. If $\Delta m^2_{23} > 0$ then $m_1 \lesssim m_2 \ll m_3$ and the lightest two neutrinos are almost degenerate, whereas for $\Delta m^2_{23} < 0$ the masses are ordered according to $m_3 \ll m1 \lesssim m2$, as indicated in figure 3.3. The first case is called *normal hierarchy* because it has the same structure as the mass hierarchy of the leptons, while the other is called *inverted hierarchy*.

Note that the oscillation probabilities all depend on the squared sine of the mass differences squared and therefore do not depend on the sign of Δm^2_{ij}. The problem of mass hierarchy can therefore not be resolved by oscillation experiments. The KATRIN experiment [45], which is presently (February 2009) being built, is designed to measure the neutrino mass directly through beta decay of tritium. It will attempt to push the electron neutrino mass down to 0.2 eV, improving the current bound [46] by one order of magnitude. The KATRIN experiment is expected to be fully operational by 2012 and run for 5 years.

Finally we mention again that our experimental data do not depend on the Dirac phase δ, and therefore we are unable to conclude how many, if any, CP violation takes place in neutrino oscillations.

3.3.2 The phases

In the preceding paragraphs we have seen that the Standard Model allows for both Majorana and Dirac mass terms to be added to its Lagrangian, although the Majorana mass of the light-handed neutrinos is not allowed (or heavily surpressed, if we consider the Standard Model as a low-energy effective theory). We have seen that for the mass eigenstates, only Majorana terms are present. However, experimentally flavour eigenstates are measured and the question is whether these are in reality Majorana or Dirac particles. In the equation for the mixing matrix (3.4) we have seen that, apart from the three mixing angles which describe the mixing between neutrinos of flavours α and β, there are also three phases which cannot be absorbed in the definitions of the fields. One of them is the Dirac phase δ which is related to the CP violation and can therefore — in theory — be measured, as we have seen in section 3.1.2. The other two phases ϕ_1, ϕ_2 are so-called Majorana phases. They are only present if the flavour neutrinos are Majorana fermions, because the Majorana mass term is not in-

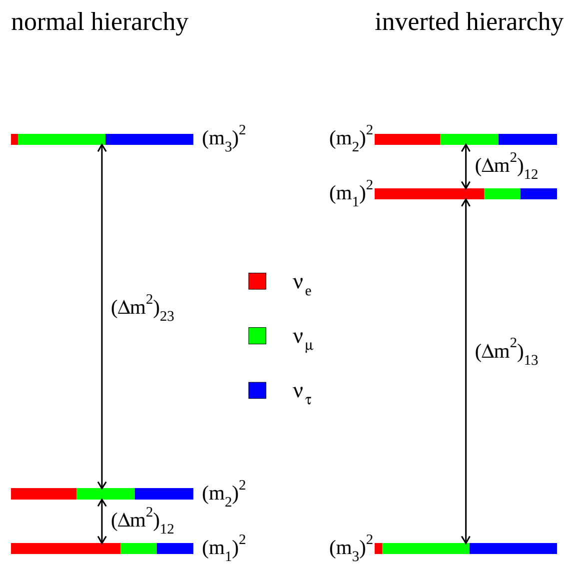


Figure 3.3: Copied from [1, fig. 12], original caption: *Cartoon of the two distinct neutrino-mass hierarchies that fit all of the current neutrino data, for fixed values of all mixing angles and mass-squared differences. The color coding (shading) indicates the fraction $|U_{\alpha i}|^2$ of each distinct flavor ν_α, $\alpha = e, \mu, \tau$ contained in each mass eigenstate ν_i, $i = 1, 2, 3$. For example, $|U_{e2}|^2$ is equal to the fraction of the $(m_2)^2$ "bar" that is painted red (shading labeled as 'ν_e').*

variant under phase transformations. They are non-physical if the Majorana term is absent though, because the remaining terms of the Lagrangian are invariant under phase rotations.

We have also shown in section 3.1.2, that neutrino oscillation experiments do not allow us to determine the Majorana phases. Even if present, they are namely irrelevant for the measurable quantities in such experiments. Therefore a non-oscillatory experiment is needed to decide between the Dirac and Majorana nature of neutrinos. The currently most sought-for is the neutrinoless double beta decay. Ordinarily, beta decay is given by the reaction

$$2\mathrm{n} \rightarrow 2\mathrm{p} + 2\mathrm{e}^- + 2\overline{\nu_\mathrm{e}}, \tag{3.62}$$

where the anti-neutrino is right-handed (as the anti-particle of a left-handed electron neutrino). Hoewever, if the left-handed neutrinos are Majorana particles, then $\overline{\nu_e} = \nu_e$ and the reaction can take place in two steps:

$$\left.\begin{aligned} \mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \overline{\nu_\mathrm{e}} \\ \nu_\mathrm{e} + \mathrm{n} \to \mathrm{p} + \mathrm{e}^- \end{aligned}\right\} \implies 2\mathrm{n} \to 2\mathrm{p} + 2\mathrm{e}^-, \tag{3.63}$$

where the neutrino which is produced in the first step, is absorbed in the second step. The Feynman diagram for this process is shown in figure 3.4. This is obviously only

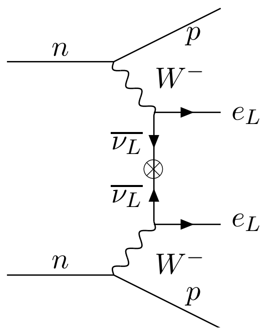


Figure 3.4: Feynman diagram of neutrinoless double beta decay (taken from [28]).

allowed when the left-handed neutrinos are Majorana particles, such that the annihilation at the crossed out vertex is possible; observation of this decay would provide conclusive evidence for the Majorana nature of the left-handed neutrino. Currently, the NEMO-3 and Cuoricino experiments are searching for this decay, while several experiments are being planned, such as* CANDLES, CUORE, EXO, GERDA, Majorana, MOON, SuperNEMO. So far the Heidelberg-Moscow experiment has claimed to have observed the neutrinoless double beta decay [49, 50], although this has not yet been confermed independently by for example NEMO [51] or other experiments [52].

