Rotating Black Holes

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NS-TP501M - Student Seminar

19 November 2008

Overview

- The Kerr-Newman family
  - The metric
  - Symmetries of the spacetime and interpretation of the parameters
  - Singularities of the Kerr metric
  - Lens-Thirring effect
  - Particle trajectories

- The Penrose Process
  - Extracting energy from a black hole
  - Limits on the extraction
  - Super radiance

The Kerr-Newman metric

- The Kerr-Newman metric in Boyer-Lindquist coordinates:

$$ds^2 = -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} dt^2 - 2a\sin^2\theta \frac{(r^2 + a^2 - \Delta)}{\Sigma} dt d\phi + \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right) \sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

Where

$$\Sigma = r^2 + a^2 \cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2 + e^2$$

- The parameters, a and e:

$$a = \frac{J}{M} \qquad e = \sqrt{Q^2 + P^2}$$

- Killing vectors: $$k = \frac{\partial}{\partial t}, \qquad m = \frac{\partial}{\partial \phi}$$

The parameters

- Thus far M, J and Q have just been given a suggestive name
- Q is verified to be the electric charge by using the fact that Q is the conserved charge associated with a conserved *vector* current, considering the asymptotic electric field (radial, ~ Q/r^2) and using Gauss' law
- What about M and J? How do we define energy, mass and angular momentum in a generic spacetime?
- Since energy and momentum are associated with a conserved tensor we cannot define them the same way as e.g. electric charge (photons carry no charge, gravitational waves do carry energy)
- For an asymptotically flat spacetime, one can however use vector fields that are asymptotically Killing to define something called the ADM energy and momentum

ADM Energy and momentum and Komar Integrals

- Linearization of Einstein's equations gives the Pauli-Fierz equation (cf. the presentation on g-waves 2 weeks ago)

- Defining the energy as usual (integral over all of space of T_{00}), and using Gauss' law this becomes:

$$E = \frac{1}{16\pi G} \oint_{\infty} dS_i \left(\partial_j h_{ij} - \partial_i h_{jj} \right)$$

- This can be shown to be equivalent to the following, using that k is asymptotically Killing,

$$E = -\frac{1}{8\pi G} \oint_{\infty} dS_{\mu\nu} \nabla^{\mu} k^{\nu}$$

- However, when the spacetime possesses genuine Killing vectors, one can (do better and) analogously associate to each such vector field (ξ) a so-called Komar integral:

$$Q_{\xi}(V) = \frac{c}{16\pi G} \oint_{\partial V} dS_{\mu\nu} \nabla^{\mu} \xi^{\nu} = \frac{c}{8\pi G} \int_{V} dS_{\mu} \nabla_{\nu} \nabla^{\mu} \xi^{\nu}$$

Komar Integrals(2)

- These charges are of course associated to a conserved current, as they should be:

$$Q_\xi\left(V\right) = \int dS_\mu J^\mu(\xi)$$

$$J^\mu(\xi) = c\left(T^\mu{}_\nu \xi^\nu - \frac{1}{2} T \xi^\mu\right)$$

- Illustration: Q_m (= J) is really angular momentum:
  - For a weak source

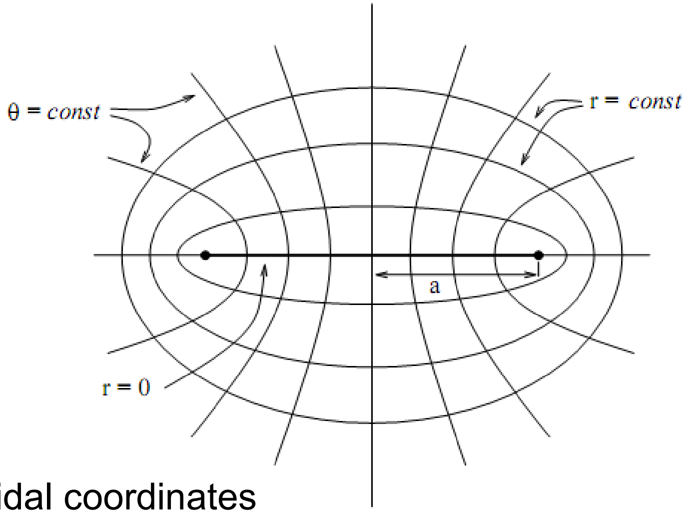
$$J\left(V\right) \approx \epsilon_{3jk} \int_V d^3 x^j T^{k0}$$

  - Exactly what it should be

- Analogously Q_k (=M) is really what we would call the mass

Singularities of the Kerr-Metric (1)

- Curvature singularity at $\Sigma = 0$, i.e. $r = 0$ and $\theta = \pi/2$
- What does this look like?
- Do not just think of r, θ, and φ as you normally would!
- For $M = 0$ the Kerr metric is just Minkowski in *ellipsoidal*[1] coordinates:



[1]In this special case aka spheroidal coordinates

Singularities of the Kerr-Metric (2)

- Change to Kerr-Schild coordinates:

$$x + iy = (r + ia)\sin\theta \exp\left[i\int (d\phi + \frac{a}{\Delta}dr)\right]$$

$$z = r\cos\theta$$

$$\tilde{t} = \int \left(dt + \frac{r^2 + a^2}{\Delta}dr\right) - r$$

- Then the metric becomes:

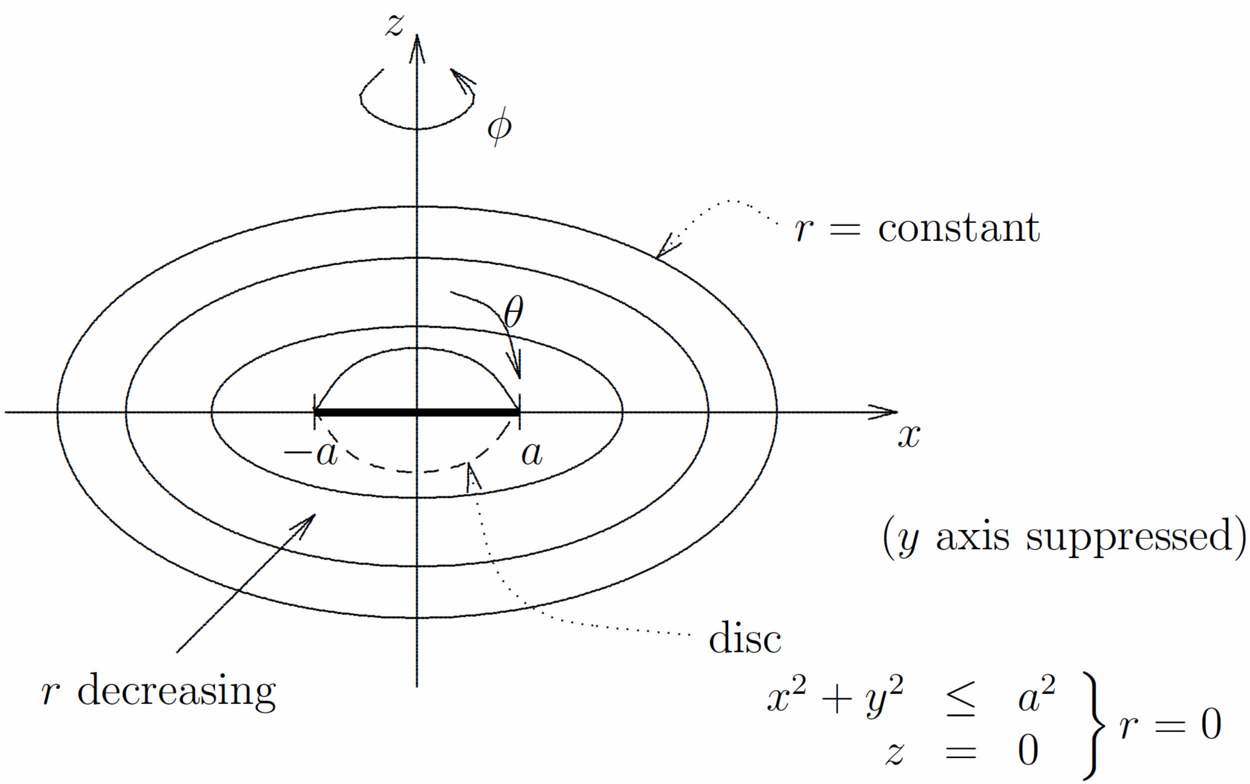
$$ds^2 = -d\tilde{t}^2 + dx^2 + dy^2 + dz^2$$

$$+ \frac{2Mr^3}{r^4 + a^2 z^2}\left[\frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r} + d\tilde{t}\right]^2$$

- Note again: the spacetime is flat for M = 0

Singularities of the Kerr-Metric (3)

The surfaces of constant $\tilde{t}, r$ are confocal ellipsoids which degenerate at $r = 0$ to the disc $z = 0$, $x^2 + y^2 \leq a^2$.



