Large Scale Structure Formation

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Outline

What is Structure Formation?

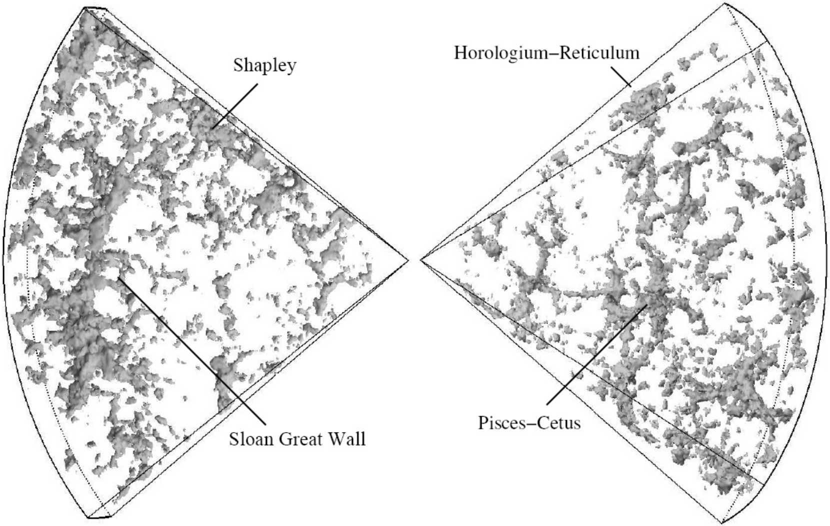
- Where do large structures in our universe come from?
- Quantum perturbations in an inflationary era.
- Initial conditions/ Today's observation

Intuitively: gravitational instability: overdense regions tend to grow.

$$\ddot{\delta} + [\mathit{Pressure} - \mathit{Gravity}]\,\delta = 0 \tag{1}$$

Typical overdensity: 1 in 10^5.

2dF Galaxy Survey



What do we want to calculate? Part 1

How do (quantum) perturbations grow during inflation?

- Obtain a classical field theory (GR + Inflation)

$$\mathcal{L}_G = \frac{1}{2}\sqrt{-g}\left[R - \partial_\mu\phi\partial^\mu\phi - 2V(\phi)\right] \qquad M_{Pl}^{-2} = 1 \quad (2)$$

- Fixing the Gauge by going to ADM Formalism (splitting space and time).
- Put restrictions on the metric (scalar mode).

What are we going to calculate? Part 2

- Find (perturbative) solutions, (up to second order).
- Different fourier modes $\delta(k, t)$
- Quantize the solutions.
- Calculate the powerspectrum.
  Powerspectrum:

$$< \delta(\vec{k})\delta(\vec{k}') >= (2\pi)^3 P(\vec{k})\delta(\vec{k} - \vec{k}') \tag{3}$$

- Hamiltonian formalism is closely related to ADM formalism
- Hamiltonian formalism naturally seperates physical/ unphysical degrees of freedom
- ADM formalism gives an intuitive interpretation.

Hamiltonian Formalism for a Field Theory 1

1) Split the Lorentzian manifold $\mathcal{M}$ into space and time, $\mathbb{R} \times S$.

- $t \in \mathbb{R}$ is the time-parameter
- Timelike vectorfield t^α *flow of time*

$$t^\alpha \nabla_\alpha t = 1 \tag{4}$$

- Spacelike submanifold $\Sigma_t \subset \mathcal{M}$, with a metric h_{ij}.

$$h_{ij} v^i v^j \geq 0 \qquad \forall v \in T_p\Sigma \tag{5}$$

Hamiltonian Formalism for a Field Theory 2

- Σ_0 and Σ_t connected by t^α *picture*
- Timelike unit-vectorfield n^α, normal to Σ

$$g_{\mu\nu} n^\mu n^\nu = -1 \tag{6}$$

$$g_{\mu\nu} n^\mu v^\nu = 0 \qquad \forall v \in T_p\Sigma \tag{7}$$

- Decomposition of any vectorfield $v \in T_p\mathcal{M}$

$$v^\alpha = \underbrace{-(g_{\mu\nu} v^\mu n^\nu) n^\alpha}_{\parallel} + \underbrace{(v^\alpha + (g_{\mu\nu} v^\mu n^\nu) n^\alpha)}_{\perp} \tag{8}$$

