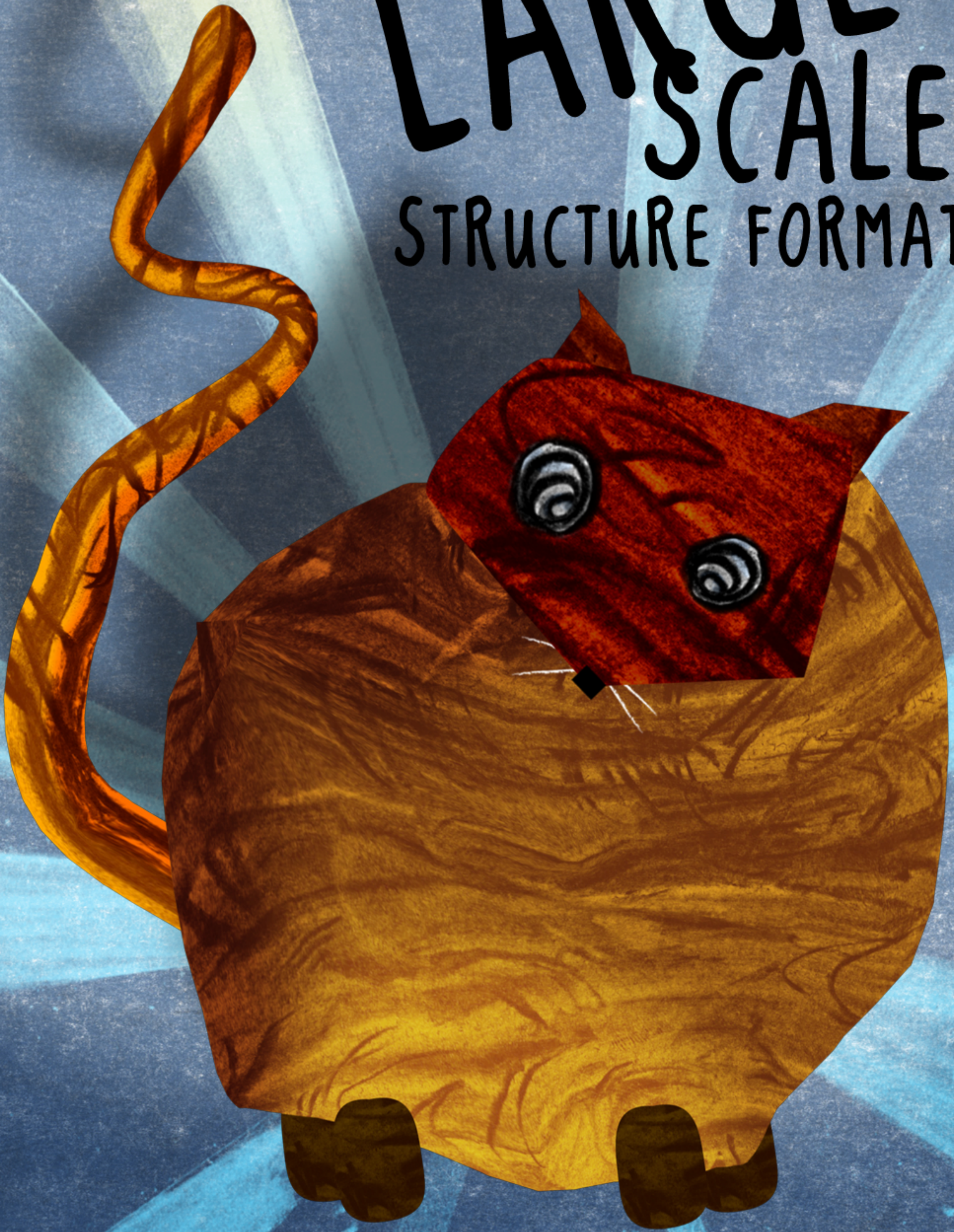
Universiteit Utrecht

LARGE SCALE STRUCTURE FORMATION



17 March 2009

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Abstract

Using the ADM formalism, and scalar field inflation, an equation of motion will be derived for perturbations in the gravitational potential. Using slow-roll approximation the power-spectrum of these perturbations will be calculated for both late and early times.

Contents

Chapter 1

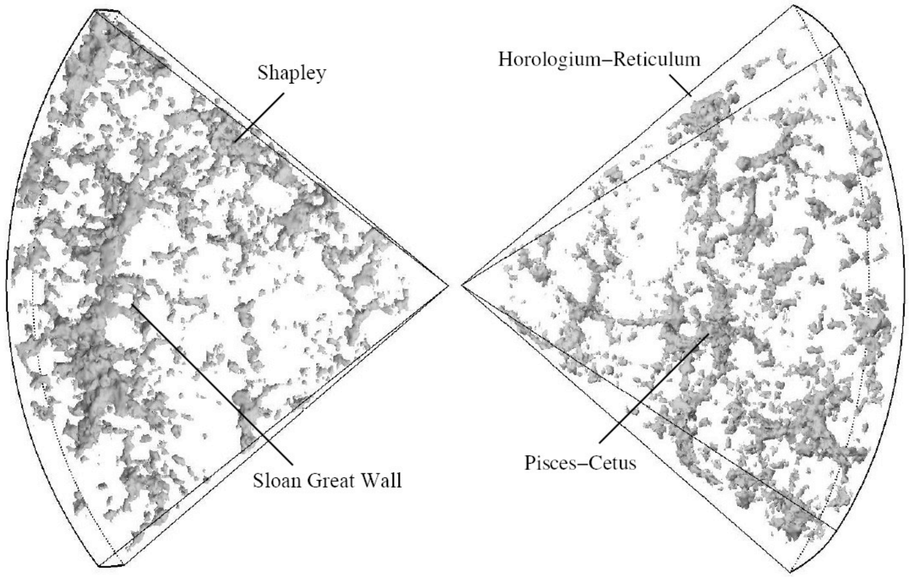
Introduction

The topic of this writeup is Large Scale Structure Formation. By large scale structures we basically mean huge clusters of galaxies. To give you an idea of *how large* these structures really are, I present you a picture, which was compiled from the data of the 2dF galaxy survey. The most prominent structure of this picture is of course *The Sloan Great Wall*. The Sloan Great Wall is the largest structure known to us these days, and it consists of many, many galaxies. To put this in perspective: the Sloan Great Wall measures 1.37 billion light years in length, while our galaxy is a mere 100 thousand light years in diameter. Now of course several interesting questions arise, for example: *How did these large scale structures come to be?* and *Can we explain the peculiar shapes of these structures?*. This writeup will focus on the first question, we will develop a model which describes the growth of structure during an inflationary epoch. The second question however, provides a nice way to check weather our model actually makes sense. Can our model predict what we see? A beautiful way to show this is to do numerical simulations. We won't go into the details of this, but it's hard to resist showing you some pictures which were made by these sort of simulations. As we can see simulations indeed give rise to *wall like* structures.

Now let us turn to the central point of this writeup. It is believed that the large scale structures we see come from quantum perturbations, which rapidly grew during an inflationary epoch. To describe these perturbations we will need to use General Relativity, combined with some form of matter which is responsible for the inflation. We will choose this form of matter to be described by a scalar field. The smoothest way to describe a combination of field theories is of course using the action principle. Let us write down the well known Hilbert Einstein Lagrangian, combined with the inflaton scalar field:

$$\mathcal{L}_G = \frac{1}{2}\sqrt{-g}\left[R - \partial_\mu\phi\partial^\mu\phi - 2V(\phi)\right], \tag{1.1}$$

where we've set the Planck mass ($M_{\rm pl}$) to one for convenience.

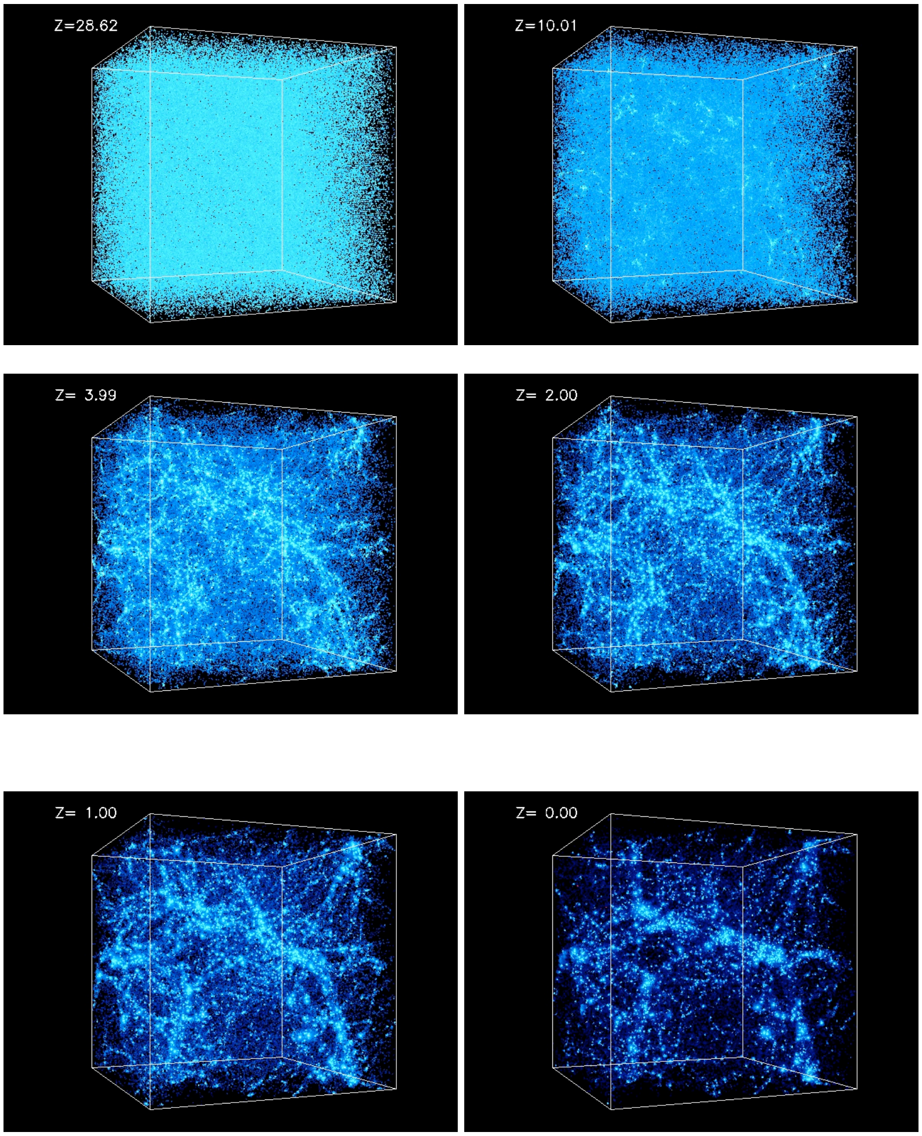


In principle General Relativity dictates the evolution of the metric tensor $g_{\mu\nu}$. This metric tensor however contains 10 different variables, from which the physical meaning is not clear a priori. Using the ADM-formalism (after Arnowitt, Deser and Misner), we will give meaning to these 10 variables, and get rid of the ones which do not interest us.
Once we've thrown away the variables that we're not interested in, we can obtain a differential equation from the Lagrangian, describing the evolution of the perturbations. Since we're dealing with perturbations, we will expand our differential equation only up to second order. A big advantage of this, is that in a second order theory, the different Fourier modes $\delta(k,t)$ don't couple, so we can describe the evolution of every Fourier mode independently.
Now as I mentioned earlier, we'd like to say things about *quantum perturbations*, therefore we need to quantize our classical theory. Luckily our differential equation up to second order will be very similar to a Harmonic oscillator, so we will quantize it in the usual way, using creation and annihilation operators. This will provide us with a natural way to calculate the amplitude of each mode as a function of time:

$$< \delta(\vec{k})\delta(\vec{k}') >= (2\pi)^3 P(\vec{k},t)\delta(\vec{k}-\vec{k}'), \tag{1.2}$$

where the function $P(\vec{k},t)$ is called the **power spectrum** of the mode. This is basically what we would like to obtain. Now the theory can provide us with a power spectrum at the end of inflation (initial data). This initial data can then be used as an input for the classical theory, which can then for instance be simulated.

In this writeup we will calculate the power spectrum of perturbations in the gravitational potential. We will specifically work out the solutions for late and early times. One very important question regarding this power spectrum is weather it is scale invariant. A scale invariant spectrum goes as k^{-3}. As we will see, inflation will predict a spectrum which deviates just slightly from scale invariance. We will express the deviation from scale invariance in terms of the slow roll parameter ϵ.



Chapter 2

Inflation

The first ingredient of the theory we're about to develop is inflation. In this chapter I will quickly review what inflation is, and how we can construct a scalar theory which drives inflation.

The metric for a flat universe, with a possibility for an homogeneous expansion is given by:

$$ds^2 = -dt^2 + a^2(t)\, dx_i dx^i, \tag{2.1}$$

where $a(t)$ is called the **scale factor**. In an era of accelerated expansion we must have:

$$\frac{\partial^2 a}{\partial t^2} > 0. \tag{2.2}$$

We will assume that the stress-energy tensor takes the form of a perfect fluid ($T^{\mu\nu} = diag(\rho, p, p, p)$). In this case, the equations which describe the evolution of the expansion coefficient, can be derived from the Einstein equation, and they are called the **Friedman equations**.

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G_N}{3c^2}\rho + \frac{\Lambda}{3}, \tag{2.3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3c^2}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.4}$$

This means that the condition for accelerated expansion (2.2) translates into:

$$p < -\frac{\rho}{3}. \tag{2.5}$$

We thus need some form of matter with a negative pressure. The most natural way to obtain this is through a (scalar)field theory.

2.1 Scalar fields in an expanding background

For our purposes, we only consider the simplest inflationary field theory, namely the scalar field theory. A scalar field action is given by:

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\left[g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + 2V\left(\phi\right)\right]. \tag{2.6}$$

Now suppose ϕ is only a function of time, then the action reduces to:

$$S = -\frac{1}{2}\int d^4x a(t)^3\left[g^{tt}(\partial_t\phi)(\partial_t\phi) + 2V(\phi)\right]. \tag{2.7}$$

Noting that $g^{tt} = -1$ and varying with respect to ϕ we find:

$$\delta_\phi S = -\frac{1}{2}\int d^4x a(t)^3\left[-2(\partial_t\phi)(\partial_t\delta\phi) + 2\frac{\partial V}{\partial\phi}\delta\phi\right]. \tag{2.8}$$

Partially integrating the first term gives:

$$\delta_\phi S = -\int d^4x a(t)^3\left[\partial_t^2\phi + 3\frac{\dot{a}(t)}{a(t)}\partial_t\phi + \frac{\partial V}{\partial\phi}\right]\delta\phi, \tag{2.9}$$

and the equation of motion thus becomes:

$$0 = \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial\phi}, \tag{2.10}$$

where H is the Hubble parameter, defined as: $H \equiv \frac{\dot{a}}{a}$. This equation of motion will be used in later derivations.

The energy and pressure of a scalar field theory can be found from the energy-momentum tensor. They're given by:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.11}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2.12}$$

It is thus possible to obtain a scalar field theory with negative pressure if the potential energy is bigger than the fields kinetic energy. If we now assume the cosmological constant to be zero, then the first Friedman equation in terms of the scalar field is given by:

$$H^2 = \frac{8\pi G_N}{3c^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = \frac{1}{3M_{Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{2.13}$$

where the parameter $M_{Pl} \equiv \frac{c}{\sqrt{8\pi G_N}}$ is called the **reduced Planck mass**.

2.2 Slow Roll Parameters

Most inflationary models are so called *slow roll* models, a name which is inspired by the condition that the zeroth order of the field should vary slowly. In practice this means that $\ddot{\phi}$ in (2.10) and $\dot{\phi}^2$ in (2.13) are negligible in comparison to the other terms in the equation. For later convenience we now define two parameters, called *slow roll parameters*, which are much smaller than one if the slow-roll condition is satisfied.

$$\epsilon \equiv \frac{{M_{Pl}}^2}{2}\left(\frac{V'}{V}\right)^2 \sim \left(\frac{\dot{\phi}}{2H^2}\right)^2, \tag{2.14}$$

$$\eta \equiv \frac{{M_{Pl}}^2 V''}{V} \sim -\frac{\ddot{\phi}}{H\dot{\phi}} + \epsilon. \tag{2.15}$$

That these relations are satisfied for slow-roll approximation is easily seen by substitution. They are however not sufficient conditions for slow-roll, since they only restrict the form of the potential. These slow-roll parameters will be used later, in the expression for the deviation of the spectrum from scale invariance.

Chapter 3

Hamiltonian Formalism

The second ingredient for our theory is of course general relativity. As stated before, general relativity can be seen as a field theory describing the dynamics of the metric tensor field. In principle this metric tensor has 10 degrees of freedom, since its a symmetric 4x4 tensor. In a vacuum the Einstein equation reads:

$$G_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu} = 0. \tag{3.1}$$

Although this is a beautiful equation from an aesthetics point of view, it's not a very transparent equation from the physics point of view. That is, you cannot easily see what the 10 different components mean and how they talk to each other. It's also not easy to see weather the 10 degrees of freedom are dynamical (describing the dynamics physical quantities), or so called gauge degrees of freedom, corresponding to constraints of the system. The precise meaning of these words will become clear in this chapter. The most elegant way to split dynamical from gauge degrees of freedom is to rewrite a Lagrangian formalism to a Hamiltonian formalism. This is what we're going to do in this chapter. First I'll describe how one generally sets up a Hamiltonian formalism for a field theory. Then, as an intermezzo, I'll show you how one applies this to electromagnetism. Finally we'll write a Hamiltonian formalism for general relativity. Upon doing this we'll note that writing a Hamiltonian formalism for general relativity is basically a carbon copy of writing a Hamiltonian formalism for electromagnetism, just a bit more involved.

3.1 Hamiltonian formalism for a field theory

In order to obtain a Hamiltonian formulation of a field theory, one needs to break up space-time into space and time. Intuitively this means that we define a 'time'-parameter t, and we slice up space-time in space-like slices of equal time. These slices of equal time are from now on referred to as Σ_t. For the moment let us assume that space-time is not curved, and in this case the breakup is trivial. Note that one can define a time-like vector field t^α with the relation:

$$t^\alpha \nabla_\alpha t = 1, \tag{3.2}$$

and the fact that t^α is time-like means that:

$$g_{\alpha\beta} t^\alpha t^\beta < 0. \tag{3.3}$$

As we see in figure 1, we can think of t^α as the flow direction of time in space-time, and it connects the subsequent spaceslices Σ_t and Σ_{t+dt}.

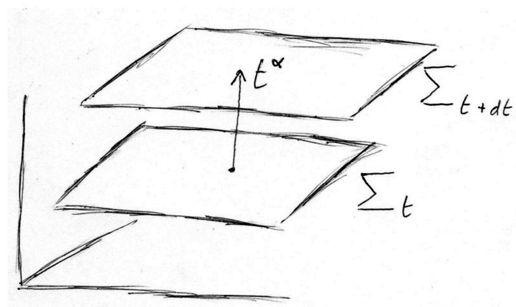


FIGURE 1: SLICING MINKOWSKI SPACE-TIME

The next step in writing a Hamiltonian formulation is to define the configuration space. This means that one needs to specify what variables (in this case fields) completely describe the system at a certain instant of time. Let us refer to these configuration space variables as q for the moment. The third step is to obtain the momenta corresponding to the configuration space variables. Let us refer to these momenta as π. Finally one needs to find a functional $H[q, \pi]$ on Σ_t, which is called the **Hamiltonian** of the system. Usually the Hamiltonian is written as an integral over the Hamiltonian density,

$$H = \int_{\Sigma_t} \mathcal{H}, \tag{3.4}$$

where $\mathcal{H}$ is a local function of q and π, and their spatial derivatives up to finite order. In case one already has a Lagrangian density $\mathcal{L}$ of the field theory, the momentum π is taken to be the **canonical momentum**, which is defined as:

$$\pi_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}, \tag{3.5}$$

where the index k is just a possible vector index. The (cannonical) Hamiltonian density then defined as:

$$\mathcal{H}(q, \pi) = \sum_i \pi_i \dot{q}_i - \mathcal{L}. \tag{3.6}$$

3.1.1 Constraint equations

In physics we have a lot of theories which actually posses more variables than degrees of freedom. With this I mean that the theory has more variables than there are needed to completely specify the system (more than the dimension of the actual phase space if you like). One could say that several degrees of freedom are truly **physical degrees of freedom** describing the dynamics of the system, while others are unphysical degrees of freedom, more commonly known as **gauge degrees of freedom**, if they're related to a **gauge transformation** . A valid question at this point would be: 'Why don't we just write down a theory without these gauge degrees of freedom?' The answer to this question is given by aesthetics. Usually the gauge degrees of freedom don't decouple nicely from the physical degrees of freedom. This means that the equations of motion for the physical degrees of freedom will contain terms proportional to gauge degrees of freedom. Trying to make them decouple is a messy business, and the equations look much nicer if one includes the gauge degrees of freedom.