$\theta = \pi/2$ corresponds to the boundary of the disc at $x^2 + y^2 = a^2$ so the curvature singularity occurs on the boundary of the disc, i.e. on the 'ring' $x^2 + y^2 = a^2, \quad z = 0$

Singularities of the Kerr-Metric (3)

- Coordinate singularities at $\theta = 0$ and at $\Delta = 0$
- Three cases:
  - $M^2 < a^2$: Naked singularity
  - $M^2 > a^2$: Physically relevant Kerr black hole
  - $M^2 = a^2$: Extreme Kerr black hole (unstable)
- Now write

$$\Delta = (r - r_+)(r - r_-)$$
$$r_\pm = M \pm \sqrt{M^2 - a^2}$$

- Coordinate singularities at $r = r_\pm$, can be removed by a coordinate transformation to Kerr coordinates analogous to ingoing EF coordinates for Schwarzschild:

Singularities of the Kerr-Metric (4)

- In Kerr coordinates,

$$dv = dt + \frac{(r^2 - a^2)}{\Delta} dr$$

$$d\chi = d\phi + \frac{a}{\Delta} dr$$

- The metric becomes:

$$ds^2 = -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} dv^2 + 2dvdr - 2a\sin^2\theta \frac{(r^2 + a^2 - \Delta)}{\Sigma} dvd\chi$$

$$- 2a\sin^2\theta d\chi dr + \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right) \sin^2\theta d\chi^2 + \Sigma d\theta^2$$

The horizons

- The hypersurfaces r = $r_\pm$ are Killing horizons of the KVFs

$$\xi_\pm = k + \left(\frac{a}{r_\pm^2 + a^2}\right) m$$

- Analogously to Schwarzschild, r_+ is the (outer) event horizon of the Kerr black hole
- I.e. beyond r_+, r becomes timelike in such a way that you have to move in the direction of decreasing r
- At the inner event horizon (r_-) however, r becomes spacelike again, and so one is not forced to move towards the singularity any further
- However *unlike* Schwarzschild, there is yet another region (outside the outer even horizon) where something interesting happens

The ergosphere

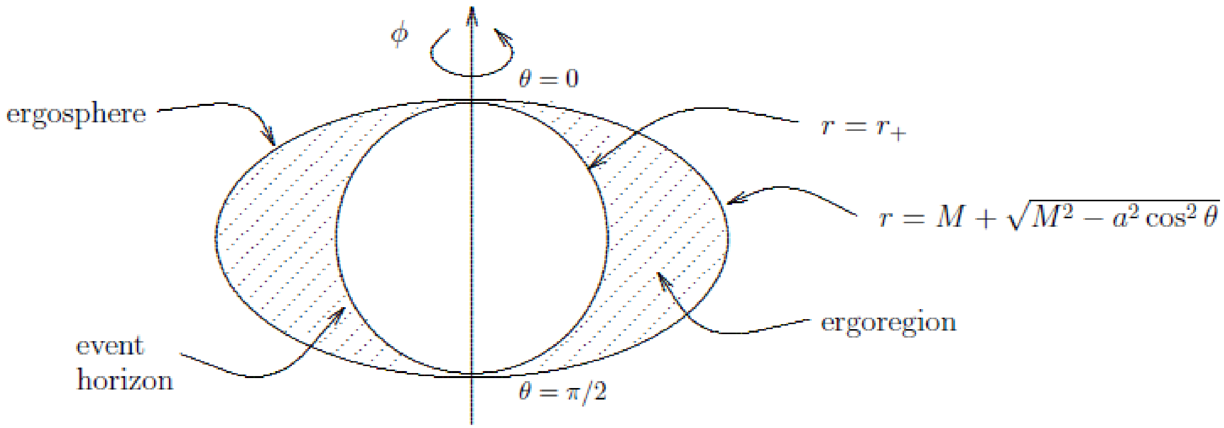
- k is timelike at infinity, but in fact not everywhere outside the event horizon:

$$k^2 = g_{tt} = -\frac{\Delta - a^2 \sin^2\theta}{\Sigma} = -\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)$$

- Outside the black hole k in fact becomes spacelike for

$$r_+ \le r \le M + \sqrt{M^2 - a^2\cos^2\theta}$$

- This region is known as the Ergosphere:



Inside the ergosphere(1)

- Behaviour perhaps most easily seen by considering a photon emitted in the φ direction at some radius r. At the instant of emission:

$$ds^2 = 0 = g_{tt}dt^2 + g_{t\phi}(dtd\phi + d\phi dt) + g_{\phi\phi}d\phi^2$$

- Solve to obtain:

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

- At the Killing horizon

$$\frac{d\phi}{dt} = 0, \qquad \frac{d\phi}{dt} = \frac{2a}{(2GM)^2 + a^2} \qquad (g_{tt} = 0)$$

- Interpret as the photon not moving (instantaneously) at all, and moving in the direction of the hole's rotation respectively

Inside the ergosphere(2)

- Massive particles which must move more slowly than photons, are thus necessarily dragged along with the hole's rotation, once inside the ergosphere
- More generally, "locally *nonrotating* observers" *rotate* with coordinate angular velocity:

$$\omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t} = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{a(r^2+a^2-\Delta)}{(r^2+a^2)^2-\Delta a^2\sin^2\theta}$$

- This is an example of the Lens-Thirring effect (frame dragging)
- The dragging continues as we approach the outer event horizon. Define the angular velocity of the horizon itself to be the minimum angular velocity of a particle at the horizon:

$$\Omega_H = \left(\frac{d\phi}{dt}\right)_-(r_+) = \frac{a}{r_+^2+a^2}$$

- Note: $\omega|_{r=r_+} = \Omega_H$

Particle Trajectories

- Analogously to the Schwarzschild case, can understand orbits around the BH through an effective potential

- Using conservation of angular momentum and energy one obtains:

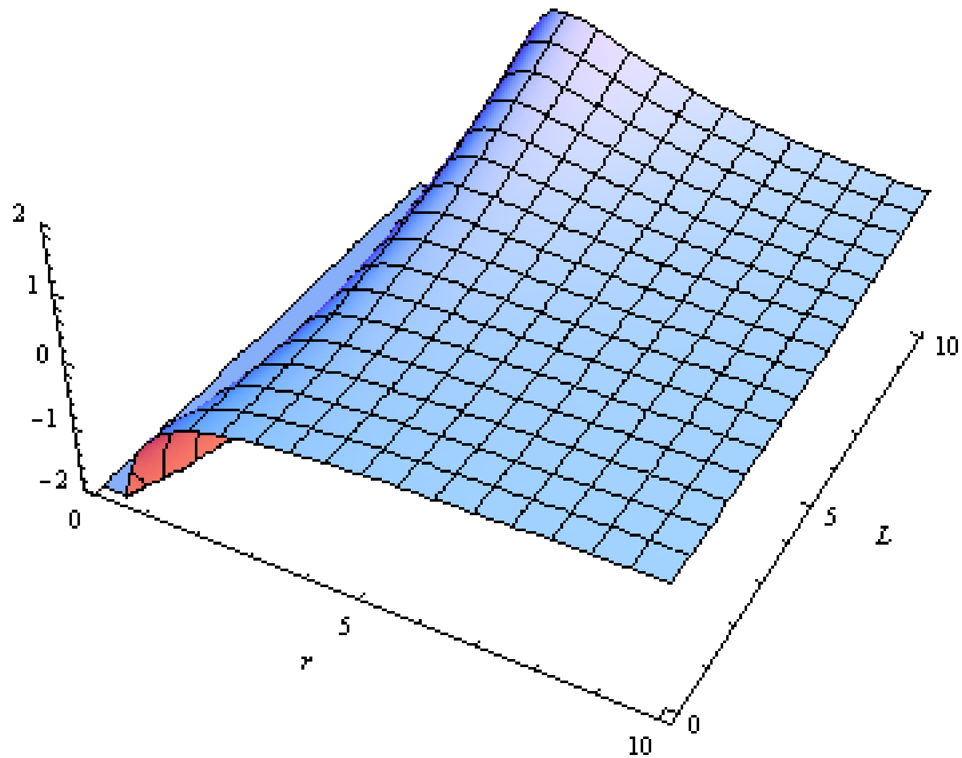
$$\frac{1}{2}\dot{r}^2 + V(E, L, r) = 0$$

Where

$$V(E, L, r) = -\epsilon\frac{M}{r} + \frac{L^2}{2r^2} + \frac{1}{2}(\epsilon - E^2)\left(1 + \frac{a^2}{r^2}\right) - \frac{M}{r^3}(L - aE)^2$$

$$\epsilon = \begin{cases} 1 & \text{for massive particles;} \\ 0 & \text{for massless particles.} \end{cases}$$

- Note that the effective potential in this case not only depends on the angular momentum but also on the energy of the particle.



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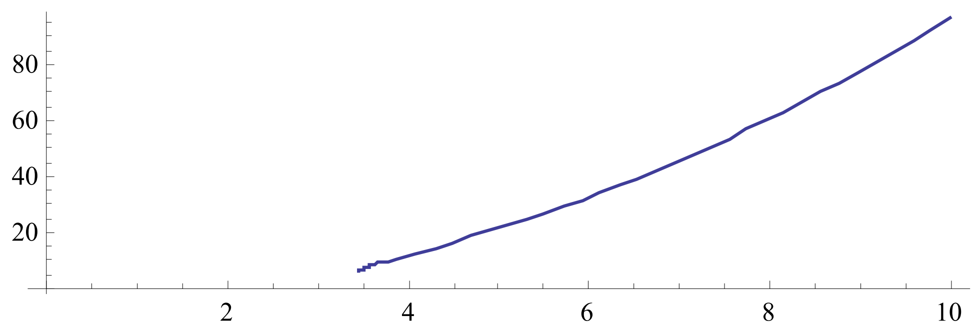

