UTRECHT UNIVERSITY
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Dark Energy and the Cosmological Constant

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1 Introduction

As much as physics has advanced in the 20th century and the beginning of the current one, reaching astounding accuracy when comparing modern theories to experimental results, we can still not account for what seems to be 75% of the energy budget of the universe (see figure 1), and hence its somewhat mystic name *Dark Energy*. If we consider for a moment the remaining 25%, about 80% of that is dark matter of which we also do not know very much yet. Actually, since the remaining 20% of that remaining energy is baryonic matter, of which a complete baryogenesis theory is yet to be developed, we are pretty much left with 5×10^{-5} of the energy budget, radiation, which is well understood. Since photons are neutral, it does not have an anti-partner, and thus no asymmetry must be explained.

The above knowledge of the division of energy between the various components of the universe has been obtained by fairly recent experiments. Specifically, the dark energy density follows from recent Type Ia Supernovae and CMBR Anisotropies observations, which show that the universe is flat (or very close to being flat) and accelerating. This, of course, requires some fundamental assumptions about the universe in which we live in:

1 It obeys Einstein's theory of General Relativity

2 It is homogeneous and isotropic on cosmological scale

These findings, it will be shown later, imply this vast amount of unexplained en-

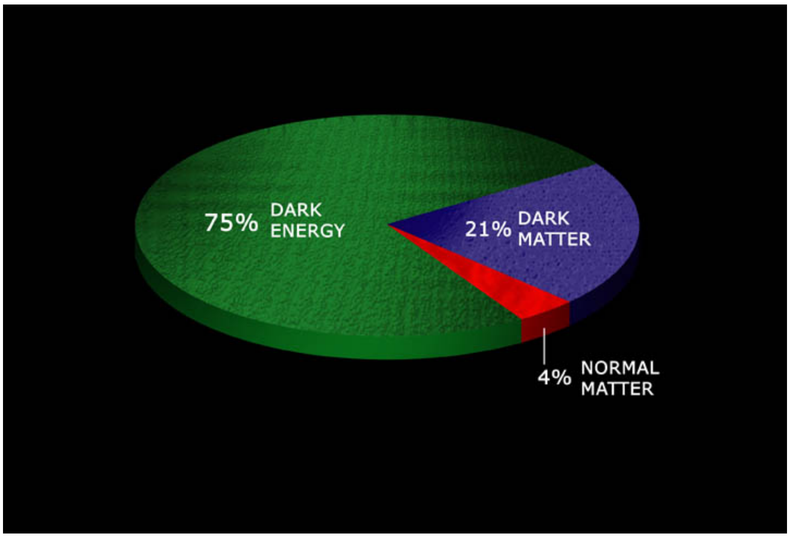


Figure 1: Energy budget of the current universe

ergy. The simplest and so far most successful explanation of dark energy is the Cosmological Constant. This is a term that can be added to Einstein's equations, in their most general form, and which manages to account for this energy and it's properties, up to the error bars of current measurements. Unfortunately, there are some fundamental problems with this constant which will be discussed later on.

After a brief review of Einstein's theory applied to cosmology, this paper will describe in detail the method and results of the aforementioned experiments; followed by a discussion on vacuum energy, a possible explanation for this energy; and lastly a short review of Quintessence, an alternative to the cosmological constant

1.1 Brief review of General Relativity

According to Einstein's theory of general relativity, the universe is a four dimensional manifold upon which a metric $g_{\mu\nu}(x)$ is defined (for a full introduction to GR see [1]). This metric is used to measure distances on the manifold, $ds^2 = g_{\mu\nu}(x)x^{\mu}x^{\nu}$. This manifold, in the general case, is curved. The curvature is characterized by Riemann's tensor, $R^{\rho}_{\mu\sigma\nu}$, from which by contraction of the first and third indices we can derive Ricci's tensor, $R_{\mu\nu}$. Further contraction of the two remaining indices gives rise to Ricci's scalar R.

Having defined this manifold, we are ready to describe the dynamics of this system. First, we must allow some matter to live on this manifold. To this end, we can postulate some Lagrangian density describing some theory of matter. For this Lagrangian density to be compatible with General Relativity, the action derived from it must be invariant to general coordinate transformations. This can be achieved by replacing the normal four dimensional measure with an invariant one:

$$d^4x \rightarrow d^4x\sqrt{-g} \tag{1}$$

where $g = detg(x)$. An action would now take the form:

$$S_{theory} = \int d^4x\sqrt{-g}\mathcal{L}_{theory} \tag{2}$$

Varying this action by the various fields which make up the Lagrangian would yield the normal equations of motion of these fields. In this case, the added $\sqrt{-g}$ term does not make a difference since it does not depend on the fields. But the metric is in itself a dynamic field, and varying the action with respect to the metric gives rise to the matter energy momentum tensor which will shortly become part of Einstein's equations:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}(x)} S_{theory} \tag{3}$$

Having described the matter in our theory using Lagrangian formalism, we seek a Lagrangian description of the curvature of the manifold. The Ricci scalar is the only independent scalar which is constructed from the metric, and has a maximum of second order derivatives of the metric. Hilbert suggested this to be the Lagrangian describing curvature. Putting this into an action and varying it gives:

$$\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}(x)} S_{Hilbert} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}(x)} \int d^4x' \sqrt{-g} \frac{1}{16\pi G} R = \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \tag{4}$$

Finally, summing up the Hilbert Lagrangian and the Lagrangian describing matter, and varying it with respect to the metric, gives Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{5}$$

Einstein's equations determine the interaction between matter described by the fields in $T_{\mu\nu}$ and the metric which describes the curvature of the manifold. Matter curves space-time, and curvature of space-time causes matter to evolve.