A consequence of having a theory with too many degrees of freedom is that the phase space we're working in is too big, and that the actual phase space is a subspace embedded in this oversized phase space. To define the subspace one needs one constraint for every gauge degree of freedom.

A nice feature of the Hamiltonian formalism is that it's easy to discover gauge degrees of freedom, and their corresponding constraint equations. The important thing to note is that it is not always possible to make a bijection from $\dot{q}$ to π through (3.5). A more formal way to put this is to say that the so called **Hessian matrix**, which is defined as:

$$H_{kl} \equiv \frac{\partial \pi_k}{\partial \dot{q}^l} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^k \partial \dot{q}^l}, \tag{3.7}$$

is not invertible. In this case there exists a relation between the different momenta, and thus not all momenta need to be known to completely specify the system.

Systems which posses such a relation between the different momenta are called **singular systems**. This relation is exactly a constraint equation on the phase space. In general the number of constraints is equal to the number of zero eigenvalues of the Hessian matrix. Now as I said, going to a Hamiltonian formalism will reveal constraints, this is because for singular systems the Hamiltonian will contain terms of the form:

$$\sum_n \chi_n \phi_n, \tag{3.8}$$

where ϕ_n is an irreducible set of constraints and χ_n are Lagrangian multipliers for the constraints. We can thus find the constraints by derriving the Hamiltonian to this Lagrangian multiplier,

$$\phi_n = \frac{\delta H}{\delta \chi_n}. \tag{3.9}$$

3.1.2 Dynamical equations

In the previous subsection we saw that relations between the different momenta corresponded to constraint equations and that these constraints reduce the size of the phase space, since we need less variables to completely specify the system. The reduced set of variables needed to specify the system will however have linearly independent and nontrivial equations of motion. These variables represent physical degrees of freedom, and their equations of motion are **dynamical**. They are obtained by:

$$\dot{q} \equiv \frac{\delta H}{\delta \pi}, \tag{3.10}$$

$$\dot{\pi} \equiv -\frac{\delta H}{\delta q}. \tag{3.11}$$

Well that's enough talking about abstracts, let us now see how this procedure applies to electrodynamics in the next section!

3.2 Electrodynamics

3.2.1 Maxwell's Equations

Let me start by reminding you of Maxwell's equations in absence of any sources:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0 \tag{3.12}$$

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B} \qquad \vec{\nabla} \times \vec{B} = \partial_t \vec{E}. \tag{3.13}$$

Now one thing you can immediately see from these equations is that the first pair (3.12) put a restriction on what type of functions can represent an electric or magnetic field. These are thus **constraint equations**. The second pair of equations (3.13) contain a time derivative, and thus tell us how the fields evolve in time. These are therefore **dynamical equations**.

In this case the separation between dynamical constraint equations is already clear from the start. This is why electrodynamics provides an excellent example to illustrate that this separation also becomes manifest upon rewriting the EM Lagrangian to a Hamiltonian.

Let's get started! Let us assume we're just working in flat Minkowski space. Naively we take the field A^μ, evaluated on a flat slice Σ_t, to completely specify the system. The Lagrangian for electrodynamics is well known:

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right). \tag{3.14}$$

It will prove to be very useful write the field variable A^μ in terms of it's projection on the plane Σ_t and perpendicular to it. For this reason let us define a time-like vector field n^α, which is normal to the surface Σ_t everywhere and has unit length. Since we're working in Minkowski space the choice is easy:

$$n^\alpha = (1, 0, 0, 0). \tag{3.15}$$

Using this vector field we can rewrite A^μ as follows:

$$A^\alpha = \underbrace{-(\eta_{\mu\nu} A^\mu n^\nu) n^\alpha}_{\perp} + \underbrace{(A^\alpha + (\eta_{\mu\nu} A^\mu n^\nu) n^\alpha)}_{\parallel} \equiv (V, \vec{A}). \tag{3.16}$$

It's not hard to show that the Lagrangian density can be rewritten in terms of these projected variables as follows:

$$\mathcal{L}_{EM} = \frac{1}{2}\left(\dot{\vec{A}} + \vec{\nabla}V\right)^2 - \frac{1}{2}\left(\vec{\nabla}\times\vec{A}\right)^2. \tag{3.17}$$

The proof follows if one writes the indices explicitly:

$$\begin{aligned}
\mathcal{L}_{EM} &= \frac{1}{2}\left(\partial_0 A_i + \partial_i A_0\right)^2 - \frac{1}{2}\left(\epsilon_{ijk}\partial^j A^k\right) \\
&= \frac{1}{2}\left(\partial_0 A_i \partial^0 A^i + \partial_0 A_i \partial^i A^0 + \partial_i A_0 \partial^0 A^i + \partial_i A_0 \partial^i A^0 - \epsilon_{ijk}\epsilon^{ilm}\partial^j A^k \partial_l A_m\right) \\
&= \frac{1}{2}\left(\partial_0 A_i \partial^0 A^i + \partial_0 A_i \partial^i A^0 + \partial_i A_0 \partial^0 A^i + \partial_i A_0 \partial^i A^0 - \partial^i A^j \partial_i A_j + \partial^i A^j \partial_j A_i\right) \\
&= \frac{1}{2}\left(\partial_\mu A_\nu \partial^\nu A^\mu - \partial_\mu A_\nu \partial^\mu A^\nu\right) = -\frac{1}{4}\left(2\partial_\mu A_\nu \partial^\mu A^\nu - 2\partial_\mu A_\nu \partial^\nu A^\mu\right) \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},
\end{aligned}$$

where we've used the identity:

$$\epsilon_{ijk}\epsilon^{ilm} = \delta_j^l \delta_k^m - \delta_j^m \delta_k^l. \tag{3.18}$$

Note in passing that yet another way to write the Lagrangian density of the electromagnetic field is:

$$\mathcal{L}_{EM} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right), \tag{3.19}$$

where we've used the following relation between the fields $\vec{E}$, $\vec{B}$ and the potentials V and $\vec{A}$:

$$\vec{E} \equiv \dot{\vec{A}} + \vec{\nabla}V, \tag{3.20}$$

$$\vec{B} \equiv \vec{\nabla}\times\vec{A}. \tag{3.21}$$

The momenta conjugate to $\vec{A}$ and V are:

$$\vec{\pi}_A = \frac{\partial\mathcal{L}}{\partial\dot{\vec{A}}} = \dot{\vec{A}} + \vec{\nabla}V \equiv -\vec{E}, \tag{3.22}$$

$$\pi_V = \frac{\partial\mathcal{L}}{\partial\dot{V}} = 0. \tag{3.23}$$

Let us pause here for a second, since this is an important result. The canonical momentum of the off-plane projection of A^μ (V) vanishes, this means that this is a non-dynamical variable. When we define our Hamiltonian density, we will therefore choose $\vec{A}$ as our vector field describing the system instead of A^μ. In this case the Hamiltonian density becomes:

$$\begin{aligned}
\mathcal{H}_{EM} &= \vec{\pi}\cdot\dot{\vec{A}} - \mathcal{L}_{EM} &(3.24)\\
&= -\vec{E}\cdot\left(-\vec{E} - \vec{\nabla}V\right) - \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 \\
&= \frac{1}{2}\vec{\pi}\cdot\vec{\pi} + \frac{1}{2}\vec{B}\cdot\vec{B} - \vec{\pi}\cdot\vec{\nabla}V \\
&= \frac{1}{2}\vec{\pi}\cdot\vec{\pi} + \frac{1}{2}\vec{B}\cdot\vec{B} + V\vec{\nabla}\cdot\vec{\pi} - \vec{\nabla}\cdot\left(V\vec{\pi}\right). &(3.25)
\end{aligned}$$

The last term in (3.25) is a total divergence, and we will therefore discard it. More importantly, the third term is of the form (3.8). We thus see that the non-dynamical variable V found in (3.23) acts as a Lagrangian multiplier, and that it's equation of motion is one of Maxwell's constraint equations! The equations of motion for the dynamical variables however, (3.10) and (3.11), provide us with two equations which are equivalent to Maxwell's dynamical equations.

$$\dot{\vec{A}} = \frac{\delta H_{EM}}{\delta \vec{\pi}} = -\vec{\pi} - \vec{\nabla} V = -\vec{E} - \vec{\nabla} V, \tag{3.26}$$

$$\dot{\vec{\pi}} = -\dot{\vec{E}} = -\frac{\delta H_{EM}}{\delta \vec{A}} = -\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right). \tag{3.27}$$

So what about Maxwell's second constraint equation? It turns out that this one simply follows from the way $\vec{B}$ is defined in terms of $\vec{A}$. It follows from taking the divergence of (3.21). So why didn't it show up with a Lagrangian multiplier in the Hamiltonian? Well the truth is, that there is much more to the story of constraint equations that that I've told you. In classical dynamics there are different classes of constraints, each discovered in seperate ways. The story I've told you is about discovering a certain class of constraints, called first class constraints. These constraints will appear in the Hamiltonian density as I described. As it turns out, all constraints of general relativity are of first class, while this is not true for electrodynamics.

3.2.2 Fixing the Gauge

The fact that electrodynamics has two constraint equations and two dynamical ones, suggests that there are only two physical degrees of freedom. This means that we should be able to choose 2 out of 4 degrees of freedom in A^μ as we like, since they don't contribute to the dynamics of the system anyway.

As we can see the dynamical equations (3.26) and (3.27) are left invariant by the following transformations:

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\kappa(\vec{x}), \qquad V \rightarrow V - \frac{\partial \lambda(t)}{\partial t}. \tag{3.28}$$

So indeed, we're free to choose two out of four degrees of freedom freely by choosing two arbitrary functions $\lambda(t)$ and $\kappa(\vec{x})$, without changing the dynamical equations of motion!

3.3 Geometrodynamics

Now that we've completed our discussion about electrodynamics, let's turn to general relativity. Because writing a Hamiltonian formalism for general relativity bears such a close resemblance to doing it for electrodynamics, this procedure was called **Geometrodynamics** by John Wheeler.