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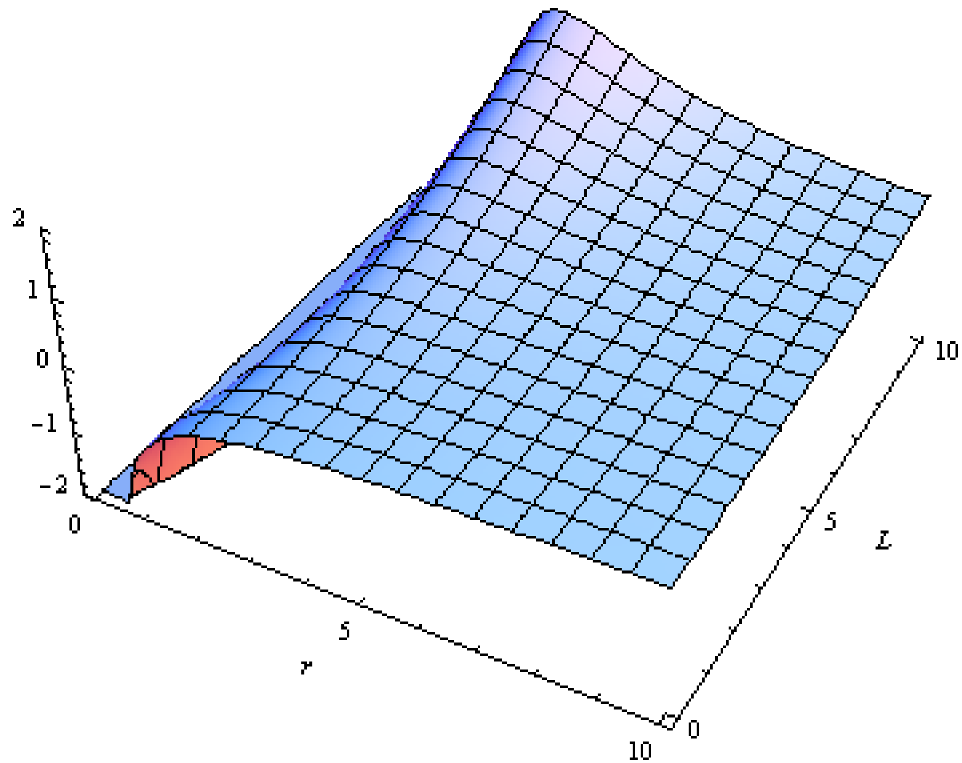
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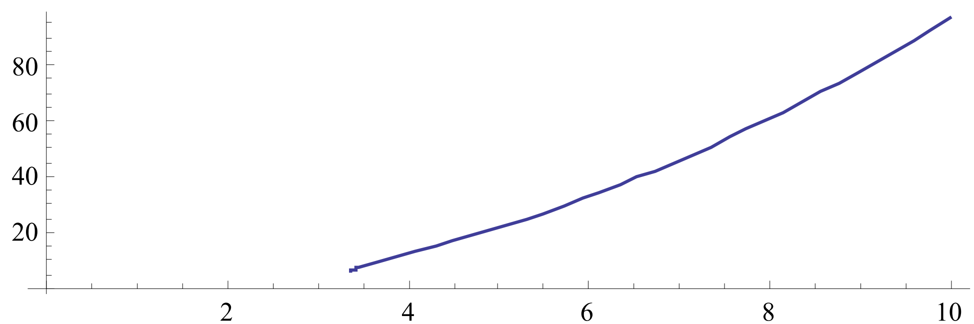
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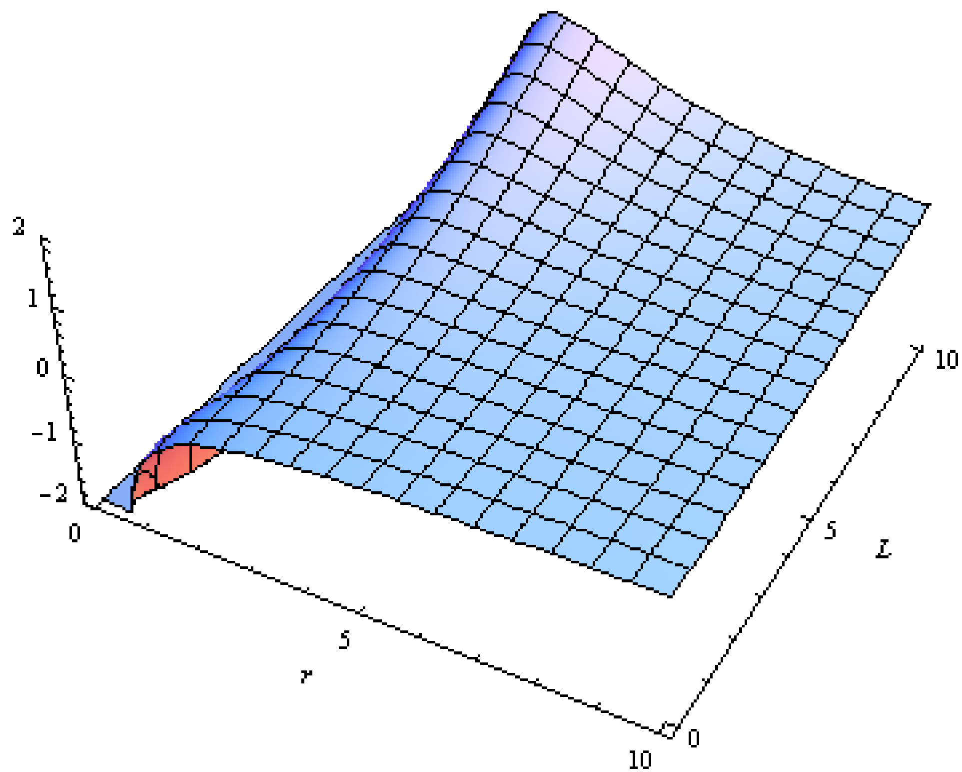
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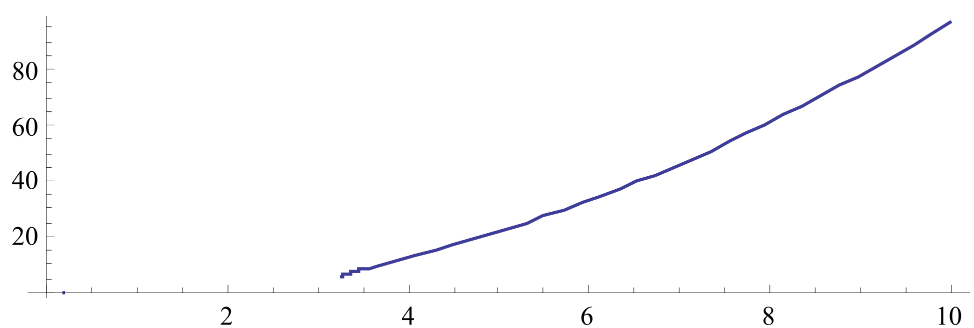
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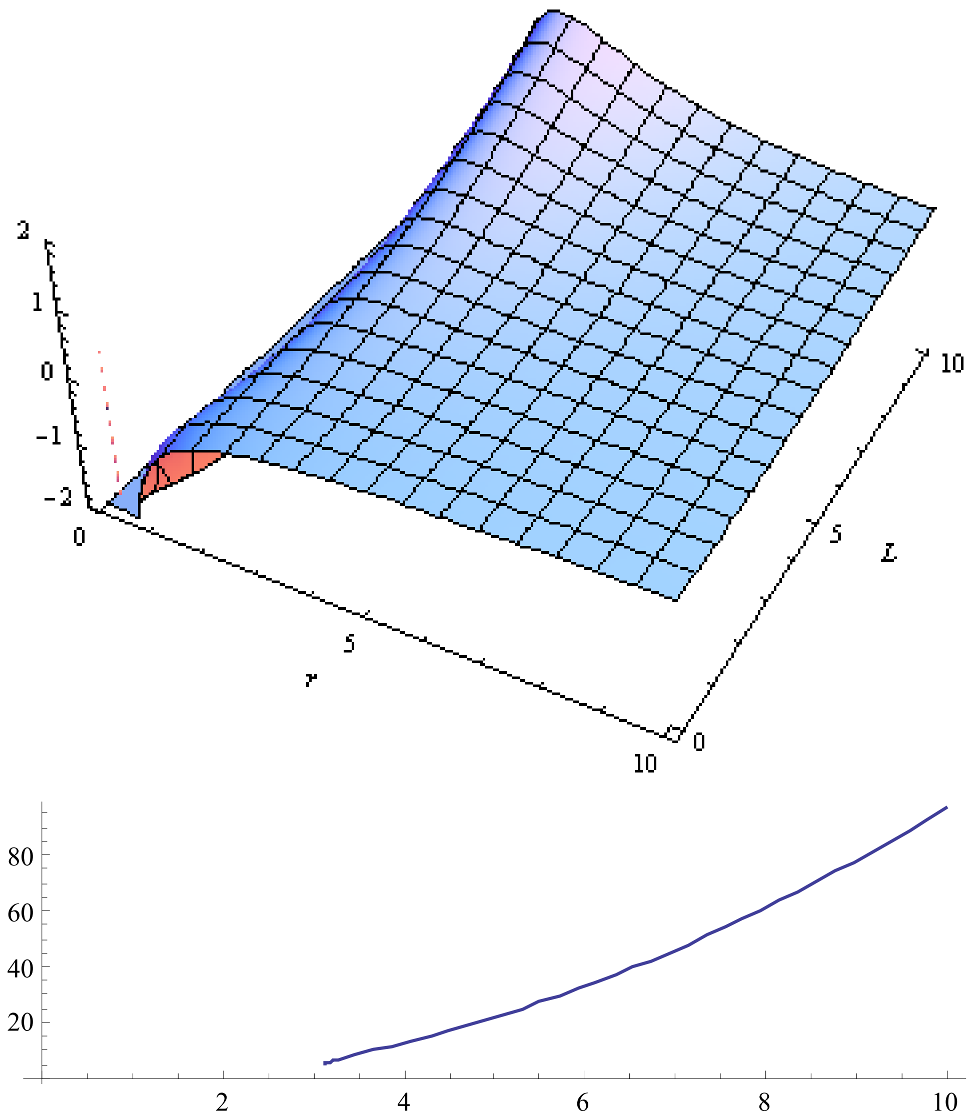