Moreover, since the amplitude for the neutrinoless beta decay process is $\mathcal{P} \simeq \langle m \rangle \stackrel{\text{def}}{=} |\sum_k U_{\mathrm{e}k}^2 m_k|$ this would give us an estimate on the Majorana mass. Unfortunately $\mathcal{P}$ is quite small ($\langle m \rangle$ is estimated to $0.39^{+0.17}_{-0.28}$ eV by Heidelberg-Moscow and IGEX [49]) which means that it is very hard to detect this decay *if* it is allowed.

3.4 Other theories

Although among current theories, the seesaw theories appear to be the most promising and they definitely receive the most attention, other attempts have also been (and are still being) made to provide an explanation for the structure of the neutrino masses.

One way to approach the problem is "bottom-up": one attempts to describe the low-energy phenomenology and tries to fit the theoretical parameters to experimental results. It is possible, for example, to consider cases where the neutrino mass matrix has a specific form because there are certain relations between its entries or certain components vanish. Naturally, it is then required to justify the assumptions

*List is by far non-exhaustive and in alphabetical order, for a more detailed overview see [47, 48] which are made, and to show that the resulting theory does not directly contradict any experimental results. The danger of such an approach is of course, that such a justification may not exist, or that an assumed equality is only an approximate equality and that the theory – however interesting — must be discarded as non-physical.

The other approach is "top-down", where an attempt is made to write down a consistent high-energy theory from which a reasonable (that is, containing the Standard Model and not contradicting any experiment) low-energy description follows as a limit. Current results can then be explained by requiring that certain symmetries be broken in that limit. Ideally a high-energy mechanism is uncovered, which creates the right-handed neutrino masses and accounts for the smallness of the left-handed neutrino masses in a natural way. An example of this is the mass-dimension 5 operator from equation (3.49), which is non-renormalisable in the Standard Model but does become renormalisable in a more symmetric theory. In this context, unifying theories with gauge groups like $SU(5)$ or $SU(10)$ and with or without supersymmetry may be considered.

Seesaw is promising in this respect, as it seems to work quite well from the "bottom up" point of view, while it also follows naturally in some "top down" theories. For a more extensive review of promising results and possible research directions for both approaches, we would like to refer the reader to [53] and the references therein.

CHAPTER 4

CONCLUSION

Many papers are currently written on neutrino oscillations, ranging from complete reviews of currently accepted ideas to wild speculations in an attempt to explain existing data with a good theory. Many authors of the sources consulted in writing this report, agree that these are exciting times, when we have recently made such good progress while still so much is left unknown. We have touched upon the matter of absolute mass scale, whether flavour neutrinos are Dirac or Majorana, whether there is CP violation (and how much), what role neutrinos have played in the development of our universe, and many smaller (but interesting) questions.

At this point we are sure (at least, as sure as one can be in physics) that the solar and atmospheric neutrino problems are solved by neutrino oscillations. The theory of neutrino oscillations is well-understood, however it is not quite clear how to incorporate it in the Standard Model. The most promising candidates seem to be the see-saw theories, which at the same time explain the smallness of left-handed neutrino masses with respect to the fermions and the absence of right-handed neutrinos in our "everyday" experience, while also hinting at some unifying theory ("GUT"). A well-known example is the theory with gauge group $SO(10)$, containing a 16-tuplet that can precisely accommodate all fifteen Standard Model fermions *plus* one extra fermion field, which would logically be the right-handed neutrino. Also many supersymmetric grand unification theories (SUSY GUTs) support the existence of right-handed neutrinos and neutrino flavour oscillations which are both not possible in the minimal Standard Model.

But also in unexpected parts of physics, neutrino physics may contribute. For example, string theory is currently a (the?) serious candidate for a "theory of everything" but it has the serious problem of requiring extra dimensions. It turns out that a see-saw mechanism would work very well if these extra dimensions are small ($\sim M_U^{-1}$ for some large mass scale M_U) while if these dimensions are large (millimeter scale) current bounds on neutrino parameters could heavily restrict our theoretical possibilities [53, § VII.D].

Moreover, since right-handed neutrinos are believed to have no interactions (except through gravity) we know very little about them. Therefore it is interesting to theorise about their role in nature, and see for example how they affect the matter content of the universe, active neutrino mixing and structure formation. This has lead to the proposal of sterile neutrinos as candidates for dark matter and as a source of the matter – anti-matter symmetry through leptogenesis [54, 55].

Although most of the extensions mentioned above are still very new and may seem unlikely at first sight, history has taught us never to discard a new idea too soon. There is a chance of falling in the trap of comparison: understanding for example quark physics very well, we are quickly inclined to project certain properties of quarks onto neutrino physics, based on points of similarity. However, the amount of possibilities is very large for us creative theoretic physicists, and it will require a great deal of ingenuity to devise experiments by which we can test our wild (and not so wild) guesses.

For the coming years, we can expect experimental determination of several parameters of the theory explained in this paper with ever improved precision, which will hopefully shed some light on the properties of neutrino physics, and physics beyond the Standard Model in general. Therefore, the reader must agree that the closing words of C. Giunti and M. Laveder from their 2003 review on neutrinos [22] are still very much applicable: "...we think that interesting years lie ahead in neutrino physics research."

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