Hamiltonian Formalism for a Field Theory 3

2) Define a configurationspace of (tensor) fields q, instantaneously describing the configuration of the field ψ.
3) Define corresponding momenta for the fields π.
4) Specify the functional $H[q, \pi]$ on Σ_t, called the Hamiltonian.

$$H = \int_{\Sigma_t} \mathcal{H} \tag{9}$$

Canonical Momentum:

$$\pi_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \tag{10}$$

Hamiltonian Density:

$$\mathcal{H}(q, \pi) = \sum_i \pi_i \dot{q}_i - \mathcal{L} \tag{11}$$

Gauge Degrees of Freedom/ Singular Systems

Non invertible Hessian matrix:

$$H_{kl} \equiv \frac{\partial \pi_k}{\partial \dot{q}^l} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^k \partial \dot{q}^l} \tag{12}$$

Hamiltonian for singular systems:

$$H = H_{can} + \sum_n \chi_n \phi_n \tag{13}$$

- Lagrangian multipliers follow from the projection of the field along the normal vector.

$$\phi_n = \frac{\delta H}{\delta \chi_n} \tag{14}$$

Dynamical Equations of Motion/ Fixing the Gauge

- Dynamical equations follow from derriving w.r.t. physical variables:

$$\dot{q} \equiv \frac{\delta H}{\delta \pi} \tag{15}$$

$$\dot{\pi} \equiv -\frac{\delta H}{\delta q} \tag{16}$$

- Choosing a gauge is equivalent to choosing a value for χ_n.

Maxwell Field Equations

Ordinary Maxwell field equations (c = 1):

- Constraint Equations (2):

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0 \tag{17}$$

- Dynamical Equations (2):

$$\vec{\nabla} \times \vec{E} = \partial_t \vec{B} \qquad \vec{\nabla} \times \vec{B} = \partial_t \vec{E} \tag{18}$$

EM Lagrangian

- EM Lagrangian:

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \tag{19}$$

- A^μ describes the system instantaneously (q).

$$n^\alpha = (1, 0, 0, 0) \tag{20}$$

$$\eta_{\mu\nu} = diag(-1, 1, 1, 1) \tag{21}$$

- Decomposing A^μ:

$$A^\alpha = \underbrace{-(\eta_{\mu\nu}A^\mu n^\nu)n^\alpha}_{\perp} + \underbrace{(A^\alpha + (\eta_{\mu\nu}A^\mu n^\nu)n^\alpha)}_{\parallel}$$

$$= (V, \vec{A}) \tag{22}$$

New Variables/ Cannonical Momenta

- The Lagrangian in the projected variables:

$$\mathcal{L}_{EM} = \frac{1}{2}\left(\dot{\vec{A}} + \vec{\nabla} V\right)^2 - \frac{1}{2}\left(\vec{\nabla} \times \vec{A}\right)^2 = \frac{1}{2}\vec{E}^2 - \frac{1}{2}\vec{B}^2 \qquad (23)$$

- Unphysical variable (normal projection):

$$\pi_V = \frac{\partial \mathcal{L}}{\partial \dot{V}} = 0 \qquad (24)$$

- Physical Variable (projection on the plane):

$$\pi_{\vec{A}} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} = \dot{\vec{A}} + \vec{\nabla} V \equiv -\vec{E} \qquad (25)$$

EM Hamiltionian/ Equations of Motion

- EM Hamiltonian:

$$\begin{aligned}\mathcal{H}_{EM} &= \vec{\pi}\cdot\dot{\vec{A}} - \mathcal{L}_{EM} = -\vec{E}\cdot\left(-\vec{E}-\vec{\nabla}V\right) - \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 \\ &= \frac{1}{2}\vec{\pi}\cdot\vec{\pi} + \frac{1}{2}\vec{B}\cdot\vec{B} - \vec{\pi}\cdot\vec{\nabla}V \\ &= \frac{1}{2}\vec{\pi}\cdot\vec{\pi} + \frac{1}{2}\vec{B}\cdot\vec{B} + V(\vec{\nabla}\cdot\vec{\pi}) - \vec{\nabla}\cdot(V\vec{\pi}) \end{aligned} \tag{26}$$