1.2 The Cosmological Constant

Actually, (5) is not the most general form we can achieve. A constant can be added to either the Hilbert Lagrangian or to the matter Lagrangian (conventionally with a factor $\frac{1}{8\pi G}$). This constant, Λ, has been christened the *Cosmological Constant*, and appears as an additive constant in the modified Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{6}$$

In this form of writing, it is as if the cosmological constant is an additional term in the Einstein tensor ($G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$). It can be interpreted as an additional constant of nature which afflicts the universe with an intrinsic curvature, since the Minkowski metric is not a solution to Einstein's equations in vacuum for a non-zero cosmological constant. An alternative interpretation would be to move this constant to the other side of the equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{7}$$

In this form, the cosmological constant becomes part of the energy-momentum tensor, and must be explained by the matter theory. It is interesting to note that in flat space physics, a constant added to a Lagrangian has no physical effects since it does not appear in the equations of motion. In curved space, because of the factor $\sqrt{-g}$ appearing in front of the Lagrangian, it does appear in Einstein's equations and contributes to the interaction between matter and space-time.

1.3 Friedmann's Equations and an Accelerating Universe

The common cosmological model assumes a homogeneous and isotropic universe. Under these assumptions, the form of the metric is highly constrained, and reduces to only one free function of the coordinate t.

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right] \tag{8}$$

$a(t)$ is called the scale factor of the universe and characterizes its size (and hence its evolution). k is the curvature of the universe. Its possible values are -1,0 or 1, corresponding to an open, flat or closed universe, respectively. This form of the metric is called the Robertson-Walker metric.

In addition, the energy-momentum tensor, when forced to be homogeneous and isotropic in its rest frame, takes the much simplified form of a perfect fluid:

$$T^{\mu}_{\nu} = diag(-\rho, p, p, p) \tag{9}$$

When inserted into Einstein's equations, one gets two independent equations, these are Friedmann's equations:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{10}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{11}$$

In addition, conservation of the energy momentum tensor yields a third equation which turns out to be dependent on the two Friedmann equations:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\dot{\rho} + p) = 0 \tag{12}$$

A quick glance at (11) shows us that in order to have an accelerating universe, we must either have a non-zero cosmological constant or some other form of a perfect fluid with negative pressure obeying $p < -\frac{1}{3}\rho$. We can choose to treat the cosmological constant as a perfect fluid with $T^{\Lambda}_{\mu\nu} = -\rho^{\Lambda} g_{\mu\nu}$, in this case we get that its pressure is $p = -\rho$. Any one of these forms of matter is known as *Dark Energy*.

As a final note on the Friedmann equations, by dividing by H^2 the first equation can be rewritten as:

$$1 = \Omega_i + \Omega_\Lambda + \Omega_k$$
$$\Omega_i = \frac{8\pi G}{3H^2}\rho_i,\ \Omega_\Lambda = \frac{\Lambda}{3H^2},\ \Omega_k = \frac{-k}{a^2 H^2}$$

Ω_k is the energy density of k divided by the critical density $\rho_{critical} = \frac{3H^2}{8\pi G}$, the energy density at which the universe is flat. Given a specific curvature k of the universe, this equation becomes a constraint on the sum of energy densities of the various matter components of the universe.

1.4 A brief history of the cosmological constant

To end this introduction we give a brief review of the history of the cosmological constant [2]. Einstein was the first to postulate the existence of such a constant. At his times, it was believed (and fitted observations at the time) that the universe was static and positively curved. Since his original equations did not allow such a universe, he added the cosmological constant and forced it to have the right value to make the right hand side of (11) vanish. The right hand side of (10) could then be brought to vanish by a proper choice of the scale factor.

1.4 A brief history of the cosmological constant

In 1929 Hubble formulated the Hubble law and showed that it fits the redshift data collected up until then. His law stated that the speed at which galaxies retreat from us is proportional to their distance from us. Assuming the Copernican principle, this pointed at an expanding universe. Einstein then was happy to drop the cosmological constant, a constant he was uncomfortable with in the first place.

Along the years, the cosmological constant was brought back into play several times to accommodate new observations. The last of these times was after the recent discovery that our universe is accelerating. The following chapter will describe the two experiments leading to this phenomenal conclusion.

2 Experimental Evidence

Recent observations of Type Ia Supernovoae and of the CMBR radiation strongly support a picture of a flat universe with a fair amount of Dark Energy causing it to accelerate. At the same time, these observations clearly leave out the possibility of a matter dominated universe as was previously believed to be the case. In this section we discuss the method used to extract this information from these observations.

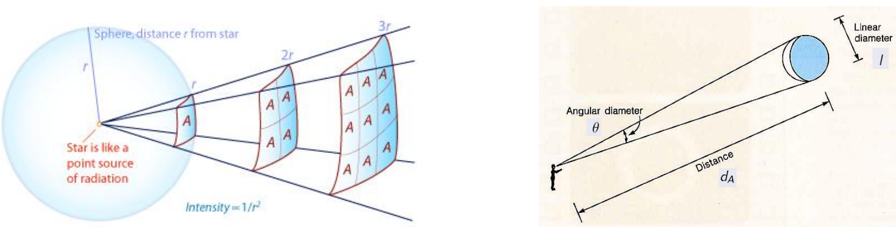
2.1 Luminosity distance and angular-diameter distance

First we must introduce two useful quantities, *luminosity distance* and *angular-diameter distance* (see for example [3]). The luminosity distance can be best understood if we first consider a static flat universe. We then define the luminosity distance d_L by (see also figure 2(a)):

$$F = \frac{S}{4\pi d_L^2} \tag{13}$$

where S is the total energy per unit time emitted by some source of radiation, and L is the flux (energy per unit time per unit area) of energy received by a detector at a distance d_L. Now we can generalize this definition to an arbitrary universe (possibly non-flat) with a scale factor $a(t)$. Recall that the scale factor of the universe at a certain time in history can be related to the redshift of radiation emitted at that time and received today by:

$$z = \frac{a(0)}{a(t)} - 1 \tag{14}$$



(a) luminosity distance (b) angular-diameter distance

Figure 2

where usually we set $a(0) = 1$. If we then use the definition (13) in the general case, then d_L is not equal to the comoving distance r anymore as it would in the above static case. We must take two factors into account: i. the photons are redshifted (for an expanding universe), and so the energy detected is scaled down by a factor $\frac{a(0)}{a(t)}$, ii. the rate of received photons is scaled down by the same factor compared to the rate of emitted photons because physical distances change with time. In total we get that:

$$d_L = \frac{r}{a(z)} = r(1+z) \tag{15}$$

where r is the comoving coordinate first appearing in (8). Its intuitive meaning is that if we could freeze the universe at a moment t and stretch a ruler between two points, the distance would be $a(t)r$.