3.3.1 ADM Formalism

Just like we did in electrodynamics, we would like to slice up space-time. This time however, space-time is curved. Without using to much fancy mathematical jargon I'd like to make this clear by drawing figure 2:

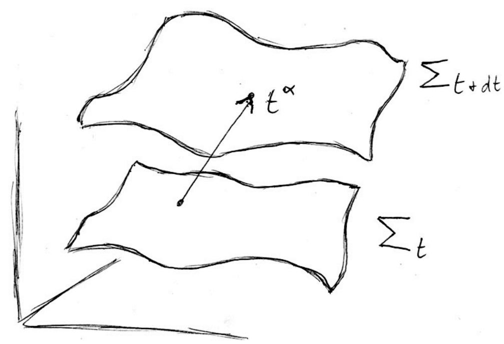


FIGURE 2: SLICING CURVED SPACE-TIME

We will again call the space-like slices Σ_t. Because our space-time is now curved, we expect Σ_t also to have a nontrivial metric. Let's call it h_{ij}. Note the use of Latin indices here, this is because the metric on the space slice is 3D.

A space time interval is defined as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{3.29}$$

Now we would like to rewrite this in terms of the 3D metric h_{ij}. Let's choose coordinates in such a way that the spatial basis-vectors ∂_i are in the tangent space of the hyper surface Σ_t. We can then say that distances on the hyper surface are given by:

$$dl^2 = h_{ij} dx^i dx^j. \tag{3.30}$$

If we now want to rewrite a general space time interval in terms of h_{ij}, we need to account for the fact that the metric of subsequent slices Σ_t and Σ_{t+dt} generally changes. We need to compensate for the fact that the vector t^α, which represents 'the flow of time', is in general not perpendicular to the hyper surfaces. We can decompose this flow-of-time vector in a part perpendicular and a part normal to Σ:

$$dt^\alpha = (\mathcal{N} dt, \mathcal{N}^i dt). \tag{3.31}$$

The function $\mathcal{N}$ is called **lapse function**, while the 3D vector $\mathcal{N}^i$ is called **shift vector**. I will be more formal on how to decompose vectors into parts normal and perpendicular to Σ later on, but for now having an intuitive notion of it suffices. Namely, as depicted in figure 3, we can now just use the Pythagorean theorem to calculate ds^2:

$$ds^2 = -(distance \perp \Sigma)^2 + (distance \| \Sigma)^2, \tag{3.32}$$

that is:

$$ds^2 = -\mathcal{N}^2 dt^2 + h_{ij}\left(dx^i + \mathcal{N}^i dt\right)\left(dx^j + \mathcal{N}^j dt\right). \tag{3.33}$$

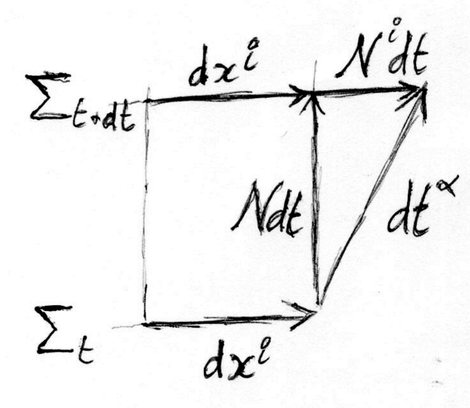


FIGURE 3: MEANING OF $\mathcal{N}^i$ AND $\mathcal{N}$

Now by comparison of (3.30) and (3.33), we can easily see that:

$$g_{\mu\nu} = \begin{bmatrix} -\mathcal{N}^2 + \mathcal{N}^k\mathcal{N}_k & \mathcal{N}_j \\ \mathcal{N}_i & h_{ij} \end{bmatrix}. \tag{3.34}$$

By the defining relation of the inverse metric

$$g_{\alpha\beta}g^{\beta\kappa} = \delta^\kappa_\alpha, \tag{3.35}$$

we can see that the inverse metric is then given by:

$$g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\mathcal{N}^2} & \frac{\mathcal{N}^j}{\mathcal{N}^2} \\ \frac{\mathcal{N}^i}{\mathcal{N}^2} & h^{ij} - \frac{\mathcal{N}^i\mathcal{N}^j}{\mathcal{N}^2} \end{bmatrix}. \tag{3.36}$$

Rewriting the metric in this particular form is also referred to as **ADM formalism**.

3.3.2 Projection of Vectors

Now that we've established an intuitive picture of slicing up space-time, it's time to become a bit more formal. First of all I'd like to introduce some nomenclature. The 4D Lorentzian manifold representing space-time we will call $\mathcal{M}$. The space-like slices Σ are sub-manifolds of $\mathcal{M}$, that is:

$$\Sigma_t \subset \mathcal{M}. \tag{3.37}$$

The fact that Σ is space-like means that all vectors in its tangent space have positive length, except for the null vector, which has zero length:

$$h_{ij}v^iv^j \geq 0, \qquad \forall v \in T_p\Sigma. \tag{3.38}$$

In the previous subsection we projected the vector t^α into a part normal to Σ, and a part perpendicular to it. To be able to do this we introduce a time-like unit vector field n^α as follows:

$$g_{\mu\nu} n^\mu n^\nu = -1, \tag{3.39}$$

$$g_{\mu\nu} n^\mu v^\nu = 0, \qquad \forall v \in T_p\Sigma. \tag{3.40}$$

Using this definition we can project any vector in the tangent space of $\mathcal{M}$ as follows:

$$v^\alpha = \underbrace{-(g_{\mu\nu} v^\mu n^\nu) n^\alpha}_{\perp} + \underbrace{(v^\alpha + (g_{\mu\nu} v^\mu n^\nu) n^\alpha)}_{\parallel}. \tag{3.41}$$

Now we're able to give a formal definition of the lapse function and the shift vector as follows:

$$t^\alpha = \underbrace{-(g_{\mu\nu} t^\mu n^\nu) n^\alpha}_{\perp} + \underbrace{(t^\alpha + (g_{\mu\nu} t^\mu n^\nu) n^\alpha)}_{\parallel} \equiv (\mathcal{N}, \vec{\mathcal{N}}). \tag{3.42}$$

3.3.3 Curvature

Let me remind you of the Lagrangian we're working with, it's a combination of the Hilbert-Einstein Lagrangian combined with the Lagrangian of a scalar matter field responsible for inflation:

$$\mathcal{L}_G = \frac{1}{2}\sqrt{-g}\left[R - \partial_\mu \phi \partial^\mu \phi - 2V(\phi)\right], \qquad M_{Pl}^{-2} = 1. \tag{3.43}$$

Now in the end we would like to rewrite this Lagrangian in terms of our new variables $(h_{ij}, \mathcal{N}, \mathcal{N}_k)$. To be able to rewrite the Ricci scalar, we need to say something about curvature. Remember that the Ricci scalar is nothing more than the contracted Riemann tensor:

$$R = R^{\alpha\beta}{}_{\alpha\beta}, \tag{3.44}$$

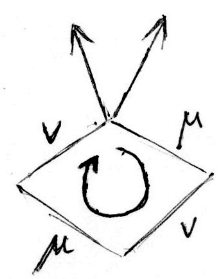


FIGURE 4: INTRINSIC CURVATURE

and that the Riemann tensor is a measure for the **intrinsic curvature** of the space, in the sense that it measures how much a vector changes when it's transported around an infinitesimal loop, as we see in figure 4. The Riemann tensor is therefore defined as the commutator of two covariant derivatives:

$$R^{\alpha}{}_{\mu\nu\beta}\partial_\alpha = [\nabla_\mu, \nabla_\nu]\,\partial_\beta, \tag{3.45}$$

where the indices μ and ν can now be thought of as defining 'the direction' of the loop. The covariant derivative is defined as:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \Gamma^\alpha_{\mu\nu}\omega_\alpha, \tag{3.46}$$

Where $\Gamma^\alpha_{\mu\nu}$ is called the Christoffel connection, and it compensates for the fact that basis vectors are slightly tilted if one compares two nearby points in space-time. A shorter way to write (3.46) is:

$$\omega_{\nu;\mu} = \omega_{\nu,\mu} + \Gamma^\alpha_{\mu\nu}\omega_\alpha. \tag{3.47}$$

Applying (3.47) on a basis vector gives us the precise definition of the Christoffel symbol:

$$\partial_{\mu;\nu} = \Gamma^\alpha_{\nu\mu}\partial_\alpha. \tag{3.48}$$

Plugging (3.47) in (3.45) yields an alternative way to write the Riemann tensor:

$$R^{\alpha}{}_{\mu\nu\beta} = \Gamma^\alpha_{\nu\beta,\mu} - \Gamma^\alpha_{\mu\beta,\nu} + \Gamma^\kappa_{\nu\beta}\Gamma^\alpha_{\mu\kappa} - \Gamma^\kappa_{\mu\beta}\Gamma^\alpha_{\nu\kappa}. \tag{3.49}$$

What we would like to do now is to relate the intrinsic curvature of the 4D metric $g_{\mu\nu}$ to the intrinsic curvature of the 3D metric of the slices h_{ij}. It is however obvious that the intrinsic curvature of the space slices contains less information than the intrinsic curvature of the whole space-time. The part of information which is missing is exactly captured in the way how the space slices Σ are embedded in the space-time manifold $\mathcal{M}$. That is to say the **extrinsic curvature** of Σ in $\mathcal{M}$. As we can see in figure 5, the extrinsic curvature tells us how the vector field n^α changes if one compares two nearby space-time points.

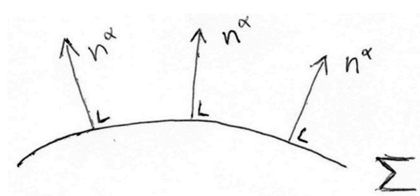


FIGURE 5: EXTRINSIC CURVATURE

Intuitively we thus expect that we can express the 4D Riemann tensor in terms of the 3D Riemann tensor and the extrinsic curvature of Σ.