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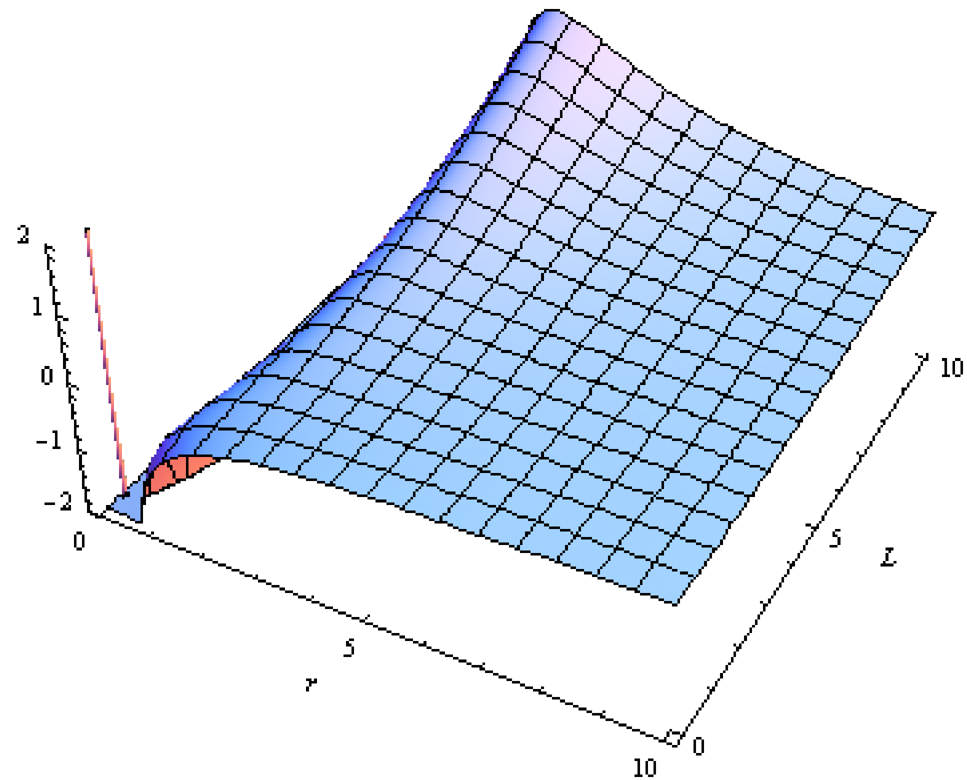
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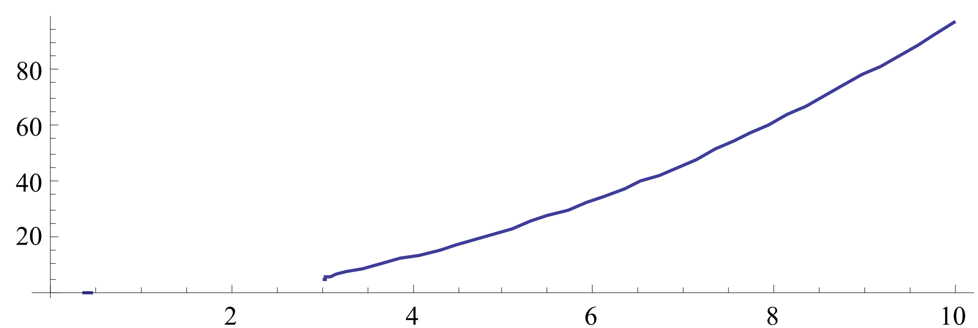
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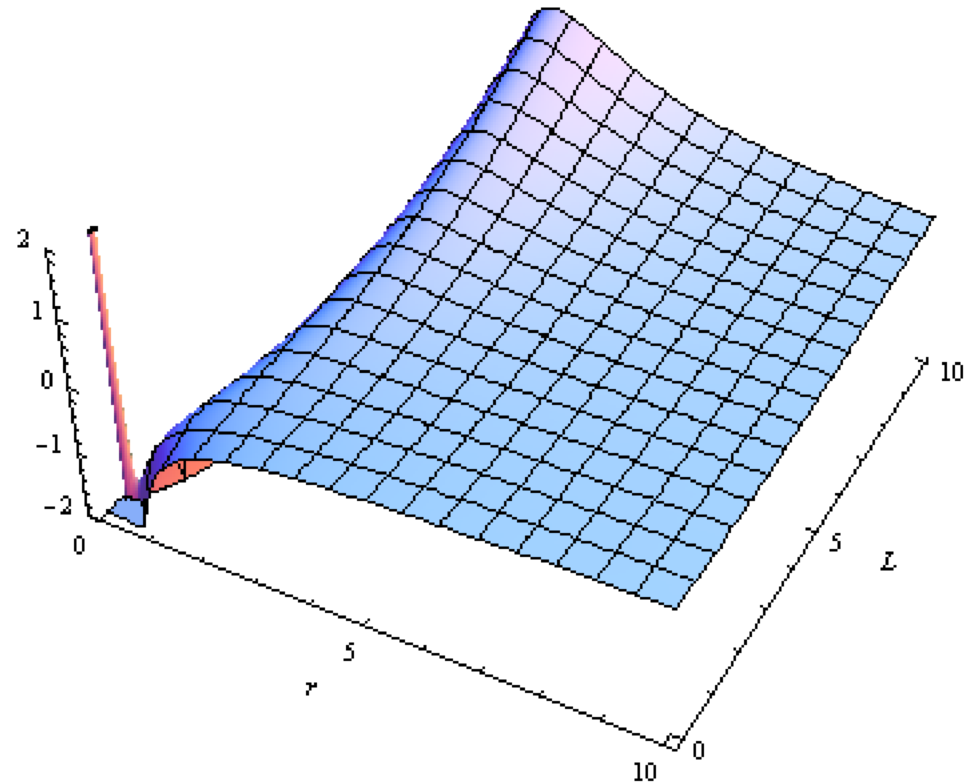
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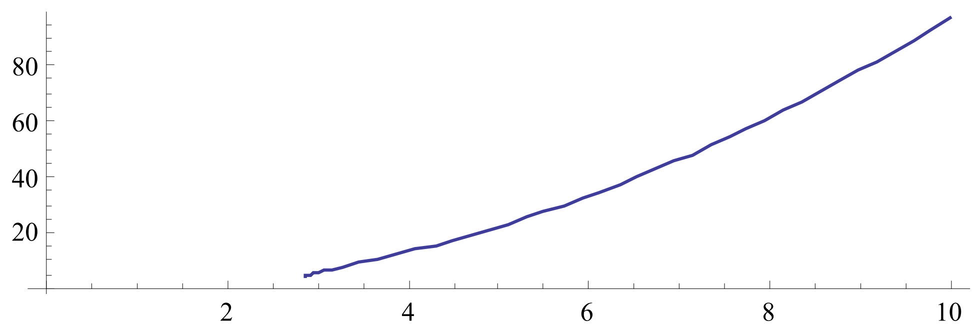
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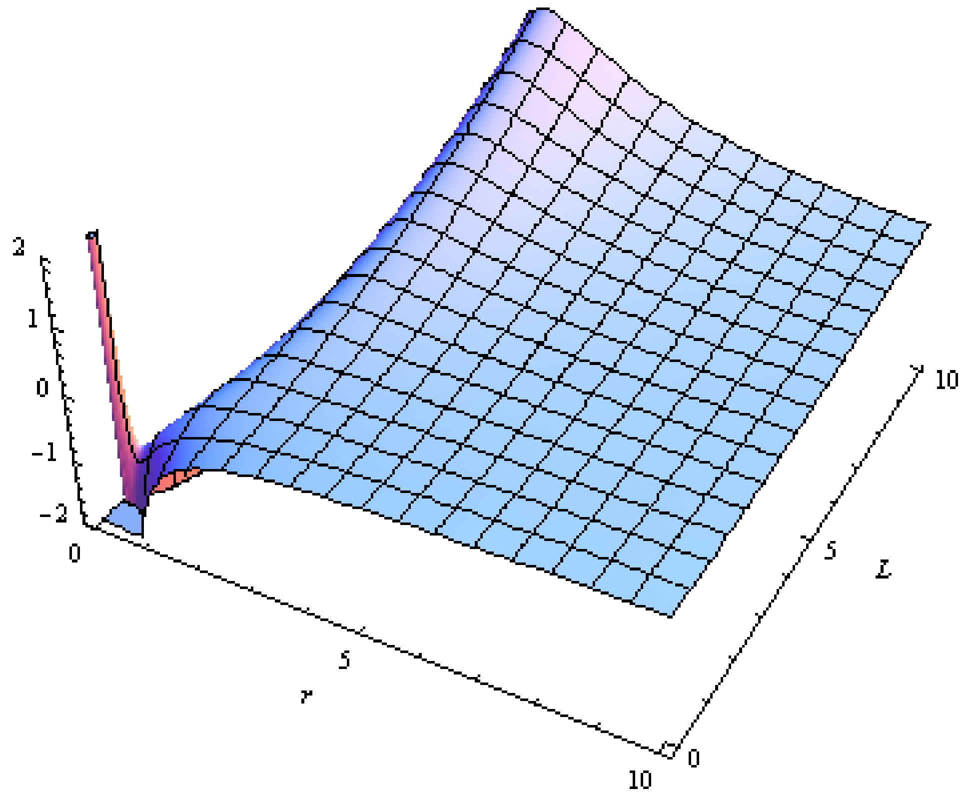