The luminosity distance is used for measuring distances between us and point objects in the sky. If, on the other hand, we measure the distance to an object with a finite measurable size, it is common to use the angular-diameter distance. In figure 2(b) we see an object of absolute length l, which is the quantity we would like to calculate. When observed, this object has an angular opening of θ. In static and flat space, these two quantities are related to the distance d_A of that object from us by:

$$\theta = \frac{l}{d_A} \tag{16}$$

Again we generalize this relation to the general case. In this case, we would just have to re-scale the comoving coordinate r by the scale factor of the universe at the time light emitted from the far object reaches us today:

$$d_A = r \times a(t) = \frac{r}{1+z} \tag{17}$$

where again we took $a(0) = 1$.

2.2 Type Ia Supernovae

In the end of the 90's, two separate teams were collecting observations of distant Type Ia Supernovae (see [4], [5], [3] for reviews) . The Supernova Cosmology Project published its results first in 1997 [6], and the High-Z Supernova Team collaboration in 1998 [7]. These observations showed with high confidence that the universe was in fact accelerating, i.e. contains a dark energy component.

Type Ia Supernovae provide a good *standard candle* for measuring distances. These occur when a white dwarf, supported mostly by electron degeneracy pressure, accretes matter (e.g. from an explosion of a partner star) and passes the Chandrasekhar limit. At this stage the gravitational force becomes too large and the star begins to collapse rapidly. Within seconds a large part of the matter in the star undergoes fusion, and the star explodes from the burst of energy released. Because of the standard explosion mechanism, these explosions have a typical light curve and can be rather easily identified and used as standard candles.

A large number of nearby supernovae lying in galaxies of known distance have been measured in the past and have been found to have an almost constant brightness. This is true up to about 40%. It has been found [8] that the decay time and the peak intensity are correlated, and this correlation can be used to reduce the 40% uncertainty to 15% (and potentially introduce a systematic error to the data). It has been more recently found [9, 10] that if the measurements are taken in the infrared region, this uncertainty is reduced to 15% intrinsically. It is then assumed that far away supernovae have the same brightness, and thus the absolute luminosity S from (13) is known, and the flux F is measured, hence d_L is known for every supernova event.

We would now like to use this data to calculate the parameters characterizing our universe. We assume that the universe is dominated by two types of energy, non-relativistic matter characterized by the density ρ_M and a cosmological constant Λ (see (13)). While other models of dark energy can be used, in this description we will suffice with a cosmological constant. Using the two Friedmann equations (13) and (11), we can calculate the evolution of the scale factor $a(t)$ for this model. We then follow the path of a photon emitted from the supernova and detected on earth today by following a null geodesic, $ds^2 = 0$. We integrate along the geodesic from emission time t to current time t_0, both along the time coordinate and along the comoving coordinate:

$$\int_t^{t_0} \frac{dt'}{a(t')} = \int_0^r \frac{dr}{(1-kr^2)^{1/2}} \tag{18}$$

The right hand side is a straightforward integral depending on the assumed curvature of the universe. The left hand side depends on the model used. The result is a function $r(z)$, z being the observed redshift. Using our model, the theoretical luminosity distance is:

$$d_L = r(1+z) = H_0^{-1}\left[z + \frac{1}{2}\left(1 + \Omega_{DE} - \frac{1}{2}\Omega_M\right)z^2\right] + \mathcal{O}(z^3) \tag{19}$$

where the right hand side was expanded in powers of z. Currently, supernovae are observed with z above 1 and thus the exact right hand expression must be used. This is done numerically. This function can be fitted to the data to obtain the optimal values for the cosmological constant and the matter density. Figure 3 shows such a fit for 42 supernova events [6] with $z < 1$. The y-axis is plotted in terms of the apparent magnitude m, a quantity proportional to the log of the angular-diameter distance, defined by:

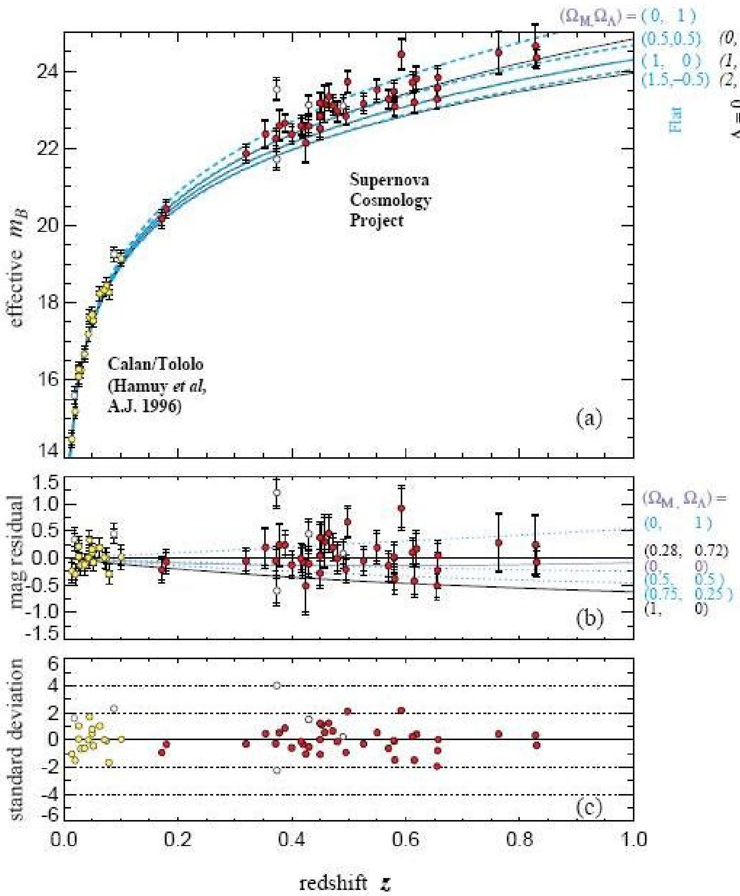


Figure 3: a model of a universe dominated by matter and cosmological constant fitted to the Supernova Cosmology Project data