Projecting the covariant derivative

Let us project the 4D covariant derivative using (3.41).

$$u^\kappa \nabla_\kappa v^\mu = \nabla_{\boldsymbol{u}} v^\mu = \underbrace{-g_{\alpha\beta}(\nabla_{\boldsymbol{u}} v^\alpha) n^\beta n^\mu}_{\perp} + \underbrace{(\nabla_{\boldsymbol{u}} v^\mu + g_{\alpha\beta}(\nabla_{\boldsymbol{u}} v^\alpha) n^\beta n^\mu)}_{\parallel}. \tag{3.50}$$

for $u \in T_p\Sigma$ and $v \in T_p\mathcal{M}$. Note that the second term term of (3.50) is the part of the covariant derivative, which 'stays' on the surface Σ, and we will therefore identify this term with the 3D covariant derivative:

$$^3\nabla_{\boldsymbol{u}} v^\mu \equiv \nabla_{\boldsymbol{u}} v^\mu + g_{\alpha\beta}(\nabla_{\boldsymbol{u}} v^\alpha) n^\beta n^\mu, \tag{3.51}$$

where the superscript 3 preceding the covariant derivative indicates that it's with respect to the 3D metric. The first part of (3.50) is exactly the part of the covariant derivative which doesn't stay on Σ, and we will therefore identify it with the extrinsic curvature:

$$K_{ij} u^i v^j \equiv -g_{\alpha\beta}(\nabla_{\boldsymbol{u}} v^\alpha) n^\beta. \tag{3.52}$$

Note that this notion of extrinsic curvature is slightly different from the intuitive idea we gave in the figure 5. We can show that this definition of the extrinsic curvature is actually the same as in the figure 5 because of metric compatibility. Metric compatibility means that:

$$\nabla_{\boldsymbol{u}} g_{\mu\nu} = 0. \tag{3.53}$$

Using this, in combination with the definition of n^α (3.40) we have:

$$0 = \nabla_{\boldsymbol{u}}(g_{\alpha\beta} n^\alpha v^\beta) = g_{\alpha\beta}(\nabla_{\boldsymbol{u}} v^\alpha, n^\beta) + g_{\alpha\beta}(v^\alpha, \nabla_{\boldsymbol{u}} n^\beta). \tag{3.54}$$

So we can define extrinsic curvature equally well as:

$$K_{ij} u^i v^j \equiv g_{\alpha\beta}(\nabla_{\boldsymbol{u}} n^\alpha) v^\beta, \tag{3.55}$$

and thus we see that (3.52) is equivalent to our intuitive idea of extrinsic curvature. To summarize, we have found the following relation:

$$\nabla_{\boldsymbol{u}} v = (K_{ij} u^i v^j) n + {}^3\nabla_{\boldsymbol{u}} v. \tag{3.56}$$

Gauss-Codazzi equations

Let us now try to express the 4D Riemann tensor in terms of the 3D Riemann tensor and the extrinsic curvature tensor by plugging (3.56) in the definition of the Riemann tensor (3.45). For this we first consider a special case of (3.56). Suppose that $u = \partial_i$ and $v = \partial_j$, then by equation (3.48) we have that:

$$\nabla_i \partial_j = K_{ij} n + {}^3\Gamma^m_{ij} \partial_m. \tag{3.57}$$

Furthermore, if we set $u = \partial_i$ and $v = dx^m$ in (3.55), and multiply it with ∂_m on the right, then we find:

$$K_i{}^m \partial_m = \nabla_i n. \tag{3.58}$$

Now we have all the tools to calculate the first term in (3.45). Let us work in a basis (n, ∂_i), such that we can write Latin indices on the tensors which are defined on the 4D manifold $\mathcal{M}$, since they coincide with the spatial part of the Greek indices.

$$\begin{aligned}\nabla_i\nabla_j\partial_k &= \nabla_i(K_{kj}n + {}^3\Gamma^m_{jk}\partial_m) \\ &= K_{jk,i}n + K_{jk}\nabla_i n + {}^3\Gamma^m_{jk,i}\partial_m + {}^3\Gamma^m_{jk}\nabla_i\partial_m \\ &= K_{jk,i}n + K_{jk}K_i{}^m\partial_m + {}^3\Gamma^m_{jk,i}\partial_m + {}^3\Gamma^m_{jk}(K_{im}n + {}^3\Gamma^l_{im}\partial_l) \\ &= (K_{jk,i} + {}^3\Gamma^m_{jk}K_{im})n + K_{jk}K_i{}^m\partial_m + ({}^3\Gamma^m_{jk,i} + {}^3\Gamma^m_{jk}{}^3\Gamma^l_{im})\partial_m,\end{aligned} \tag{3.59}$$

and thus the Riemann tensor becomes:

$$\begin{aligned}R^\alpha{}_{ijk}\partial_\alpha = &\quad \nabla_i\nabla_j\partial_k - \nabla_j\nabla_i\partial_k \\ = &\quad (K_{jk,i} - K_{ik,j} + {}^3\Gamma^m_{jk}K_{im} - {}^3\Gamma^m_{ik}K_{jm})n + \\ &\quad (K_{jk}K_i{}^m - K_{ik}K_j{}^m)\partial_m + \\ &\quad ({}^3\Gamma^m_{jk,i} - {}^3\Gamma^m_{ik,j} + {}^3\Gamma^l_{jk}{}^3\Gamma^m_{il} - {}^3\Gamma^l_{ik}{}^3\Gamma^m_{jl})\partial_m.\end{aligned} \tag{3.60}$$

Note that the first line is just:

$$({}^3\nabla_i K_{jk} - {}^3\nabla_j K_{ik})n, \tag{3.61}$$

and comparing with (3.49), we see that the last term is:

$${}^3R^m{}_{ijk}\partial_m, \tag{3.62}$$

and thus we arrive at the **Gauss-Codazzi equations**:

$$R^\alpha{}_{ijk}\partial_\alpha = ({}^3\nabla_i K_{jk} - {}^3\nabla_j K_{ik})n + ({}^3R^m_{ijk} + K_{jk}K_i{}^m - K_{ik}K_j{}^m)\partial_m. \tag{3.63}$$

For our purposes however we're only interested in the second term, the **Codazzi equation**. We obtain it by taking an inner product of (3.63) and dx^m:

$$R^m{}_{ijk} = {}^3R^m_{ijk} + K_{jk}K_i{}^m - K_{ik}K_j{}^m. \tag{3.64}$$

Now performing the proper contractions, one finds:

$$\begin{aligned}R &= R^{ab}{}_{ab} = {}^3R^{ab}{}_{ab} + K^{ab}K_{ab} - (K^a{}_b)^2 \\ &= {}^3R + K^{ab}K_{ab} - (TrK)^2,\end{aligned} \tag{3.65}$$

where ${}^3R$ is the Ricci scalar of the hypersurface Σ.

Extrinsic curvature in terms of $h_{ij}, \mathcal{N}, \mathcal{N}_i$

We have expressed the 4D Ricci scalar in terms of the 3D Ricci scalar plus some terms proportional to the extrinsic curvature. Beforehand we said that we would like to express the Lagrangian (3.43) in terms of $(h_{ij}, \mathcal{N}, \mathcal{N}_i)$, and not in terms of the extrinsic curvature. We will therefore rewrite the extrinsic curvature in a proper way.

Using (3.55), and plugging in $u = \partial_i$ and $v = \partial_j$, we find that:

$$K_{ik} = n_{i;k}. \tag{3.66}$$

Now the vector n_α should have length -1, and thus it's given by:

$$n_\alpha = (-\mathcal{N}, 0, 0, 0). \tag{3.67}$$

Calculating the covariant derivative we thus obtain:

$$\begin{aligned}
K_{ik} &= -\mathcal{N}\Gamma^0_{ik} \\
&= -\frac{1}{2}g^{0\lambda}(\partial_i g_{\lambda k} + \partial_k g_{\lambda i} - \partial_\lambda g_{ik}) \\
&= -\frac{1}{2}g^{00}(\partial_i g_{0k} + \partial_k g_{0i} - \partial_0 g_{ik}) + \frac{1}{2}g^{0j}(\partial_i g_{jk} - \partial_k g_{ji} - \partial_j g_{ik}) \\
&= \frac{1}{2\mathcal{N}}(\partial_i \mathcal{N}_k + \partial_k \mathcal{N}_i - \partial_t h_{ik}) + \frac{\mathcal{N}_l}{\mathcal{N}} h_{lp} {}^3\Gamma^p_{ik} \\
&= \frac{1}{2\mathcal{N}}({}^3\nabla_i \mathcal{N}_k + {}^3\nabla_k \mathcal{N}_i - \dot{h}_{ik}),
\end{aligned} \tag{3.68}$$

where we've used (3.34) and (3.36).

Rewriting $\sqrt{-g}$

Now there is only one thing left to do still. In the Lagrangian (3.43) also a term $\sqrt{-g}$ appears. We still need to rewrite this in terms of $(h_{ij}, \mathcal{N}, \mathcal{N}_i)$. Basically we just need to take the determinant of (3.34). Note that since $g_{\mu\nu}$ is a symmetric tensor, so will be h_{ij}. Let's say h_{ij} is this:

$$h_{ij} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, \tag{3.69}$$

and its determinant is:

$$\det h = adf - a^2 e - b^2 f + bce + cbe - c^2 d. \tag{3.70}$$

The determinant of g becomes:

$$\begin{aligned}
&\det g = -\mathcal{N}^2 \det h + \mathcal{N}^k \mathcal{N}_k \det h \\
&- \mathcal{N}_1 \det \begin{bmatrix} \mathcal{N}_1 & b & c \\ \mathcal{N}_2 & d & e \\ \mathcal{N}_3 & e & f \end{bmatrix} + \mathcal{N}_2 \det \begin{bmatrix} \mathcal{N}_1 & a & c \\ \mathcal{N}_2 & b & e \\ \mathcal{N}_3 & c & f \end{bmatrix} - \mathcal{N}_3 \det \begin{bmatrix} \mathcal{N}_1 & a & b \\ \mathcal{N}_2 & b & d \\ \mathcal{N}_3 & c & e \end{bmatrix}.
\end{aligned} \tag{3.71}$$

Note that the spatial indices can be raised and lowered by the metric h_{ij} and therefore the last four terms of (3.71) can be written as:

$$\mathcal{N}^k\left(\mathcal{N}_k \det h \underbrace{- h_{1k}\det\begin{bmatrix}\mathcal{N}_1 & b & c\\ \mathcal{N}_2 & d & e\\ \mathcal{N}_3 & e & f\end{bmatrix} + h_{2k}\det\begin{bmatrix}\mathcal{N}_1 & a & c\\ \mathcal{N}_2 & b & e\\ \mathcal{N}_3 & c & f\end{bmatrix} - h_{3k}\det\begin{bmatrix}\mathcal{N}_1 & a & b\\ \mathcal{N}_2 & b & d\\ \mathcal{N}_3 & c & e\end{bmatrix}}_{(*)}\right). \tag{3.72}$$

We can work this out for the different values of $k \in \{1,2,3\}$ for instance, in case $k=1$ the term (*) becomes:

$$\begin{aligned}
&- a\left[\mathcal{N}_1\left(df - e^2\right) - \mathcal{N}_2\left(bf - ce\right) + \mathcal{N}_3\left(be - cd\right)\right]\\
&+ b\left[\mathcal{N}_1\left(bf - ce\right) - \mathcal{N}_2\left(af - c^2\right) + \mathcal{N}_3\left(ae - bc\right)\right]\\
&- c\left[\mathcal{N}_1\left(be - cd\right) - \mathcal{N}_2\left(ae - bc\right) + \mathcal{N}_3\left(ad - b^2\right)\right]\\
=&\mathcal{N}_1\left[-adf + ae^2 + b^2 f - bce - bce + c^2 d\right]\\
=&\mathcal{N}_1 \det h.
\end{aligned}$$