- Constraint Equations:

$$\frac{\delta H_{EM}}{\delta V} = \vec{\nabla}\cdot\vec{E} \tag{27}$$

- Dynamical Equations

$$\dot{\vec{A}} = \frac{\delta H_{EM}}{\delta\vec{\pi}} = \vec{\pi} - \vec{\nabla}V = -\vec{E} - \vec{\nabla}V \tag{28}$$

$$\dot{\vec{\pi}} = -\dot{\vec{E}} = -\frac{\delta H_{EM}}{\delta\vec{A}} = -\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) \tag{29}$$

Fixing the Gauge

- The Gauge invariance of the theory is:

$$\vec{A} \to \vec{A} + \vec{\nabla}\lambda \qquad V \to V - \frac{\partial \lambda}{\partial t} \tag{30}$$

- Fixing the Gauge is done by fixing the Lagrangian multiplier!

What is Inflation?

Consider perturbations in an expanding background.

$$ds^2 = -dt^2 + a^2(t)\, dx_i dx^i \tag{31}$$

The condition for an accelerated inflation is:

$$\frac{\partial^2 a}{\partial t^2} > 0 \tag{32}$$

Second Friedman equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3c^2}(\rho + 3p) + \frac{\Lambda}{3} \tag{33}$$

Condition becomes:

$$p < -\frac{\rho}{3} \tag{34}$$

Scalar Field Inflation 1

Action for a time dependent Scalar field:

$$S = -\frac{1}{2}\int d^4x a(t)^3 \left[g^{tt}(\partial_t\phi)(\partial_t\phi) + 2V(\phi)\right] \tag{35}$$

$$\delta_\phi S = -\frac{1}{2}\int d^4x a(t)^3 \left[-2(\partial_t\phi)(\partial_t\delta\phi) + 2\frac{\partial V}{\partial\phi}\delta\phi\right] \tag{36}$$

$$\delta_\phi S = -\int d^4x a(t)^3 \left[\partial_t^2\phi + 3\frac{\dot{a}(t)}{a(t)}\partial_t\phi + \frac{\partial V}{\partial\phi}\right]\delta\phi \tag{37}$$

$$0 = \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial\phi} \tag{38}$$

Scalar Field Inflation 2

Energy and Pressure:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \qquad p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{39}$$

First Friedmann equation:

$$H^2 = \frac{1}{3M_{Pl}^2}(\rho) = \frac{1}{3M_{Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{40}$$

Slow Roll parameters, satisfied for slow roll approximation:

$$H^2 \simeq \frac{V}{3M_{pl}^2} \qquad 3H\dot{\phi} \simeq V' \tag{41}$$

$$\epsilon \equiv \frac{{M_{Pl}}^2}{2}\left(\frac{V'}{V}\right)^2 \qquad \eta \equiv \frac{{M_{Pl}}^2 V''}{V} \tag{42}$$

Einstein Field Equations

- The Vacuum Einstein Field Equations read:

$$G_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu} = 0 \tag{43}$$

- Hard to extract physics from it.
- Field variable is $g_{\mu\nu}$, 10 equations, 6 dynamical, 4 constraints, space-time splitting makes this easy to see.
- Reparameterising $(g_{\mu\nu}) \Rightarrow ({}^{(3)}h_{ij}, \mathcal{N}_i, \mathcal{N})$

ADM Formalism/ How to split the Metric

- Spacetime interval:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{44}$$

- In the new variables

$$ds^2 = -\mathcal{N}^2 dt^2 + h_{ij}\left(dx^i + \mathcal{N}^i dt\right)\left(dx^j + \mathcal{N}^j dt\right) \tag{45}$$

- Metric:

$$g_{\mu\nu} = \begin{bmatrix} -\mathcal{N}^2 + \mathcal{N}^k\mathcal{N}_k & \mathcal{N}_j \\ \mathcal{N}_i & h_{ij} \end{bmatrix} \tag{46}$$