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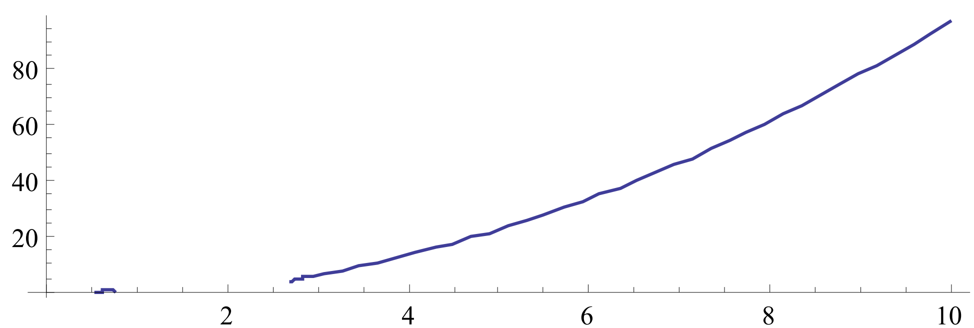

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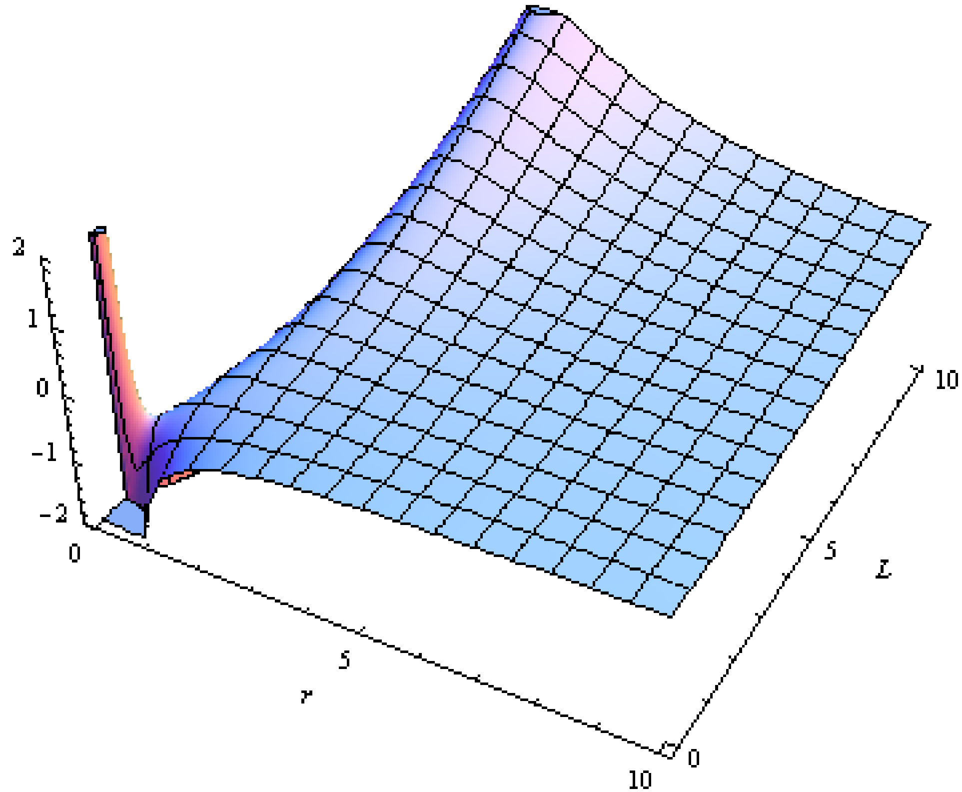
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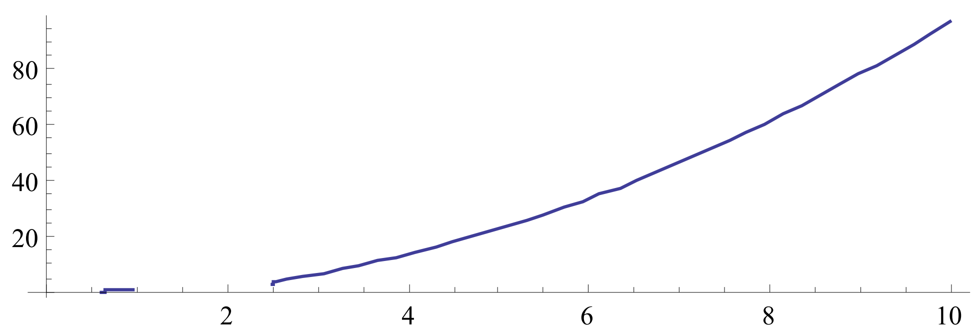
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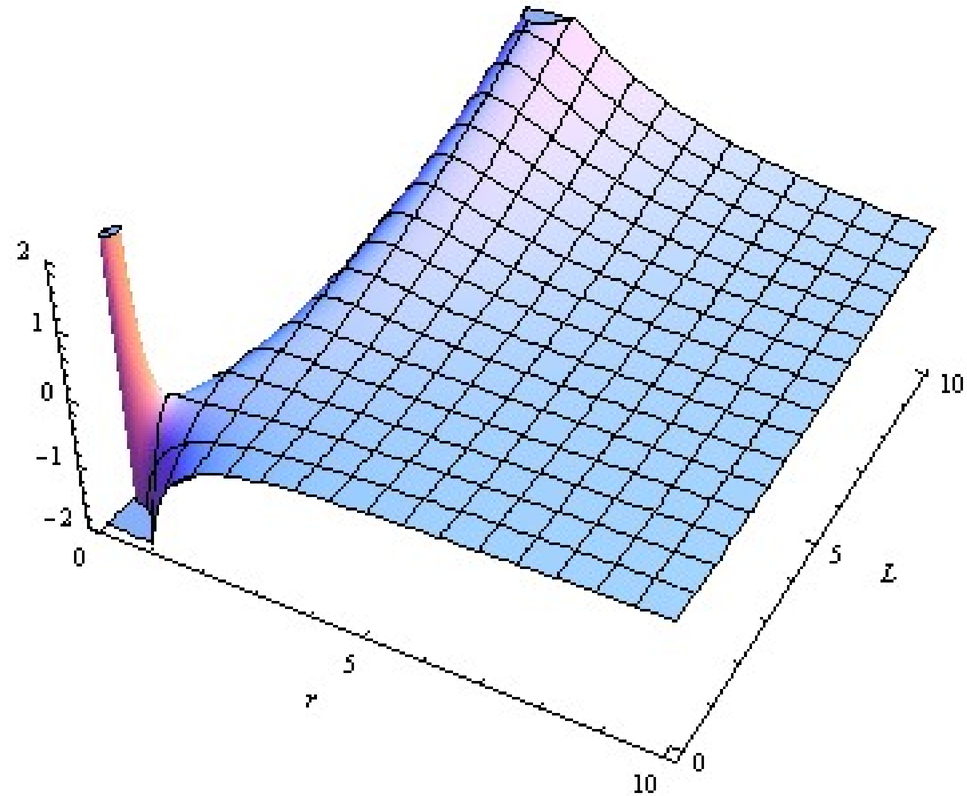
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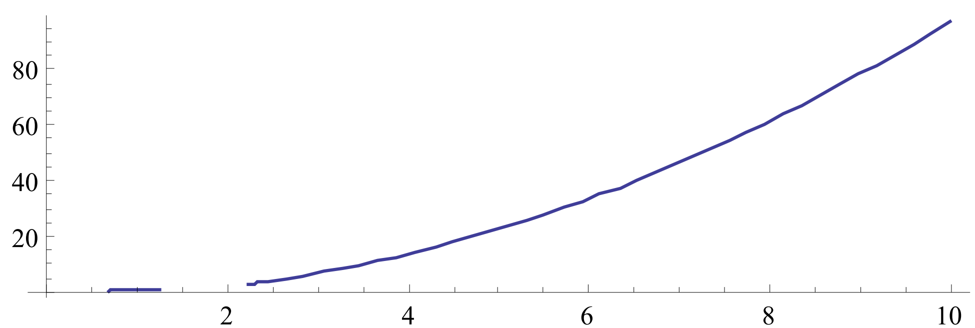
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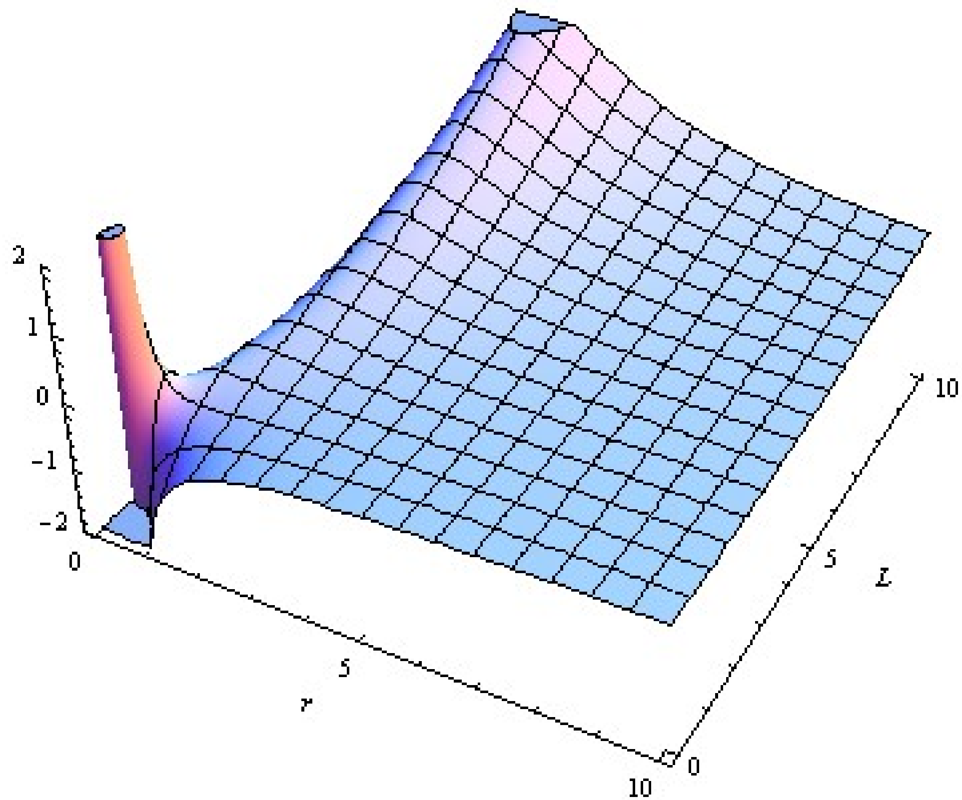

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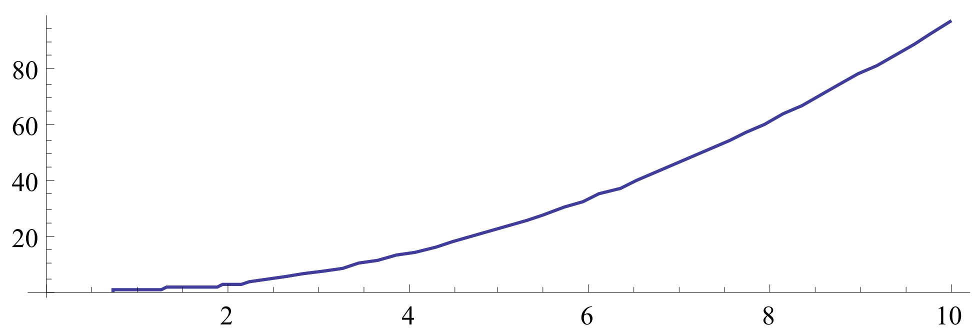

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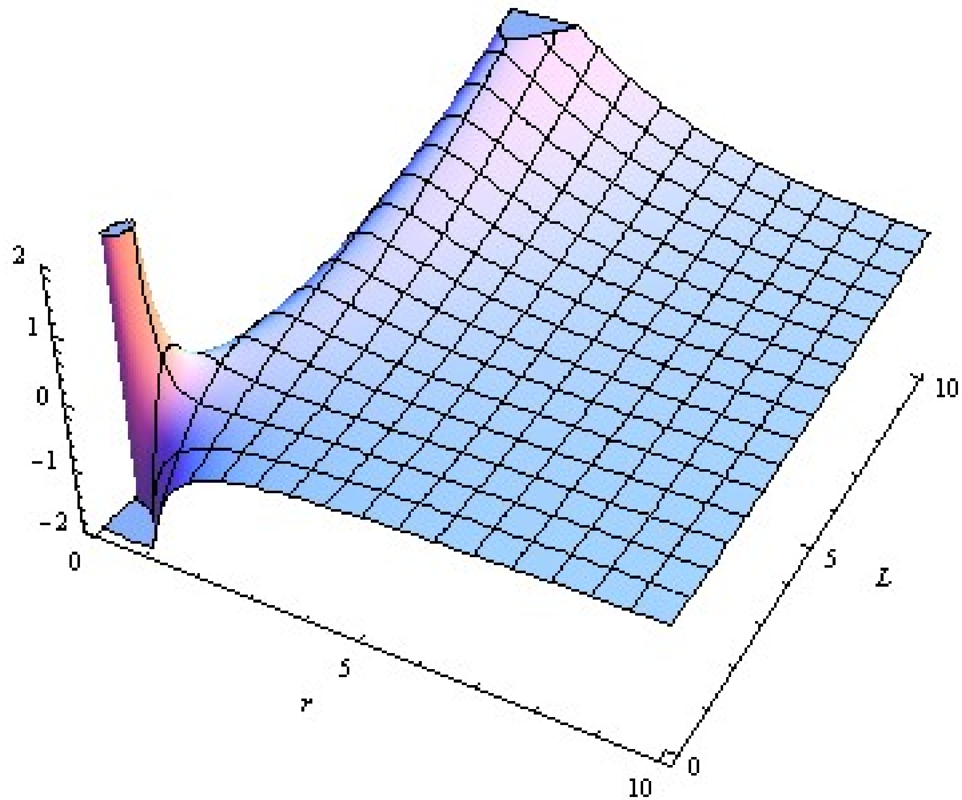