This means the contribution of the $k=1$ term in (3.72) vanishes. Similarly one can show that also for $k=2$ and $k=3$ the contribution vanishes. Plugging this result into (3.71), we thus find:

$$\sqrt{-g} = \sqrt{\mathcal{N}^2 h} = \mathcal{N}\sqrt{h}. \tag{3.73}$$

3.3.4 Rewriting the Lagrangian

We've done all the calculations needed for rewriting the Lagrangian (3.43), so let's do it, using (3.65) and (3.73) we find:

$$\begin{aligned}
\mathcal{L}_G &= \frac{1}{2}\sqrt{-g}\left[R - \partial_\mu\phi\partial^\mu\phi - 2V(\phi)\right]\\
&= \frac{1}{2}\sqrt{h}\mathcal{N}\left[{}^3R + K_{ij}K^{ij} - K^2 + \mathcal{N}^{-2}\dot{\phi}^2 - 2V(\phi)\right],
\end{aligned} \tag{3.74}$$

where we assumed $\phi = phi(t)$. For the sake of compactness we won't plug in (3.68) yet, but let's keep in mind that K_{ij} contains a term proportional to $\dot{h}_{ij}$. We see that this reparameterized Lagrangian has a very important feature, namely that it doesn't contain time derivatives to the lapse function and the shift vector. This is very nice, since we've therefore proven that they are non-dynamical variables, and thus that their equations of motion should correspond to constraint equations, just like in the case of Electrodynamics. We will see that $\mathcal{N}$ and $\mathcal{N}_i$ act has Lagrangian multipliers in Hamiltonian formalism.

3.3.5 GR Hamiltonian

For the sake of simplicity, let us consider just the rewritten Hilbert-Einstein Lagrangian:

$$\mathcal{L}_G = \frac{1}{2}\sqrt{h}\mathcal{N}\left[{}^3R + K_{ij}K^{ij} - K^2\right]. \tag{3.75}$$

The canonical momenta are given by:

$$\pi_{\mathcal{N}} = \frac{\partial \mathcal{L}_G}{\partial \dot{\mathcal{N}}} = 0, \tag{3.76}$$

$$\pi_{\mathcal{N}_i} = \frac{\partial \mathcal{L}_G}{\partial \dot{\mathcal{N}}_i} = 0, \tag{3.77}$$

$$\pi^{ab} = \frac{\partial \mathcal{L}_G}{\partial \dot{h}_{ab}} = \sqrt{h}(K^{ab} - Kh^{ab}), \tag{3.78}$$

where we've made use of the (contracted version of) equation (3.68). Indeed we see that the momenta corresponding to the lapse function and the shift vector vanish. The Hamiltonian density thus becomes:

$$\begin{aligned} \mathcal{H}_G &= \pi^{ab}\dot{h}_{ab} - \mathcal{L}_G \\ &= -\sqrt{h}\mathcal{N}{}^3R + \frac{\mathcal{N}}{\sqrt{h}}\left[\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2\right] + 2\pi_{ab}{}^3\nabla^a\mathcal{N}^b \\ &= \sqrt{h}\left\{\mathcal{N}\left[-{}^3R + \frac{\pi_{ab}\pi^{ab}}{h} - \frac{\pi^2}{2h}\right] - 2\mathcal{N}_b\left[{}^3\nabla_a\left(\frac{\pi^{ab}}{\sqrt{h}}\right)\right] + {}^3\nabla_a\left(\frac{2\mathcal{N}_b\pi^{ab}}{\sqrt{h}}\right)\right\}, \end{aligned} \tag{3.79}$$

where we've made use of (3.68) and (3.78), and with π we mean $\pi_a{}^a$. Note that the last term of (3.79) is a total divergence, so it's only a boundary term, and we will discard it. Note that $\mathcal{N}$ and $\mathcal{N}_a$ indeed act as Lagrangian multipliers, and that the corresponding constraint equations are:

$$-{}^3R + \frac{\pi_{ab}\pi^{ab}}{h} - \frac{\pi^2}{2h} = 0, \tag{3.80}$$

and

$${}^3\nabla_a\left(\frac{\pi^{ab}}{\sqrt{h}}\right) = 0. \tag{3.81}$$

This brings us to the end of the discussion on the Hamiltonian formalism. We've identified the constraint equations of general relativity, and therefore we're now able to solve these equations to get rid of the nonphysical degrees of freedom.

Chapter 4

The Power spectrum

As was shown in the previous chapter, the 10 degrees of freedom of general relativity split up in 6 dynamical degrees of freedom and 4 gauge degrees of freedom. We found four constraint equations (3.80), (3.81) corresponding to the gauge degrees of freedom. The remaining 6 degrees of freedom are written as a symmetric 3x3 tensor called $h(t)_{ij}$, which is the metric of a space slice Σ_t. We're interested in how perturbations of this metric h_{ij} behave. The power spectrum for these perturbations will be calculated during inflation ($\epsilon << 1$). As we will find out the solution simplifies when we write down approximations for late and early times.

4.1 Decomposition theorem

Generally all 6 degrees of freedom of h_{ij} couple to one another, this means that the equation of motion of every degree of freedom contains terms proportional to all the other degrees of freedom. However, if one only writes the equations up to second order there will be decoupling. This is sometimes referred to as **decomposition theorem**. The proof of this theorem is, although very laborious, quite straightforward. It is done for instance in Weinbergs book Cosmology (2008). Essentially it boils down to this: One can decompose the perturbations of the metric h_{ik} in scalar perturbations, vector perturbations and tensor perturbations.

$$\delta h = \delta S + \delta V + \delta T. \tag{4.1}$$

Each of these types of perturbations have their own physical meaning. The tensor perturbations for instance correspond to gravitational waves, while the scalar perturbations are interpreted as gravitational potentials. Because the equations of motion of the different types of perturbations don't talk to each other, they can be treated separately. That is to say, we can safely set every type of perturbation to zero without changing the equation of motion of the others. Note that setting certain variables to zero is **not** the same as fixing a gauge! I emphasize this point since in most of the literature out there does call this gauge fixing.

Anyway, for our purposes we're only interested in the gravitational potential, and thus we set all perturbations to zero except for the scalar mode:

$$h_{ij} = a(t)^2(1 + 2\zeta(t))\delta_{ij}, \tag{4.2}$$

where $a(t)$ is the expansion coefficient as defined in (2.1). Note that we've just removed 5 out of 6 dynamical degrees of freedom!

4.2 Calculating the Spectrum

4.2.1 Solving constraint equations

Now it's time to also remove the four gauge degrees of freedom. If we include the inflationary scalar field in the constraint equations, then they can be written as:

$${}^3\nabla_i(K^{ij} - h^{ij}K) = 0, \tag{4.3}$$

and

$${}^3R - 2V - K_{ij}K^{ij} + K^2 - \frac{\dot{\phi}^2}{\mathcal{N}^2} = 0. \tag{4.4}$$

Solving these constraint equations to first order and plugging them back in the action corresponding to Lagrangian (3.74) gives us:

$$S = \frac{1}{2}\int d^4x\, a\, e^{\zeta}(1 + \frac{\dot{\zeta}}{H})\left[-4\partial^2\zeta - 2(\partial\zeta)^2 - 2Va^2e^{2\zeta}\right] + a^3e^{3\zeta}\frac{1}{1+\frac{\dot{\zeta}}{H}}\left[-6(H+\dot{\zeta})^2 + \dot{\phi}^2\right]. \tag{4.5}$$

Now by taylor expanding and using the equations of motion for the scalar field (2.10), this can be written up to second order as follows:

$$S = \frac{1}{2}\int d^4x \frac{\dot{\phi}^2}{H^2}\left[a^3\,\dot{\zeta}^2 - a\,(\partial\zeta)^2\right]. \tag{4.6}$$

4.2.2 Equation of Motion

Now we need to get the equation of motion from this action Using the slow roll approximation as stated in section 2.2, and the definition of ϵ (2.14) we find that:

$$\frac{\dot{\phi}^2}{H^2} = \left(\frac{dV}{d\phi}\right)^2\frac{1}{9H^4} = M_{\text{pl}}^4\left(V\frac{dV}{d\phi}\right)^2 = 2M_{\text{pl}}^2\epsilon. \tag{4.7}$$

In conformal time ($dt = ad\eta$) the action becomes:

$$S = \frac{1}{2}\int d^3x(ad\eta)(2M_{\text{pl}}^2\epsilon)a\left\{(\zeta')^2 - (\vec{\partial}\zeta)^2\right\}, \tag{4.8}$$

where $'$ means differentiation with respect to conformal time ∂_η. We can now make the following substitution:

$$\zeta = \frac{v}{\sqrt{2\epsilon} a M_{\rm pl}}, \tag{4.9}$$

and assuming that ϵ is constant:

$$\dot{\zeta} = \frac{1}{\sqrt{2\epsilon} M_{\rm pl}} \left(\frac{v'}{a} - \frac{va'}{a^2} \right) = \frac{1}{\sqrt{2\epsilon} M_{\rm pl} a} \left(v' - vH_c \right), \tag{4.10}$$

where and H_c is the Hubble parameter defined through a derivative to conformal time. After performing this substitution, the action becomes:

$$\begin{aligned} S &= \frac{1}{2} \int d^3x d\eta \left\{ (v' - H_c v)^2 - (\vec{\partial} v)^2 \right\} && (4.11) \\ &= \frac{1}{2} \int d^3x d\eta \left\{ (v')^2 - 2H_c v v' + {H_c}^2 v^2 - (\vec{\partial} v)^2 \right\}, && (4.12) \end{aligned}$$

then by noting that:

$$\int d\eta \left\{ -2H_c v v' \right\} = -\int d\eta \left\{ \frac{d}{d\eta}(v^2) H_c \right\} = \underbrace{-\int d\eta \frac{d}{d\eta} \left\{ v^2 H_c \right\}}_{\text{boundary term}} + \int d\eta v^2 H_c', \tag{4.13}$$

the action can be written as:

$$S = \frac{1}{2} \int d^3x d\eta \left\{ (v')^2 - (\vec{\partial} v)^2 + ({H_c}^2 + H_c') v^2 \right\}, \tag{4.14}$$

where we've discarded the boundary term. Varying this action with respect to v gives us the equation of motion:

$$\frac{\delta S}{\delta v} = 0 \Rightarrow -v'' + \vec{\partial}^2 v + (H_c^2 + H_c') v = 0. \tag{4.15}$$