- $\mathcal{N}$ and $\vec{\mathcal{N}}$ will be the nonphysical parameters, and appear as Largrangian multipliers. (like V)

Decomposition of t^α/ Shift Vector/ Lapse Function

- Decomposition of t^α is non-trivial:

$$t^\alpha = \underbrace{-(g_{\mu\nu} t^\mu n^\nu) n^\alpha}_{\perp} + \underbrace{(t^\alpha + (g_{\mu\nu} t^\mu n^\nu) n^\alpha)}_{\parallel} \tag{47}$$

- Lapse Function

$$\mathcal{N} = -(g_{\mu\nu} t^\mu n^\nu) n^\alpha \tag{48}$$

- Shift Vector

$$\vec{\mathcal{N}} = (t^\alpha + (g_{\mu\nu} t^\mu n^\nu) n^\alpha) \tag{49}$$

Rewriting the Lagrangian

- Lagrangian:

$$\mathcal{L}_G = \frac{1}{2}\sqrt{-g}\,[R - \partial_\mu\phi\partial^\mu\phi - 2V(\phi)] \qquad M_{Pl}^{-2} = 1 \tag{50}$$

What do we need to do?

- Rewrite the Ricci scalar, in terms of the Ricci scalar of the sub-manifold Σ and the way it's embedded in $\mathcal{M}$.
- Rewrite the determinant of g in terms of the determinant of h and the lapse function.

Covariant Derivative/ Intrinsic/ Extrinsic Curvature

- Covariant Derivative:

$$\nabla_{\mathbf{u}} v^{\mu} = \underbrace{-g_{\alpha\beta}(\nabla_{\mathbf{u}} v^{\alpha}, n^{\beta}) n^{\mu}}_{\perp} + \underbrace{(\nabla_{\mathbf{u}} v^{\mu} + g_{\alpha\beta}(\nabla_{\mathbf{u}} v^{\alpha}, n^{\beta}) n^{\mu})}_{\parallel} \tag{51}$$

$u, v \in Vect(\Sigma)$

- Extrinsic Curvature $\Rightarrow$ Variation of tensor field. n^{α} normal to Σ_t

$$\nabla_{\mathbf{u}} v = K(u, v) n +^{3} \nabla_{\mathbf{u}} v \tag{52}$$

- Intrinsic Curvature $\Rightarrow$ Riemann tensor $\Rightarrow [\nabla_{\alpha}, \nabla_{\beta}]$
- Intrinsic Curvature of Σ and $\mathcal{M}$ are related through the Extrinsic curvature!

Extrinsic curvature

- Extrinsic curvature was not what we expected:

$$-g_{\alpha\beta}(\nabla_{\boldsymbol{u}} v^{\alpha}, n^{\beta})n^{\mu} = (K_{ij}u^i v^j)n^{\mu} = K(u,v)n^{\mu} \tag{53}$$

- Metric compatebility:

$$0 = \nabla_{\boldsymbol{u}}(g_{\alpha\beta}n^{\alpha}v^{\beta}) = g_{\alpha\beta}(\nabla_{\boldsymbol{u}} v^{\alpha}, n^{\beta}) + g_{\alpha\beta}(v^{\alpha}, \nabla_{\boldsymbol{u}} n^{\beta}) \tag{54}$$

- Intuitive picture of Extrinsic Curvature:

$$K(u,v) = g_{\alpha\beta}(v^{\alpha}, \nabla_{\boldsymbol{u}} n^{\beta}) \tag{55}$$

Intrinsic Curvature

- Take a point $p \in \Sigma$, local coordinates (x^0, x^1, x^2, x^3)
- $x^0 = t, \partial_0 = \partial_t$ and $\partial_1, \partial_2, \partial_3$ are tangent to Σ at p.
- Riemann tensor:

$$R^{\alpha}{}_{ijk} = R(\partial_i, \partial_j)\partial_k dx^{\alpha} = \left[\nabla_i, \nabla_j\right] \partial_k dx^{\alpha} \qquad (56)$$