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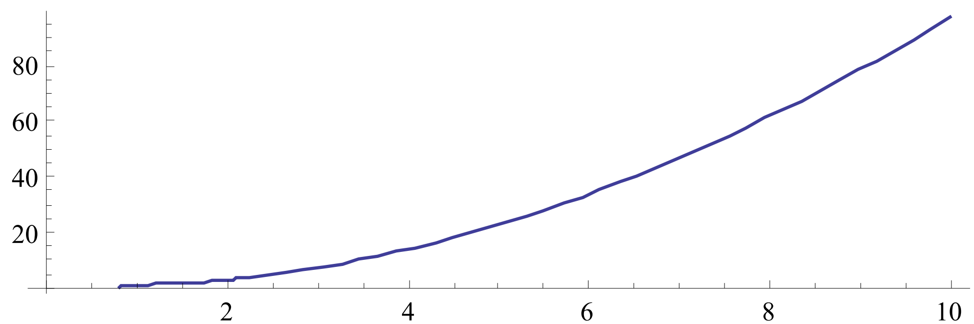

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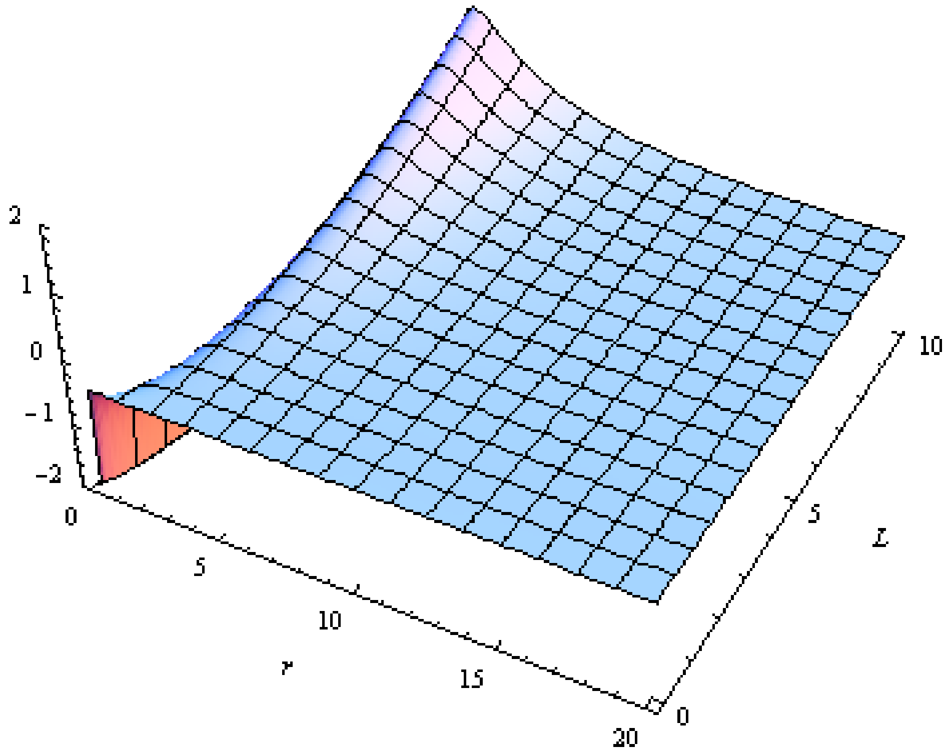
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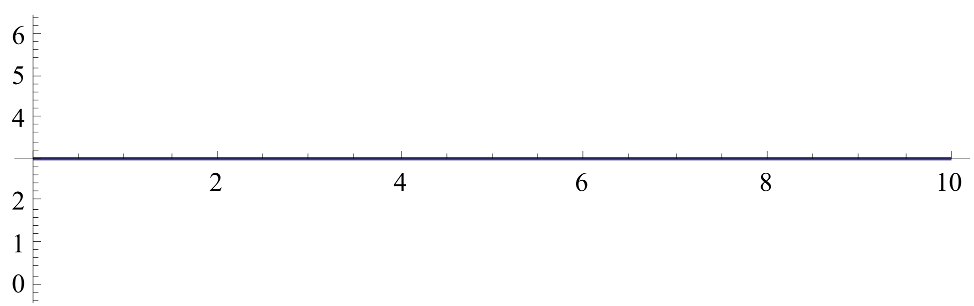
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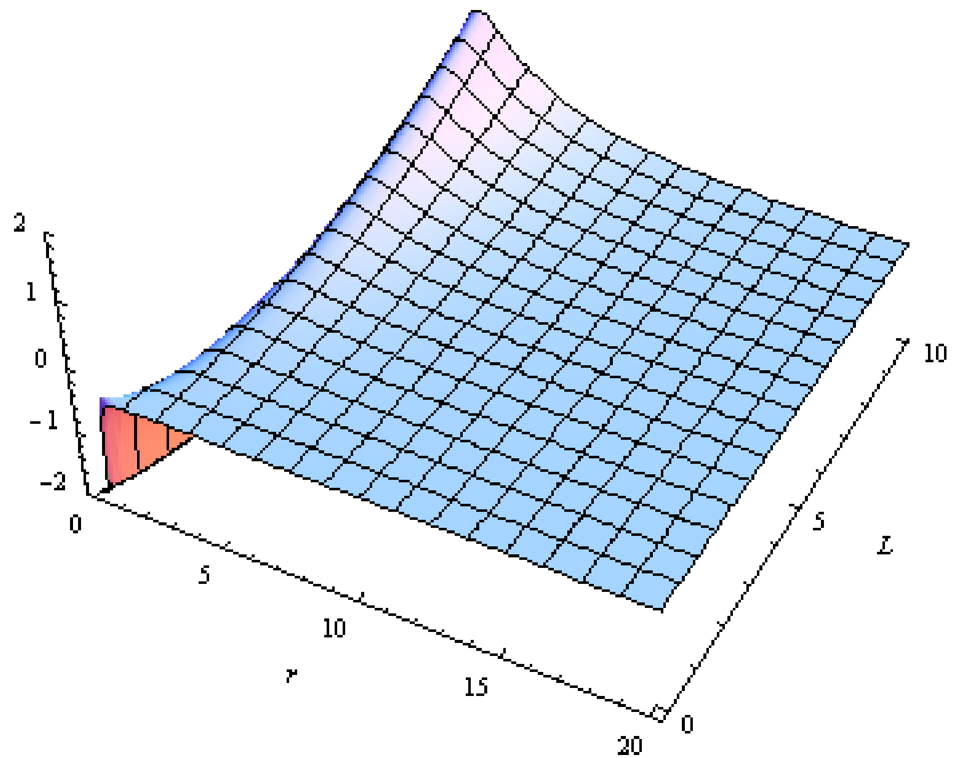

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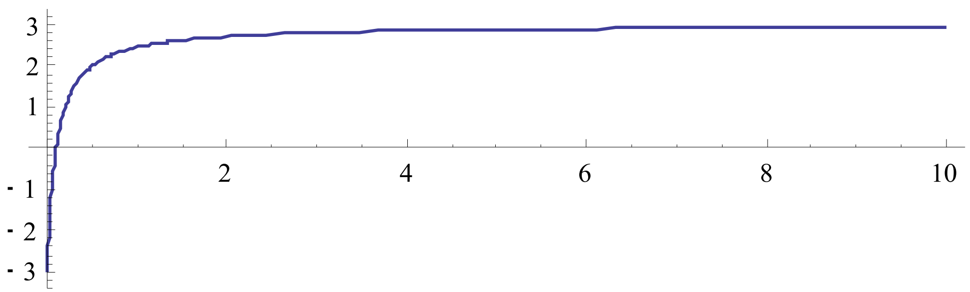