4.2.3 Solving the EOM in terms of ϵ

We assume we can write the scale factor as follows:

$$a(\eta) = a_0 \eta^\beta, \tag{4.16}$$

and it thus follows that:

$$H_c \equiv \frac{a'}{a} = \beta \eta^{-1}. \tag{4.17}$$

Now we'd like to write β in terms of the slow roll parameter ϵ. Using the fact that we can write the slow roll parameter ϵ as follows:

$$\epsilon = -\frac{\dot{H}}{H^2}, \tag{4.18}$$

we can make the following calculation:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{-a^2}{H_c^2}\frac{1}{a}\left(\frac{H_c'}{a} - \frac{H_c a'}{a^2}\right) = -\frac{H_c'}{H_c^2} + 1. \tag{4.19}$$

Using (4.17) we can now relate β to ϵ as follows:

$$-\frac{H_c'}{H_c^2} = \frac{\beta\eta^2}{\beta^2\eta^2} = \beta^{-1} \Rightarrow \beta = \frac{1}{\epsilon - 1}. \tag{4.20}$$

The equation of motion for v, as given in (4.15) can now be written as follows:

$$-v'' + \vec{\partial}^2 v + \left(\frac{2-\epsilon}{\eta^2(1-\epsilon)^2}\right) v = 0. \tag{4.21}$$

Since this differential equation is linear, the different modes don't couple, and we can write the equation for every k-mode seperately:

$$v_k'' + \left(k^2 - \frac{2-\epsilon}{\eta^2(1-\epsilon)^2}\right) v_k = 0. \tag{4.22}$$

This differential equation is of the form:

$$\frac{d^2y}{dx^2} + \frac{2p+1}{x}\frac{dy}{dx} + (a^2x^{2r-2} + \beta^2x^{-2})y = 0, \tag{4.23}$$

which is a transformed version of the Bessel differential equation given by Bowman (1958). It's solution is given by:

$$y = x^{-p}\left[C_1 J_{q/r}\left(\frac{a}{r}x^r\right) + C_2 Y_{q/r}\left(\frac{a}{r}x^r\right)\right], \tag{4.24}$$

where $q \equiv \sqrt{p^2 - \beta^2}$; C_1, C_2 are constants and $J_n(x)$, $Y_n(x)$ are Bessel functions of the first and second kind. The solution for (4.22) is found by setting:

$$p = -\frac{1}{2}, \quad r = 1, \quad a = k, \quad \beta = \pm\frac{\sqrt{\epsilon - 2}}{1-\epsilon}, \quad q^2 - \frac{1}{4} = \frac{2-\epsilon}{(1-\epsilon)^2}. \tag{4.25}$$

Furthermore for an expanding universe we have to choose $x = -\eta$. Upon doing so we find:

$$v_k(\eta) = \sqrt{-\eta}\left[\alpha_k J_q(-k\eta) + \beta_k Y_q(-k\eta)\right]. \tag{4.26}$$

where α_k and β_k are constants. Now it's convenient to rewrite (4.26) in terms of Hankel functions of the first and second kind, which are defined as follows:

$$H_n^{(1)}(z) \equiv J_n(z) + iY_n(z), \tag{4.27}$$

$$H_n^{(2)}(z) \equiv J_n(z) - iY_n(z). \tag{4.28}$$

Since these functions are linearly independent, they form an equally good basis for the solution of (4.22) as the Bessel functions. We thus write:

$$v_k(\eta) = \sqrt{-\eta}\left[\alpha_k H_q^{(1)}(-k\eta) + \beta_k H_q^{(2)}(-k\eta)\right]. \tag{4.29}$$

4.2.4 The Power Spectrum

The Power Spectrum of quantum perturbations is related to the two-point correlation function as follows:

$$\xi(r) \equiv < 0|\hat{\zeta}(\vec{x},\eta)\hat{\zeta}(\vec{y},\eta)|0 > \equiv \int_0^\infty \frac{dk}{k} P(k,\eta) \frac{\sin(kr)}{kr}. \tag{4.30}$$

So first of all we need to quantize the solution that we found in the previous section.

The Lagrangian density corresponding to (4.14) is given by:

$$\mathcal{L} = \frac{1}{2}(v')^2 - \frac{1}{2}(\vec{\partial}v)^2 + ({H_c}^2 + H_c')v^2. \tag{4.31}$$

Using the definition of canonical momentum (3.5) we find:

$$\pi_v \equiv \frac{\partial \mathcal{L}}{\partial v'} = v', \tag{4.32}$$

and therefore the the Hamiltonian becomes:

$$H = \int \{\pi_v v' - \mathcal{L}\} \tag{4.33}$$

$$= \int d^3x \left\{ \frac{1}{2}{\pi_v}^2 + \frac{1}{2}(\vec{\partial}v)^2 - (H_c' + {H_c}^2)v^2 \right\}. \tag{4.34}$$

Now we're going to quantize this system using canonical quantization. As usual we demand the following commutation relation:

$$[\hat{v}(\vec{x},\eta), \hat{\pi}_v(\vec{y},\eta)] = i\hbar\delta^3(\vec{x}-\vec{y}). \tag{4.35}$$

As mentioned before, the different k-modes don't couple, since the EOM is linear. It is therefore useful to expand the operators $\hat{v}(\vec{x},\eta)$ and $\hat{\pi}_v(\vec{x},\eta)$ in terms of creation and annihilation operators $\hat{a}_k^\dagger$ and $\hat{a}_k$:

$$\hat{v}(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^3} \left[e^{i\vec{k}\cdot\vec{x}} v(k,\eta)\hat{a}(\vec{k}) + h.c. \right], \tag{4.36}$$

and

$$\hat{\pi}_v(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^3} \left[e^{i\vec{k}\cdot\vec{x}} v'(k,\eta)\hat{a}(\vec{k}) + h.c. \right], \tag{4.37}$$

where the h.c. stands for "hermitean conjugate", and $v(k,\eta)$ is a fourier coeficient, determined by the classical solution. For the creation and annihilation operators we demand the following commutation relation:

$$\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')\right] = (2\pi)^3\delta^3(\vec{k}-\vec{k}') \tag{4.38}$$

Now to make the commutators $[v, \pi_v]$ and $\left[a, a^\dagger\right]$ consistent, we need to put a restriction on the normalization of $v(k,\eta)$. This can be seen as follows, we demand that:

$$i\hbar\delta^3(\vec{x}-\vec{y}) = < 0| \left[\hat{v}(\vec{x},\eta), \hat{\pi}_v(\vec{y},\eta)\right] |0 > . \tag{4.39}$$

Upon plugging in the expansions for $\hat{v}$ and $\hat{\pi}_v$, we find:

$$=< 0| \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{i(\vec{k}_1\cdot\vec{x}-\vec{k}_2\cdot\vec{y})} v(k_1,\eta)v'^*(k_2,\eta) \left[\hat{a}(\vec{k}_1), \hat{a}^\dagger(\vec{k}_2)\right] - e^{-i(\vec{k}_1\cdot\vec{x}-\vec{k}_2\cdot\vec{y})} v'(k_1,\eta)v^*(k_2,\eta) \left[\hat{a}(\vec{k}_2), \hat{a}^\dagger(\vec{k}_1)\right] |0>,$$

where we're using the fact that the part of the commutator where $\hat{a}(\vec{k})$ acts first on the vacuum vanishes. Plugging in the demanded commutation relations for $\hat{a}$ and $\hat{a}^\dagger$, we find:

$$=< 0| \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{i(\vec{k}_1\cdot\vec{x}-\vec{k}_2\cdot\vec{y})} v(k_1,\eta)v'^*(k_2,\eta)(2\pi)^3\delta^3(\vec{k}_1-\vec{k}_2) - e^{-i(\vec{k}_1\cdot\vec{x}-\vec{k}_2\cdot\vec{y})} v'(k_1,\eta)v^*(k_2,\eta)(2\pi)^3\delta^3(\vec{k}_1-\vec{k}_2)|0>,$$

$$=< 0| \int \frac{d^3k_1}{(2\pi)^3} e^{i\vec{k}_1\cdot(\vec{x}-\vec{y})} v(k_1,\eta)v'^*(k_1,\eta) - e^{-i\vec{k}_1\cdot(\vec{x}-\vec{y})} v'(k_1,\eta)v^*(k_1,\eta)|0>,$$

$$= \delta^3(\vec{x}-\vec{y}) < 0| \int \frac{d^3k}{(2\pi)^3} W\left[v(k,\eta), v^*(k,\eta)\right]|0> . \tag{4.40}$$

Here $W[a,b]$ is called the "Wronskian". In case of two functions (there are also Wronskians for more than two) it's defined as follows:

$$W[a,b] \equiv (ab' - a'b) \tag{4.41}$$

We see that we need:

$$W\left[v(k,\eta), v^*(k,\eta)\right] = i, \tag{4.42}$$

for the commutators $[v, \pi_v]$ and $\left[a, a^\dagger\right]$ to be consistent. Now we're able to calculate the spectrum. We change variables back, from $\hat{v}$ to $\hat{\zeta}$, and obtain:

$$\begin{aligned}
\xi(r) &= < 0|\hat{\zeta}(\vec{x},\eta)\hat{\zeta}(\vec{y},\eta)|0 >= \frac{1}{2\epsilon M_p^2 a^2} < 0|\hat{v}(\vec{x},\eta)\hat{v}(\vec{y},\eta)|0 > \\
&= \frac{1}{2\epsilon M_p^2 a^2} \int \frac{d^3k d^3k'}{(2\pi)^6} < 0| \left[v(k,\eta)e^{i\vec{k}\cdot\vec{x}}\hat{a}(k) + h.c.\right] \left[v(k',\eta)e^{i\vec{k}'\cdot\vec{y}}\hat{a}(k') + h.c.\right]|0> \\
&= \frac{1}{2\epsilon M_p^2 a^2} \int \frac{d^3k d^3k'}{(2\pi)^6} < 0| \underbrace{\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')\right]}_{=(2\pi)^3\delta^3(\vec{k}-\vec{k}')} |0 > v(k,\eta)v^*(k',\eta)e^{i(\vec{k}\cdot\vec{x}-\vec{k}'\cdot\vec{y})} \\
&= \frac{1}{2\epsilon M_p^2 a^2} \int \frac{d^3k}{(2\pi)^3} |v(k,\eta)|^2 e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \\
&= \frac{1}{2\epsilon M_p^2 a^2} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \int_0^\infty \frac{d|k|}{(2\pi)^3} |k|^2 |v(k,\eta)|^2 e^{i|k|r\cos\theta} \\
&= \frac{2}{2\epsilon M_p^2 a^2} \frac{(2\pi)}{(2\pi)^3} \int_0^\infty d|k||k|^2|v(k,\eta)|^2 \int_{-1}^1 d(\cos\theta) e^{i|k|r\cos\theta} \\
&= \frac{1}{2\epsilon M_p^2 a^2} \frac{1}{(2\pi)^2} \int_0^\infty d|k||k|^2|v(k,\eta)|^2 \frac{\sin(kr)}{kr}
\end{aligned} \tag{4.43}$$

And we thus recognize the powerspectrum to be:

$$P(k,\eta) = \frac{|k|^3|v(k,\eta)|^2}{4\pi^2 a^2 M_p^2 \epsilon}, \tag{4.44}$$

in terms of the slow roll parameter, and the solution to the equation of motion. The solution for the equation of motion was given in terms of Hankel functions of the first and second kind. Now as it stands, equation (4.44) isn't particularly illuminating, since Hankel functions aren't easy to work with. It is therefore usefull to consider the asymptotic behaviour of the Hankel functions, meaning either it's behaviour at late times ($|k\eta| << 1$) or at early times ($|k\eta| \to \infty$). We'll see that in both these limits we can greatly simplify the solutions $v_k(\eta)$.