- *Gauss-Codazzi equations* follow from taking the commutator:

$$R(\partial_i, \partial_j)\partial_k = ({}^3\nabla_i K_{jk} - {}^3\nabla_j K_{ik})n + ({}^3R^m{}_{ijk} + K_{jk}K_i{}^m - K_{ik}K_j{}^m)\partial_m \qquad (57)$$

- Codazzi equation follows from taking an innerproduct with dx^m:

$$R^m{}_{ijk} = {}^3R^m{}_{ijk} + K_{jk}K_i{}^m - K_{ik}K_j{}^m \qquad (58)$$

The Lagrangian

- Lagrangian:

$$\mathcal{L}_G = \frac{1}{2}\sqrt{-g}\left[R - \dot{\phi}^2 - 2V(\phi)\right] \qquad M_{Pl}^{-2} = 1 \tag{59}$$

$$\sqrt{-g} = \mathcal{N}\sqrt{h} \tag{60}$$

- Contracting the Codazzi equation:

$$R =^3 R + K_{ij}K^{ij} - K^2 \tag{61}$$

- Action becomes:

$$S = \frac{1}{2}\int d^4x\sqrt{h}\mathcal{N}\left[{}^3R + K_{ij}K^{ij} - K^2 + \mathcal{N}^{-2}\dot{\phi}^2 - 2V(\phi)\right] \tag{62}$$

Constraints/ Dynamical EOM's

$$S = \frac{1}{2}\int d^4x\sqrt{h}\mathcal{N}\left[{}^3R + K_{ij}K^{ij} - K^2 + \mathcal{N}^{-2}\dot{\phi}^2 - 2V(\phi)\right]$$

Where:

$$K_{ij} = \frac{1}{2}\mathcal{N}^{-1}\left[\dot{h}_{ij} -{}^3\nabla_i\mathcal{N}_j -{}^3\nabla_j\mathcal{N}_i\right] \tag{63}$$

- $\mathcal{N}$ and $\mathcal{N}_i$ are indeed unphysical and correspond to constraints:

$$\pi_{\mathcal{N}} = \frac{\delta\mathcal{L}}{\delta\dot{\mathcal{N}}} = 0 \Rightarrow \frac{\delta\mathcal{L}}{\delta\mathcal{N}} ={}^3 R + K_{ij}K^{ij} - K^2 - \mathcal{N}^{-1}\dot{\phi}^2 - 2V(\phi) = 0 \tag{64}$$

$$\pi_{\mathcal{N}_i} = \frac{\delta\mathcal{L}}{\delta\dot{\mathcal{N}}_i} = 0 \Rightarrow \nabla_i\left[K_j{}^i = \delta^i_j E\right] = 0 \tag{65}$$

Canonical Momentum/ Hamiltonian Density

$$S = \frac{1}{2}\int d^4x\sqrt{h}\mathcal{N}\left[{}^3R + K_{ij}K^{ij} - K^2 + \mathcal{N}^{-2}\dot{\phi}^2 - 2V(\phi)\right]$$

Where:

$$K_{ij} = \frac{1}{2}\mathcal{N}^{-1}\left[\dot{h}_{ij} -^3 \nabla_i\mathcal{N}_j -^3 \nabla_j\mathcal{N}_i\right] \tag{66}$$

- Canonical momentum to h_{ij}:

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = \sqrt{h}(K^{ij} - Kh^{ij}) \tag{67}$$

- Hamiltonian Density (Quantum Gravity):

$$\mathcal{H}_G = \pi^{ij}\dot{h}_{ij} - \mathcal{L}_G \tag{68}$$

Fixing the Gauge/ Solving Constraints

- Fixing the Gauge, modes decouple in second order:

$$h_{ij} = a^2 \left[(1+2\zeta)\delta_{ij} + \gamma_{ij}\right] \qquad \partial_i \gamma_{ij} = 0 \qquad \gamma_{ii} = 0 \tag{69}$$