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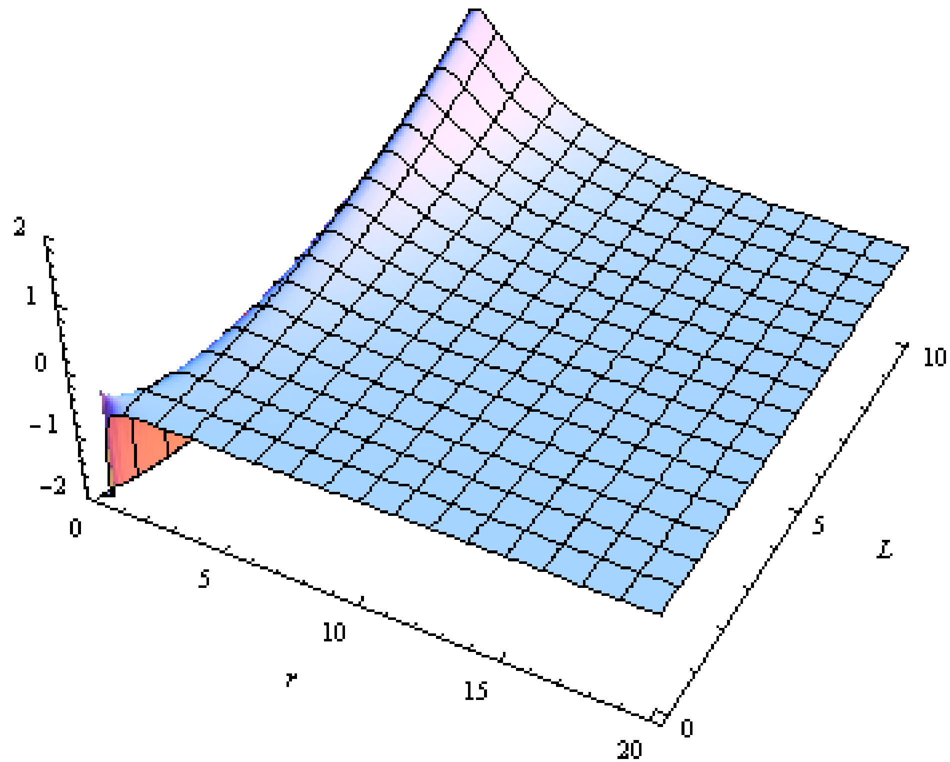
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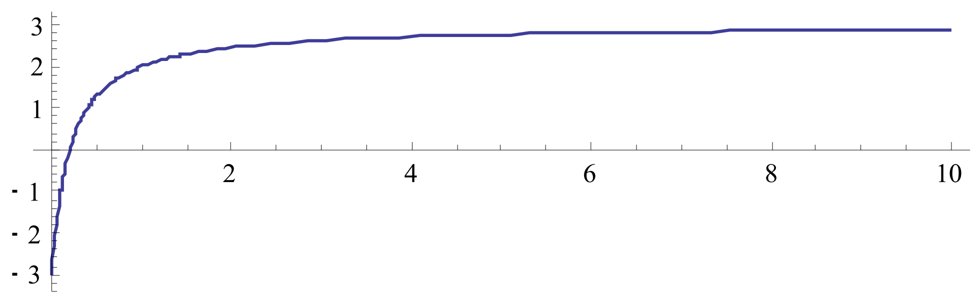
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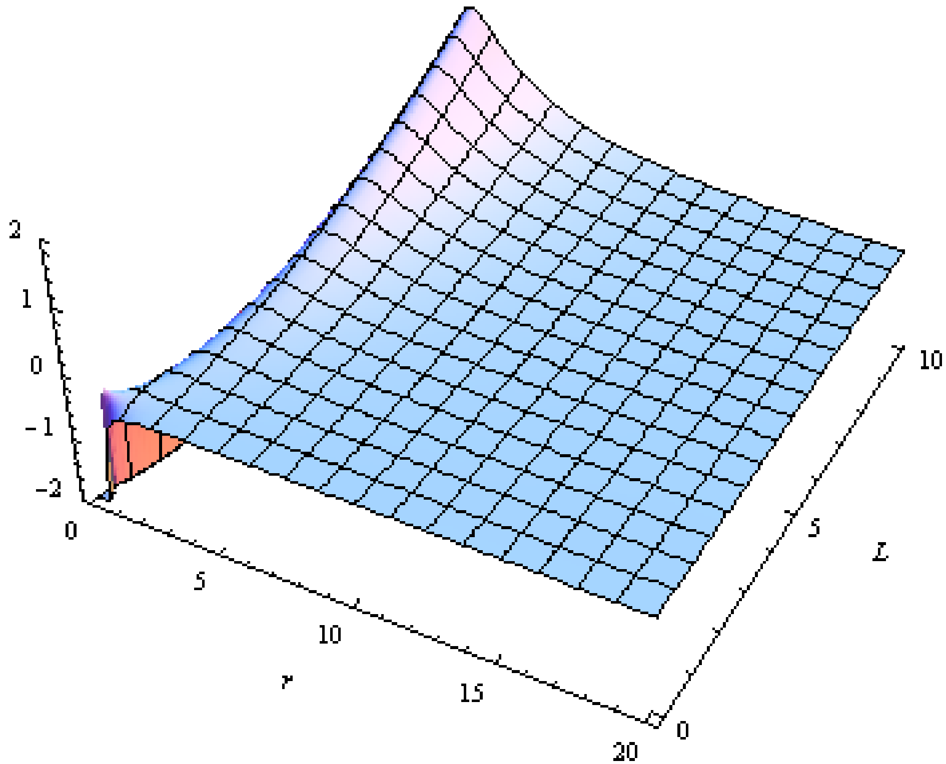
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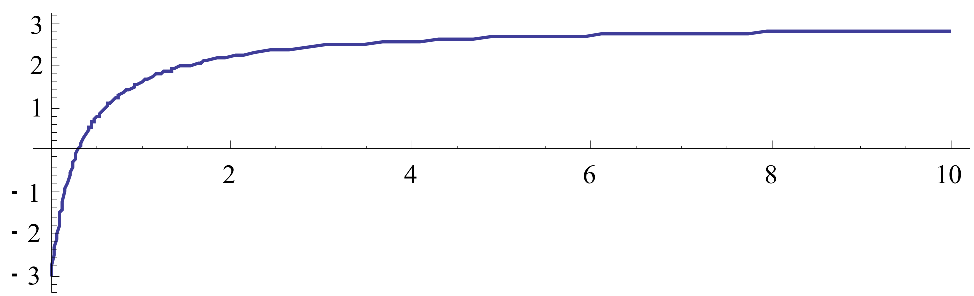
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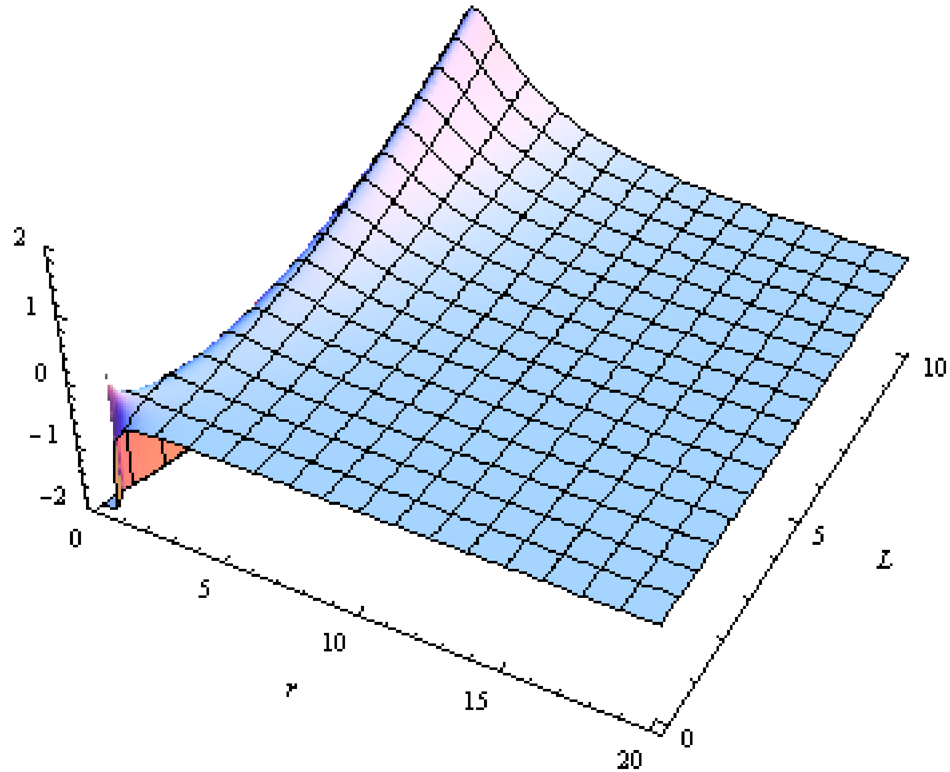
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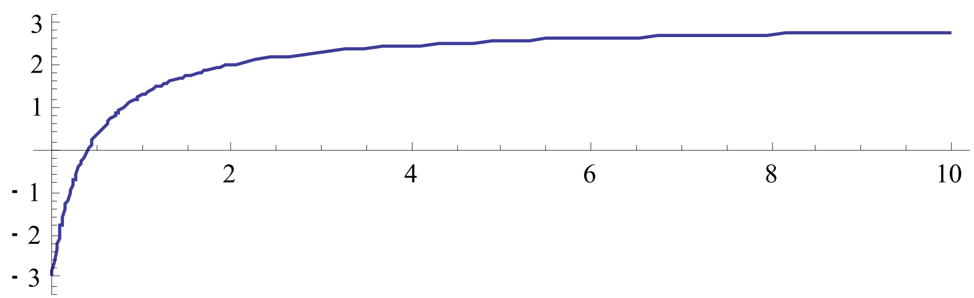
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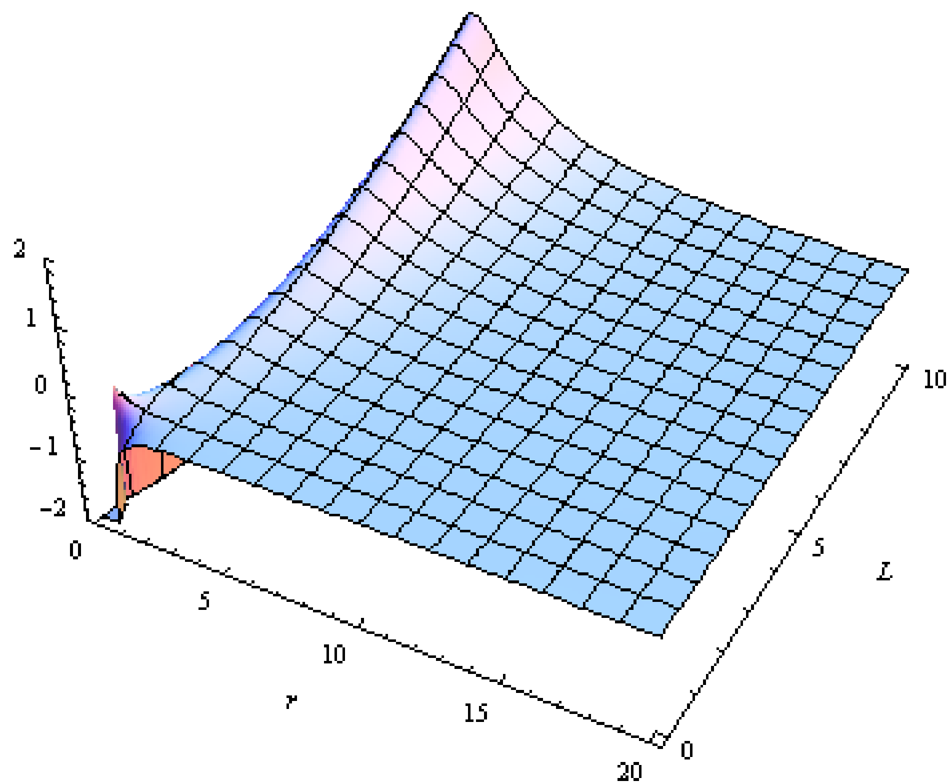