4.2.5 Early times

We are going to consider solutions for early times, or "subhubble limit" ($|k\eta| \to \infty$). In this limit the Bessel functions $J_m(x)$ and $Y_m(x)$ have the following asymptotic behavior:

$$J_m(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \qquad x >> |m^2 - \frac{1}{4}|, \tag{4.45}$$

$$Y_m(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \qquad x >> 1. \tag{4.46}$$

Using our knowledge of the asymptotic behaviour of the Bessel functions, we can write the asymptotic behavior of the Hankel funtions as follows:

$$H_n^{(1),(2)}(z) \sim \sqrt{\frac{2}{\pi z}} \exp\left\{\pm i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right\}. \tag{4.47}$$

So, using (4.29), in early times the solution of (4.22) can be written as:

$$v_k(\eta) \to \frac{\alpha_k}{\sqrt{2k}} e^{-ik\eta} + \frac{\beta_k}{\sqrt{2k}} e^{ik\eta} \qquad (k\eta \to \pm\infty), \tag{4.48}$$

where by convention, the constants changed as follows: $\alpha_k \to (-\pi)^{-1/2}\alpha_k, \beta_k \to (-\pi)^{-1/2}\beta_k$. We choose the constants to be as follows: $\alpha_k = 1$,$\beta_k = 0$, which corresponds to the Bench Davis vacuum. We thus find:

$$v(k,\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}, \tag{4.49}$$

and accordingly, the power spectrum as defined in (4.44) now reduces to:

$$P(k,\eta) = \frac{k^2}{8\pi^2 M_p^2 a^2 \epsilon}. \tag{4.50}$$

4.2.6 Late times

Now we're going to deal with the other limit, namely the "super hubble limit" ($|k\eta| << 1$). We thus need to know how Hankel functions behave in this regime. First of all let me introduce the Neumann function, which expresses the Bessel function of the second kind in terms of the Bessel function of the first kind:

$$Y_\alpha(x) = \frac{J_\alpha \cos(\alpha\pi) - J_{-\alpha}}{\sin(\alpha\pi)}. \tag{4.51}$$

Using the Neumann function and relations (4.27) and (4.28), we can write the following identity for the Hankel fuctions:

$$H_\alpha^{(1)(2)}(x) = J_\alpha(x) \pm \frac{i}{\sin(\alpha\pi)} \left[\cos(\alpha\pi) J_\alpha(x) - J_{-\alpha}(x)\right]. \tag{4.52}$$

The asymptotic behavior ($|k\eta| << 1$) of the Bessel function of the first kind we can obtain from it's Taylor expansion around $x = 0$:

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}, \tag{4.53}$$

where $\Gamma(x)$ is the Euler gramma function. If we now only take the first (leading order) term into account, we can write for the Hankel functions:

$$\begin{aligned} H_\alpha^{(1)(2)}(z) \rightarrow (1 \pm i\cot(\alpha\pi)) \left(\frac{z}{2}\right)^{\alpha} &\left[\frac{1}{\Gamma(\alpha+1)} - \ldots\right] \\ &\mp \frac{i}{\sin(\alpha\pi)} \left(\frac{z}{2}\right)^{-\alpha} \left[\frac{1}{\Gamma(1-\alpha)} - \ldots\right]. \end{aligned} \tag{4.54}$$

So, what is the value of α? Well, if we look at (4.29) and (4.25), we see that we can write it in terms of the slow roll parameter ϵ:

$$\alpha = q = \frac{3-\epsilon}{2(1-\epsilon)}. \tag{4.55}$$

We thus see that:

$$\begin{aligned} \alpha \in [3/2, \infty) \quad &for \quad \epsilon \in [0, 1) \\ \alpha \in (-\infty, -1/2) \quad &for \quad \epsilon \in (1, \infty) \end{aligned} \tag{4.56}$$

Let us assume now that $\epsilon \approx 0$, which corresponds to in inflationary epoch, and that α is thus positive. Because we're also assuming that the argument of the Hankel function, $z = -k\eta$, is very small, we can neglect the first term in (4.54). The approximation then becomes:

$$H_\alpha^{(1)(2)}(-k\eta) \approx \mp \frac{i}{\sin(\alpha\pi)} \left(\frac{2}{-k\eta}\right)^{\alpha} \left[\frac{1}{\Gamma(1-\alpha)}\right], \tag{4.57}$$

and thus:

$$H_\alpha^{(1)} = H_\alpha^{(2)*}. \tag{4.58}$$

Now that we've found a workable expression for the Hankel function, we can plug it in the solution (4.29), and calculate the corresponding $|v_k|^2$

$$\begin{aligned}|v_k(\eta)|^2 &= -\eta\left[|\alpha_k|^2|H_q^{(1)}(-k\eta)|^2 + |\beta_k|^2|H_q^{(2)}(-k\eta)|^2\right] \\ &\quad + \eta\left[\alpha_k\beta_k^* H_q^{(1)}(-k\eta)H_q^{(2)*}(-k\eta) + \alpha_k^*\beta_k H_q^{(1)*}(-k\eta)H_q^{(2)}(-k\eta)\right] \qquad (4.59)\end{aligned}$$

$$\begin{aligned}|v_k(\eta)|^2 &= -\eta\left[|\alpha_k|^2 + |\beta_k|^2 - \alpha_k^*\beta_k - \alpha_k\beta_k^*\right]|H_q^{(1)}(-k\eta)|^2 \\ &= -\eta|\alpha_k - \beta_k|^2|H_q^{(1)}(-k\eta)|^2 \qquad (4.60)\end{aligned}$$

The power spectrum becomes:

$$P(k,\eta) \approx \frac{-|k|^3\eta}{4\pi^2 a^2 M_p^2\epsilon}|\alpha_k - \beta_k|^2 \frac{1}{\sin(\alpha\pi)^2}\left(\frac{2}{-k\eta}\right)^{2\alpha}\left[\frac{1}{\Gamma(1-\alpha)^2}\right] \qquad (4.61)$$

This equation can be simplified using the Euler reflection formula:

$$\Gamma(1-\alpha)\Gamma(\alpha) = \frac{\pi}{\sin(\pi\alpha)}, \qquad (4.62)$$

and the relation:

$$\eta^{-1} = -aH(1-\epsilon), \qquad (4.63)$$

which can be deduced from (4.17) and (4.20). We obtain:

$$P(k,\eta) \approx \frac{|k|^2|\alpha_k - \beta_k|^2}{2\pi^4 a^2 M_p^2\epsilon}\left(\frac{2Ha(1-\epsilon)}{k}\right)^{2\alpha-1}\Gamma(\alpha)^2 \qquad (4.64)$$

We thus see that the k-dependance of the power spectrum goes like:

$$P \sim k^{3-2\alpha}. \qquad (4.65)$$

Now it's conventional to define a parameter n_s as follows:

$$P \sim k^{n_s-1}. \qquad (4.66)$$

We'd like to know n_s to first order in ϵ. Taylor expanding $\alpha(\epsilon)$ as defined in (4.55), we find that:

$$\alpha(\epsilon) \approx \frac{3}{2} + \epsilon, \qquad (4.67)$$

so for n_s we find:

$$n_s = 1 - 2\epsilon. \qquad (4.68)$$

We'd also like to know how the power spectrum itself behaves to leading order in ϵ. We expand terms of (4.64) separately:

$$(1-\epsilon)^{2\alpha-1} = 1 - 2\epsilon + O(\epsilon^2), \qquad (4.69)$$

$$\Gamma(\alpha) \approx \Gamma\left(\frac{3}{2}+\epsilon\right) \approx \Gamma\left(\frac{3}{2}\right)\left[1+\epsilon\psi\left(\frac{3}{2}\right)\right]$$
$$= \frac{1}{2}\sqrt{\pi}\left[1+\epsilon(2-2\ln(2)-\gamma_E)\right], \quad (4.70)$$

where ψ is called the **digamma function** and γ_E is the Euler constant. Now the fact that all these expansions have constant terms, shows that for small ϵ we have that:

$$P \sim \frac{1}{\epsilon} \quad (4.71)$$

Chapter 5

Conclusion

As promised in the Introduction, I've showed you how to calculate the power spectrum of perturbations of the gravitational potential during inflation in terms of the slow-roll parameter ϵ. We calculated that for early times ($|k\eta| \to \infty$) the power spectrum is given by:

$$P(k,\eta) = \frac{k^2}{8\pi^2 M_p^2 a^2 \epsilon}. \tag{5.1}$$

For late times we found:

$$P(k,\eta) \approx \frac{|k|^2|\alpha_k - \beta_k|^2}{2\pi^4 a^2 M_p^2 \epsilon} \left(\frac{2Ha(1-\epsilon)}{k}\right)^{2\alpha-1} \Gamma(\alpha)^2. \tag{5.2}$$

After Taylor expanding for small ϵ, we found that to leading order it behaves as:

$$P \sim \frac{1}{\epsilon} \tag{5.3}$$

. Also the deviation from scale invariance for late times was calculated:

$$n_s = 1 - 2\epsilon. \tag{5.4}$$

Chapter 6

Literature List

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