$$m - M = 5\log_{10}\left(\frac{d_L}{Mpc}\right) + 25 \tag{20}$$

where M is related to the absolute luminosity S in a logarithmic fashion. In this figure, different models of a flat universe ($\Omega_M + \Omega_\Lambda$) are plotted. Figure 4, in turn, shows the preferred values of Ω_M and Ω_Λ. We see that a flat universe with a zero cosmological constant is ruled out with over 99% confidence. Actually it is fairly safe to assume a non-zero cosmological constant. The data also allows a flat universe, in this case the universe is dominated by the cosmological constant by a factor between 1 and 3. These are quite extraordinary results.

Before turning our attention to the CMBR result, which will prove to strengthen this results and place further constrains, we will note a few drawbacks of these

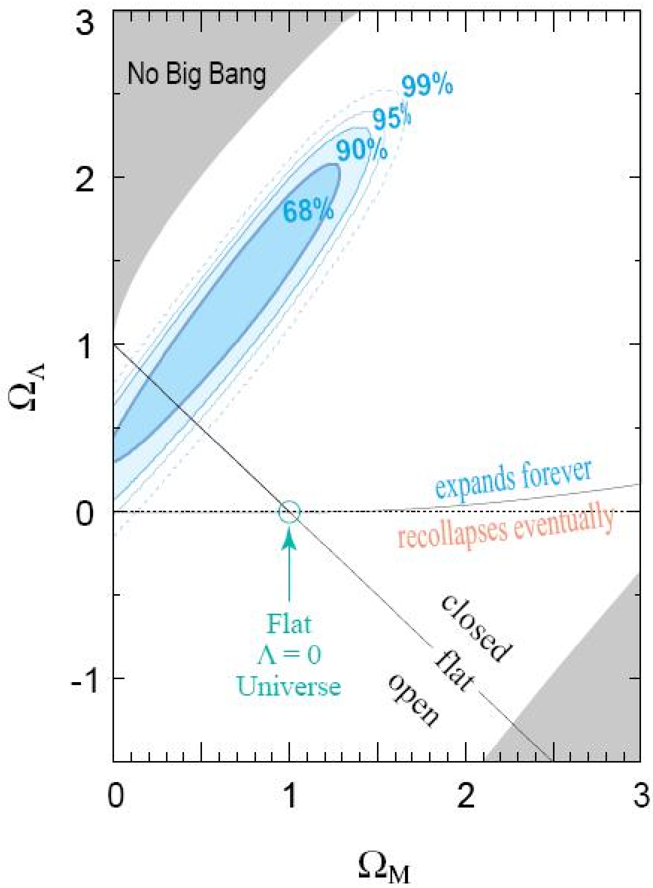


Figure 4: range of values for cosmological constant density and matter density with confidence level contours

measurements. First, as explained above, the supernova are standard candles only up to about 15%. This uncertainty is taken into account in the Supernova Cosmology Project data. Above that, all supernova used to calibrate the standard candle are nearby supernovae, hence ”old” supernovae that occurred at a rather aged universe. It is possible that there are evolution effects (mostly related to metallic composition affecting brightness), and that younger supernovae observed at higher redshifts will have different light curves, thus biasing our results. Currently there is no evidence to support this possibility, and it has been argued that if such evolution effects exist, we should statistically see such younger supernovae also in nearby galaxies.

One more drawback is possible obscuration by dust. Photons traveling through long distances could be absorbed by dust particles on the way. This has been seen to cause dimming and reddening of the incoming signals. Reddening is caused by preferred absorption of the blue light. This effect could lead us to mistake the reddening for redshift. This possibility has been disproved by recent observations of supernovae at a distance of about $z = 2$. According to our model of matter and cosmological constant, at $z \simeq 1$ the matter and cosmological constant are equally abundant. We would then expect, according to the model, that at earlier times, there will be a change in the curve. This change has been observed, and would not have occurred if the reddening would have been caused by dust. So at this point, these two effects are believed to be unimportant.

2.3 CMBR Anisotropies

The *Cosmological Microwave Background Radiation* is a remnant of the far history. After inflation the universe was a thick fluid of ionized matter and radiation, thick enough to have radiation be at thermal equilibrium with matter. As the universe expanded and cooled down, at a calculated value of $z = 1089$, the electrons and protons recombined and the radiation decoupled from the matter. We believe this story to be true because the radiation observed in the sky, almost perfectly isotropic, has the most perfect black body radiation spectrum ever observed (or produced).

In 1992, NASA’s Cosmic Background Explorer (COBE satellite) was first to observe deviations from perfect isotropy. More recent observation have been collected (and still being collected) by NASA’s Wilkinson Microwave Anisotropy Probe (WMAP) satellite [11], [12]. These are shown in figure 5. The graph shows the amplitude of the l-th order Legendre function, when expanding the map of the sky (a sphere) in spherical harmonics and averaging out the angle ϕ. This, then, gives us a measure of the size of structures in the sky, where structures of angular size θ

are related to the l-th moment by $\theta \sim \frac{\pi}{l}$. Peaks at moment l point at the abundance of large structures with an angular opening of θ.