- Solving the constraints and expanding upto second order:

$$S = \frac{1}{2}\int d^4x\, a\, e^{\zeta}\left(1 + \frac{\dot{\zeta}}{H}\right)\left[-4\partial^2\zeta - 2(\partial\zeta)^2 - 2Va^2e^{2\zeta}\right] + a^3 e^{3\zeta}\frac{1}{1+\frac{\dot{\zeta}}{H}}\left[-6(H+\dot{\zeta})^2 + \dot{\phi}^2\right] \tag{70}$$

- Using background EOM's:

$$S = \frac{1}{2}\int d^4x \frac{\dot{\phi}}{H^2}\left[a^3\,\dot{\zeta}^2 - a\,(\partial\zeta)^2\right] \tag{71}$$

Equation of motion

- Free Field Theory
- Fourier Expansion:

$$\zeta(t,x) = \int \frac{d^3k}{(2\pi)^3} \zeta_k(t) e^{i\vec{k}\cdot\vec{x}} \tag{72}$$

- Equation of Motion:

$$\frac{\delta L}{\delta \zeta} = -\frac{d\left(a^3 \frac{\dot{\phi}}{H^2} \dot{\zeta}_k\right)}{dt} - a\frac{\dot{\phi}}{H^2} k^2 \zeta_k = 0 \tag{73}$$

- Quantization:

$$\hat{\zeta}_{\vec{k}}(t) = \zeta_k^{cl}(t)\hat{a}^\dagger_{\vec{k}} + \zeta_k^{cl*}(t)\hat{a}_{-\vec{k}} \tag{74}$$

Solving the EOM

$$\frac{\delta L}{\delta \zeta} = -\frac{d\left(a^3 \frac{\dot{\phi}}{H^2}\dot{\zeta}_k\right)}{dt} - a\frac{\dot{\phi}}{H^2}k^2\zeta_k = 0$$

- Early Times $\Rightarrow$ Large k $\Rightarrow$ WKB approximation.
- Late Times $\Rightarrow$ Small k $\Rightarrow$ Solutions go to a constant.
- Example in de Sitter space (conformal time):

$$S = \frac{1}{2}\int \frac{1}{\eta^2 H^2}\left[(\partial_\eta f)^2 - (\partial f)^2\right] \tag{75}$$

- Normalized Solution, $\eta \in (-\infty, 0)$.

$$f_k^{cl} = \frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta} \tag{76}$$

The Correlation Function

$$f_k^{cl} = \frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta}$$

- Correlation Function:

$$\begin{aligned} < 0|\hat{f}_{\vec{k}}(\eta)\hat{f}_{\vec{k}'}(\eta)|0 > &= (2\pi)^3\delta^3(\vec{k} + \vec{k}')|f_k^{cl}(\eta)|^2 \\ &= (2\pi)^3\delta^3(\vec{k} + \vec{k}')\frac{H^2}{2k^3}(1 + k^2\eta^2) \\ &\sim (2\pi)^3\delta^3(\vec{k} + \vec{k}')\frac{H^2}{2k^3} \qquad (77) \end{aligned}$$

In Slow Roll Inflation

- We can approximate the solution in inflation, near horizon crossing by the de Sitter solution. We let:

$$f = \frac{\dot{\phi}}{H}\zeta \tag{78}$$

- Substitution in the previously obtained solution:

$$< 0|\zeta_{\vec{k}}(t)\zeta_{\vec{k}'}(t)|0 > \sim \frac{1}{2k^3}\frac{H_*^4}{\phi_*^2} \tag{79}$$

(Deviation from) Scale Invariance

- Spectrum is nearly scale invariant ($P \sim k^{-3}$).
- Deviation from scale invariance is measured by n_s:

$$< 0|\zeta_{\vec{k}}(t)\zeta_{\vec{k}'}(t)|0 >\sim \frac{1}{2k^3}\frac{H_*^4}{\phi_*^2} \sim k^{-3+n_s} \tag{80}$$

- At horizon crossing we have: $aH \sim k$ so $ln(k) = ln(a) + ln(H)$.
- Calculation on the board leads to the deviation of scale invariance in slow roll parameters:

$$n_s = 2(\eta - 3\epsilon) \tag{81}$$

Questions?

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