We will now try to better understand the origin of these deviations (see [3], [5], [13]), and thus the structures which appear to have a typical size corresponding to $l \sim 220$ and higher harmonics (as seen from the resonances of the plot in figure 5). It is believed that inflation, by an exponential expansion of quantum fluctuations, induced perturbations in the post-inflation fluid of matter and radiation that are linear in the wave vector k. This means that if we Fourier transform the spatial density of that fluid, we will see that all Fourier components have the same amplitude. We will treat this state as our initial state.

Perturbations in the energy density of this fluid mean that some parts are denser than others. These denser areas can be viewed as potential wells attracting the matter around them. This attractive force has a tendency to make denser areas yet denser. As a result, the pressure in these areas increases and opposes the gravitational force. The compression and decompression of denser areas propagates through the fluid in the form of sound waves with a speed of sound equal approximately to:

$$c_s = \sqrt{\frac{dP}{d\rho}} \approx c[3(1 + 3\Omega_b/4\Omega_R)]^{-1/2} \tag{21}$$

Since the fluid is composed of non relativistic pressureless baryonic matter and relativistic photons with an equation of state $P = \frac{1}{3}\rho$, most of the contribution to

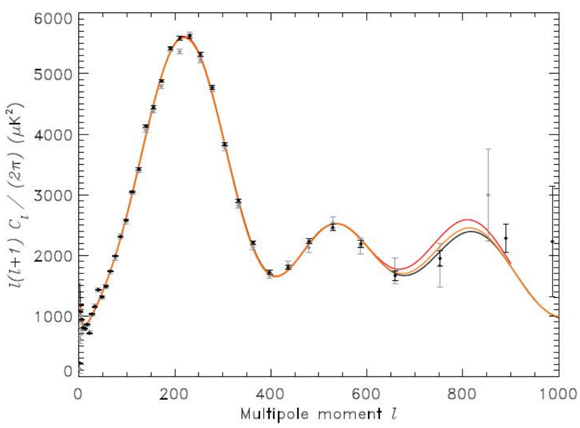


Figure 5: WMAP observations of the CMBR anisotropies

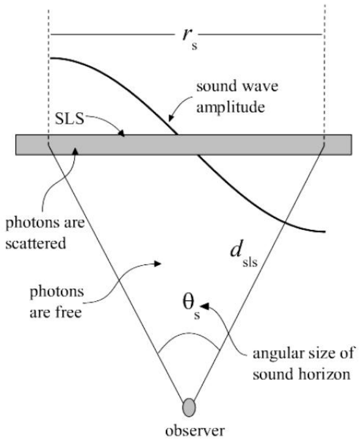


Figure 6: an illustration of how an observer on earth would see the first fundamental mode (picture courtesy of [13])

the speed of sound is attributed to the photons.

As argued above, inflation excited all modes (wavelengths) in the density perturbation, these modes oscillate with time. For simplicity, let us first consider the case of a static fluid. The period of oscillation is related to the speed of sound and the wave length by $T = \frac{\lambda}{c_s}$. Those modes with a period satisfying $T = 2(t_{dec} - t_{infl})n$ ("dec" for decoupling and "infl" for end of inflation), for n being any positive integer, will be at a maximum of oscillation at the time of recombination. Let us consider the fundamental mode corresponding to $n = 1$. Assuming $t_{infl} = 0$, the period of this mode is $T = 2t_{dec}$, and the wavelength is $\lambda_{dec} = 2t_{dec}c_s$. Half the wavelength of this mode is the farthest information could have traveled in the period of time t_{dec}. This is called the sound horizon. Below we will calculate the more realistic sound horizon.

The CMBR anisotropies we observe in the sky today are a frozen image of that time. The first peak (lowest l) observed in the spectrum, corresponds to the largest structure in the sky, which in turn corresponds to this fundamental mode (see figure 6). Since we know the time of decoupling, we know the size of this structure.

Recall now the angular-diameter distance defined in (16):

$$d_A = \frac{r_s}{\theta_s} \tag{22}$$

r_s is the sound horizon in the realistic (dynamic universe) case, and can be calculated by propagating an acoustic wave through the fluid from the end of inflation until recombination. The integral over time can be replaced by an integral over the redshift by solving Friedmann's equations for the fluid. The integral will then also depend on the densities of the constituents of the fluid. Assuming a matter dominated universe, the sound horizon is:

$$r_s(z_{dec}, \Omega_i) \approx \int_0^{t_{dec}} c_s dt \approx \frac{c/\sqrt{3}}{H_0\sqrt{\Omega_M}} \int_{z_{dec}}^{\infty} (1+z)^{-5/2} dz \tag{23}$$

θ_s is what we observe, and thus what we would like to predict theoretically. To accomplish that, we are only left with calculating the angular-diameter distance. To do that, we first recall its relation to the comoving coordinate (17). The distance in terms of the comoving coordinate can be calculated by following a path of a photon, similarly to what was done in the case of the supernova, only now integrating from z_{dec} until today $z = 0$. The angular-diameter distance (denoted as d_{sls} in figure 6) is then:

$$d_A = \frac{r_A}{1 + z_{dec}} \approx \frac{c\Omega_M^{-m}}{H_0} \tag{24}$$

where $m = 1$ for a model with a zero cosmological constant, and $m \approx 0.4$ for a flat universe obeying $\Omega_M + \Omega_\Lambda = 1$. We can now calculate the angular size of the object using (22), or rather calculate the moment l of the fundamental corresponding to this angle:

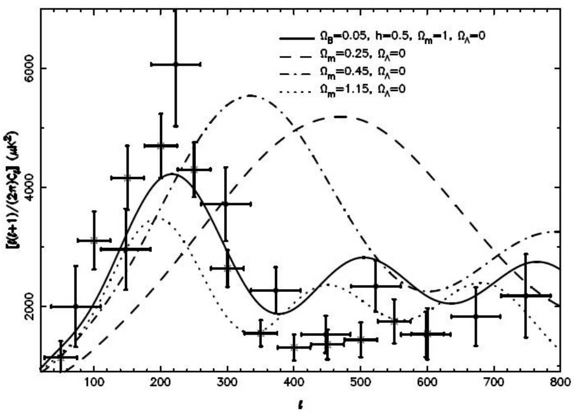


Figure 7: data points are from first results of MAXIMA and BOOMERANG experiments, curves correspond to various models.

$$l_{peak} \approx \frac{\pi}{\theta_s} \approx \frac{d_A}{r_s} \quad \propto \quad \Omega_M^{-1/2} \text{ if } \Omega_\Lambda = 0$$
$$\propto \quad \Omega_M^{0.1} \text{ if } \Omega_k = 0 \tag{25}$$

We see that if the universe is flat, the moment l is roughly constant, while for a universe with zero cosmological constant there is a strong dependence on the matter density.

A slightly more exact calculation would give us the actual value of l and that in the case of a flat universe l is indeed constant:

$$\begin{aligned} l_{peak} &\approx 220\Omega_M^{-1/2} \text{ for } \Omega_{DE} = 0 \\ &\approx 220 \text{ for } \Omega_k = 0. \end{aligned} \tag{26}$$

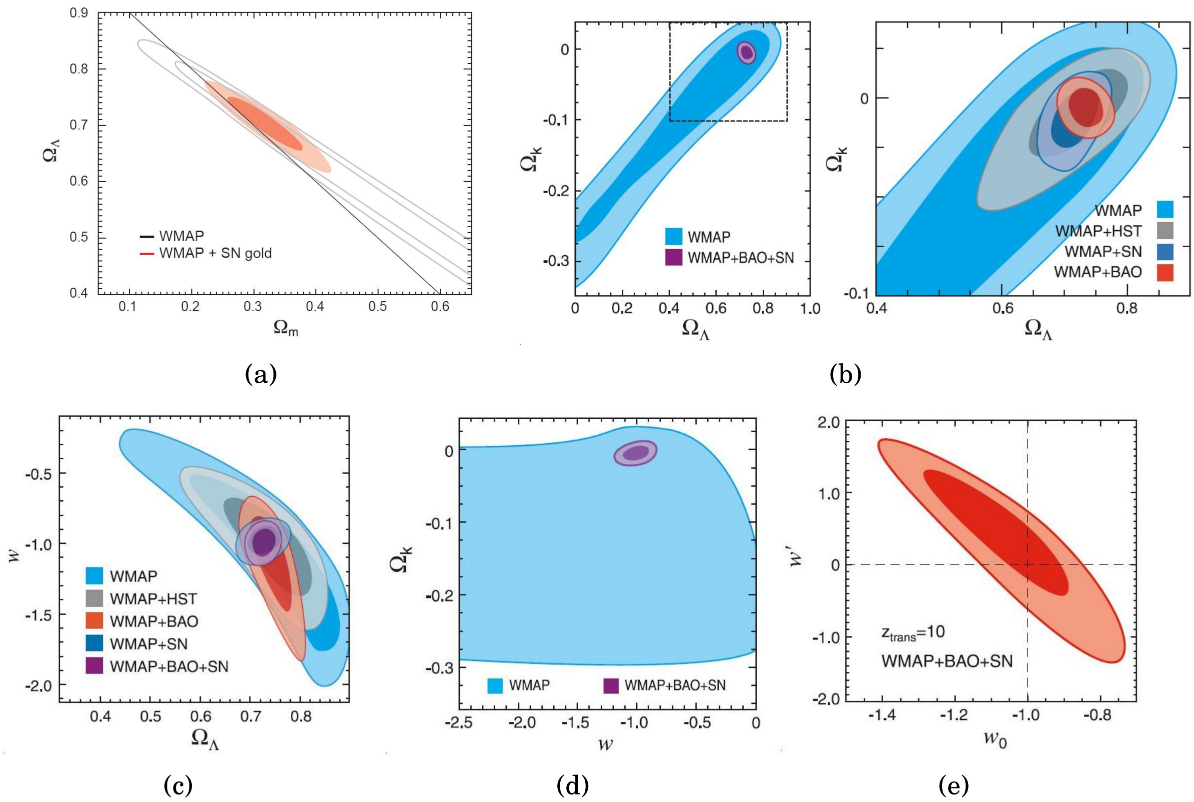


Figure 8: WMAP 5-year results [12], except for 8(a) which is WMAP 3-year results [11]

Figure 7 shows us a comparison of different models to primary data collected by MAXIMA and BOOMERANG data. We see how the first peak, corresponding to the fundamental mode, moves to the left as the matter density increases for the case of a zero cosmological constant universe. The case of a flat universe gives the best fit. We have seen that in this case, the peak location will be the same for all models as long as the sum of densities is 1.

This was an example of how information about the content of the universe can be extracted from the measurements of the CMBR anisotropies. We saw that the CMBR mostly constrains the universe to be flat. WMAP provides us with higher quality results. Figures 8 show us constraints on the parameter space for different combinations of parameters. Each figure contains the constraints based on WMAP only results and a combination of WMAP with other experiments: Type Ia Supernova, Hubble Space Telescope and Baryon Acoustic Oscillations. In figure 8(a) we see that the WMAP favors a flat universe, and together with the Superonova results we also get a good constraint on the densities of the cosmological constant (around 0.7) and matter (around 0.3). Figures 8(c), 8(d), 8(e) allow the equation of state of dark energy ($P = w\rho$) to differ from that of the cosmological constant ($w = -1$). Wee see that in all cases, w is constrained to be very close to -1. Figure 8(e) also constrains the derivative of w in the current universe. We see that a zero derivative, which would then agree with a cosmological constant, fits the constraints.

3 Vacuum Energy

We have seen that the cosmological constant is consistent with experimental evidence. Although being a simple and elegant solution to the problem of dark energy, a few conceptual problems tag along with it. The main one being the question of its origin. We have seen that this constant can be absorbed in the energy-momentum tensor of Einstein's equations, and if this tensor is taken to describe a perfect fluid as in (9), then its equation of state will be:

$$p_\Lambda = w_\Lambda \rho_\Lambda = -\rho_\Lambda \tag{27}$$

Experiments show that the cosmological constant energy density is positive, implying a type of energy with negative pressure. Let us examine what this means. Consider the equation for the relative geodesic acceleration:

$$\nabla \cdot \mathbf{g} = -4\pi G(\rho + 3p) \tag{28}$$

This equation measures the gravitational attraction between two test particles traveling along two nearby geodesics in the presence of matter with density ρ and pressure p. We see that if the pressure becomes less than $-\frac{1}{3}\rho$, i.e. dark energy, the sign of the right hand side of (28) changes. Dark energy is thus a type of energy with repulsive gravity. (28) is a Poisson equation (if g is conservative field), and a positive sign on the right hand side is analogous to the electrostatic case of two identically charged particles.

Since we try to derive the energy-momentum tensor from some field theory, a constant term in a theory's Lagrangian can be attributed to the contribution of the vacuum energy [4]. Let us consider a few simple cases. If we take a classical scalar field Lagrangian:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \tag{29}$$

The energy density derived from this Lagrangian, under the assumption of a homogeneous and isotropic universe, is:

$$\rho(\phi(t)) = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{30}$$

The minimum value of this potential can be identified with the cosmological constant:

$$\rho_\Lambda = \Lambda/(8\pi G) = V_{min} \tag{31}$$

In a quantum field theory, a constant contribution can be extracted from the zero point energy $\frac{1}{2}\hbar\omega_k$ each mode k of the field contributes. The following summation over all modes would give us the total energy contribution of the zero point energies:

$$\rho_{vac} = \frac{1}{2}\hbar \int_{IRcutoff}^{UVcutoff} \frac{d^3\mathbf{k}}{(2\pi)^3}\omega_k \Rightarrow \rho_{vac} = \hbar\frac{k_{cutoff}^4}{16\pi^2} \tag{32}$$

A UV-cutoff is necessary to prevent the integral from blowing up, and hints at the fact that our theory can only be trusted up to certain energies above which we would need a new theory (e.g. GUT). An IR-cutoff is necessary because the low energy (or large distance) modes are affected by the dynamics of the scale factor of the universe and are thus not constant. In simple models, it can be shown that these modes do not contribute significantly to the above sum. It is not generally known, though, what consequences the dynamics of the universe have on these modes. We see then that the vacuum energy is proportional to the fourth power of the cutoff momentum.

3.1 The Smallness Problem

We now reach the problem of the vacuum energy interpretation of the cosmological constant. If, for example, we choose to trust our quantum theory up to the energy corresponding to the reduced Planck mass $M_{Pl} = (8\pi G)^{-1/2} \approx 10^{18} GeV$, normally believed to be the point where a quantum gravity theory becomes significant, then vacuum energy takes the value:

$$\rho_{vac}^{(Pl)} \approx 10^{109} J/m^3 \tag{33}$$

Compare this value to the measured value of the cosmological constant energy density:

$$\rho_\Lambda^{(obs)} \approx 10^{-11} J/m^3 \tag{34}$$

There is a 120 orders of magnitude discrepancy! To reduce our cut-off sufficiently, we would have to assume that our theories are trustworthy only up to an energy of $10^{-12} GeV$. This is an unreasonably low value, considering that particle accelerators today work at energies of hundreds of GeV. This is known as the Smallness problem of the cosmological constant. Supersymmetry is one possible theory that could resolve this problem. Fermions have a negative contribution to the total vacuum energy. In non-supersymmetric theories it is unreasonable that the fermions would exactly cancel the contributions of the bosons to give this small number. In supersymmetry, every particle has a supersymmetric partner which cancel each other. A breaking of this symmetry could theoretically explain the small number. So far, though, no such symmetry has been found in nature.

3.2 The Casimir Effect

In classical physics and in quantum field theories in flat space it is known that the vacuum energy does not have physical meaning since the it drops out of the equations of motion. It has been commented by many authors, though, that the vacuum energy does indeed exist as has been experimentally observed in the *Casimir Effect*. Sean Carroll says in [4], "...And the vacuum fluctuations themselves are very real, as evidenced by the Casimir effect." Steven Weinberg in turns says in [14], "Perhaps surprisingly, it was a long time before particle physicists began seriously to worry about (quantum zero-point fluctuation contributions in Λ) despite the demonstration in the Casimir effect of the reality of zero-point energies".

R.L. Jaffe argues in [15] that the Casimir Effect is not a proof of the reality of vacuum energy fluctuations. The Casimir Effect is a force calculated and observed between two parallel conducting plates in vacuum (see figure 9). The vacuum energy is summed over all space with the presence of the plates (which place boundary conditions on the wave functions). The derivative of this energy with respect to the distance between the plates is non-zero and gives an attractive force between the plates:

$$\frac{F_c}{A} = -\frac{\hbar c \pi^2}{240 a^4} \tag{35}$$

Jaffe argues that Casimir's derivation, as described above, assumes that the plates are ideal conductors and the boundary conditions are such that the waves vanish on the edge of the plates. This is equivalent to taking the electromagnetic coupling constant to infinity, and that is why the coupling constant does not appear in the expression of the force. A more careful calculation shows that the force does depend on the coupling constant and thus cannot be a result of only vacuum fluctuations which are a result of the free particle Lagrangian. In his paper, Jaffe shows how this same result can be reached by summing over only Feynman diagrams which are non-vacuum (have external legs). The above force is indeed reached when taking the limit $e \to \infty$.

3.3 The Coincidence Problem

One last remark on a conceptual problem with the cosmological constant, or in this case, a conceptual problem with dark energy. This is known as the Coincidence Problem. We have seen that the current cosmological model describes an evolving universe where matter energy scales as a^{-3} and dark energy, if taken to be a cosmological constant, is constant. It seems, then, that currently $\Omega_M \approx \Omega_{DE}$. Considering the very different time dependence of both densities, it is quite surprising that we live in this epoch where they are almost equal. A complete theory of dark energy should address this question as well.

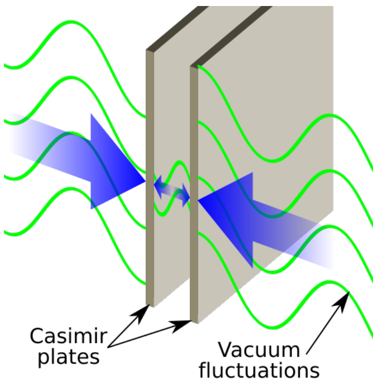


Figure 9: Casimir Effect - force between two parallel plates in vacuum

4 Quintessence

We have so far seen the cosmological constant as a possible description of dark energy. We have seen, though, that any form of energy with an equation of state $p = w\rho$ obeying $w < -\frac{1}{3}$, could explain an accelerating universe. In fact, the w could depend on time in the general case. People have suggested many alternative theories to the cosmological constant until this day. As an example, we will look at the simplest one known as Quintessence [16]. In quintessence we assume a scalar field obeying the following Lagrangian:

$$\mathcal{L}_{quin} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{36}$$

By varying the action with respect to the metric to get the energy-momentum tensor, and assuming a homogeneous and isotropic universe, we get the following energy density and pressure of the scalar field:

$$\begin{aligned} \rho_q(t) &= \frac{1}{2}\dot{\phi}^2 + V \\ p_q(t) &= \frac{1}{2}\dot{\phi}^2 - V \end{aligned} \tag{37}$$

Taking the ratio of these two quantities we can get the coefficient of the equation of state w:

$$w_q = \frac{1 - (2V/\dot{\phi}^2)}{1 + (2V/\dot{\phi}^2)} \tag{38}$$

We immediately notice that if the field obeys $\dot{\phi}^2 \ll V$, then the scalar field reduces to the behavior of a cosmological constant to meet observations. In [17], T. Padmanabhan showed that in fact given any functional evolution of the scale factor $a(t)$, a potential can be produced that will reproduce the required behavior of the universe.

If we take the case of a flat universe with an energy contribution from only the the scalar field, we get the following potential and field:

$$V(t) = \frac{3H^2}{8\pi G}\left[1 + \frac{\dot{H}}{3H^2}\right]$$
$$\phi(t) = \int dt \left[-\frac{\dot{H}}{3\pi G}\right]^{1/2} \tag{39}$$

where $H = \frac{\dot{a}}{a}$. So, if we are interested in a power law expanding universe, $a(t) = a_0 t^n$, solving the above equation for $V(t)$ would give the required potential:

$$V(\phi) = V_0 exp\left(-\sqrt{\frac{2}{n}}\sqrt{8\pi G}\phi\right) \tag{40}$$

If we would like an exponentially expanding universe, $a(t) \propto exp(\alpha t^f)$, $f = \beta/4 + \beta$, $0 < f < 1$, $\alpha > 0$, as a cosmological constant dominated universe will behave, the required potential is:

$$V(\phi) \propto (\sqrt{8\pi G}\phi)^{-\beta}\left(1 - \frac{\beta^2}{6}\frac{1}{8\pi G\phi^2}\right) \tag{41}$$

So we see that quintessence has very little predictive power since every evolution of $a(t)$ can be modeled with an appropriate $V(\phi)$, and these potentials do not follow from a symmetry of nature as for example we derive the potentials in QED or QCD.

Another problem with this theory is that it is not unique. There are other Lagrangians that could produce the same equation of state as quintessence. One such

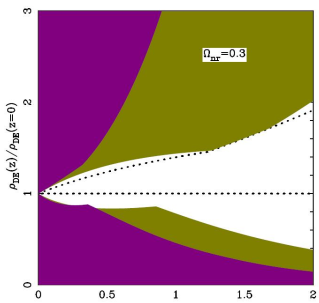


Figure 10: constraints on the possible variation of the dark energy density with redshift [16]

well known Lagrangian is the Tachyonic Lagrangian, which is a field generalization of a relativistic particle:

$$L_{tach} = -V(\phi)[1 - \partial_\mu \phi \partial^\mu \phi]^{1/2} \tag{42}$$

An additional problem is that although quintessence exhibits the flexibility to fit a theory to the observed data, it does not solve the need to explain a cosmological constant, since the potential can have an added term. We are then still left with the task of explaining why this constant must vanish or take on a specific value.

Lastly, there is no current justification for having a time dependent equation of state. Figure 10 shows us constraints on variations of the dark energy density up to $z = 2$. We see there is yet no reason to believe that the dark energy is time dependent, yet this is not excluded.

5 Conclusion

We have seen that two large observational experiments performed in the 90s, Type Ia Supernova and CMBR anisotropies (supported by more recent observations of large scale structure growth) , when combined, strongly favor a (nearly) flat and accelerating universe. These observations point at the existence of a yet unknown form of energy known as Dark Energy, with an odd character of being gravitationally repulsive.

The simplest explanation for these observations so far is a cosmological constant, a tool that has been brought into play a few times in the past in order to resolve misalignments between theory and experiment. So far, the cosmological constant fits all experimental evidence, but these experiments do not rule out other more complicated solutions. Problems with the cosmological constant are mainly conceptual ones, leading to a difficulty in identifying the cosmological constant with vacuum energy for example.

To resolve these conceptual problems and to try to offer deeper understanding, answers are sought in supersymmetric theories, string theories or *dark gravity* theories attempting to modify Einstein's General Relativity theory and to remove the need for dark energy.

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