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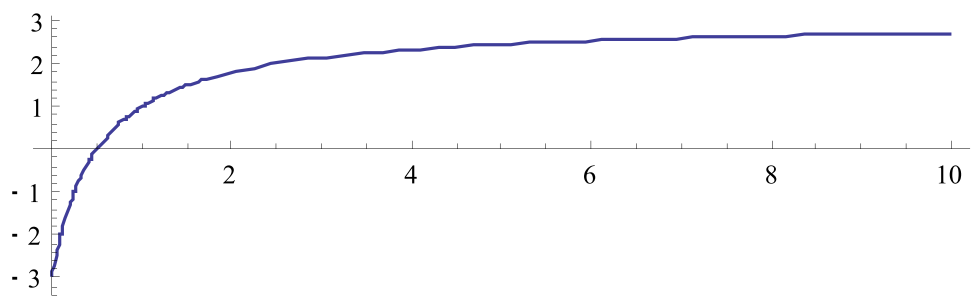

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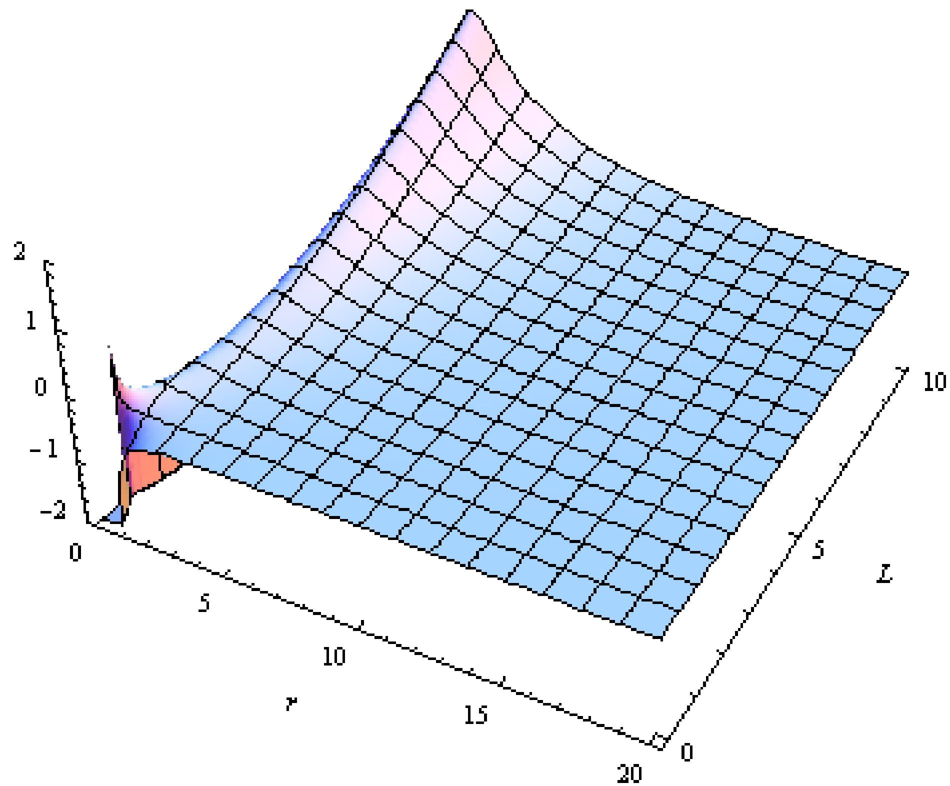

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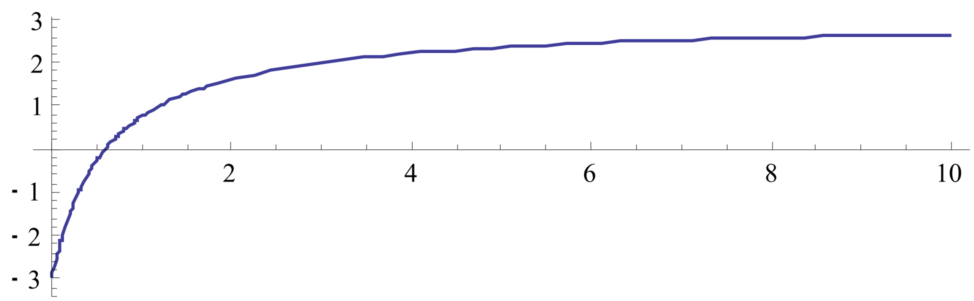

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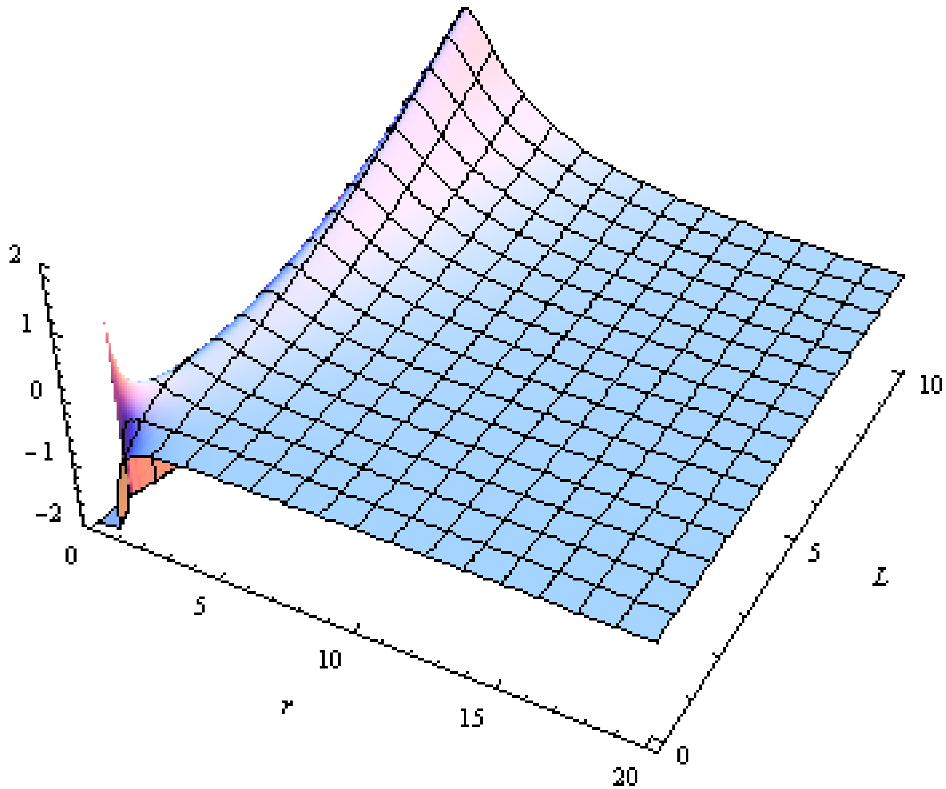
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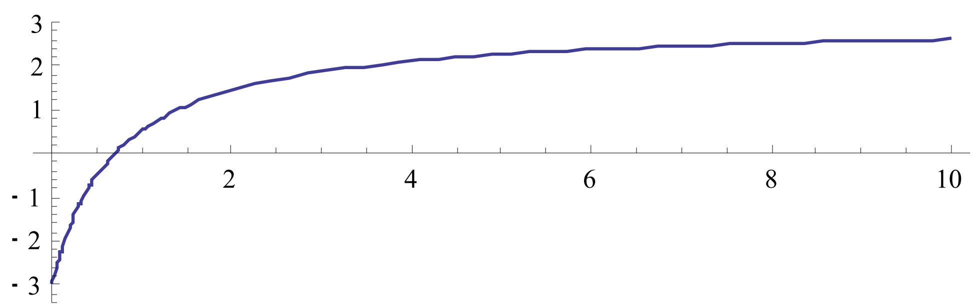
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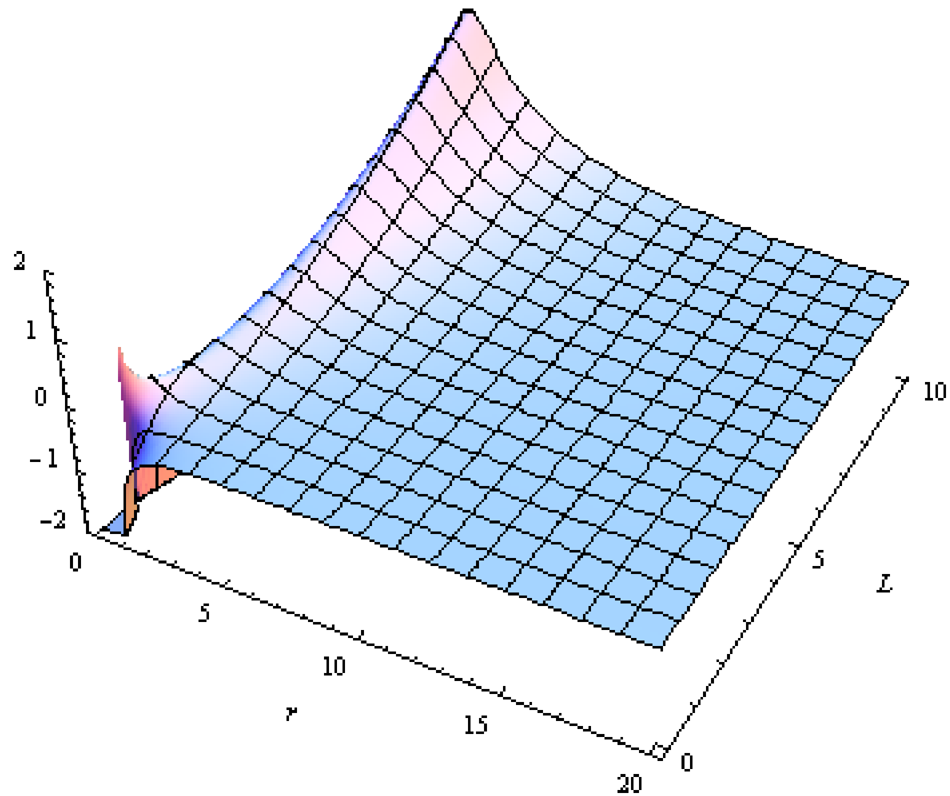

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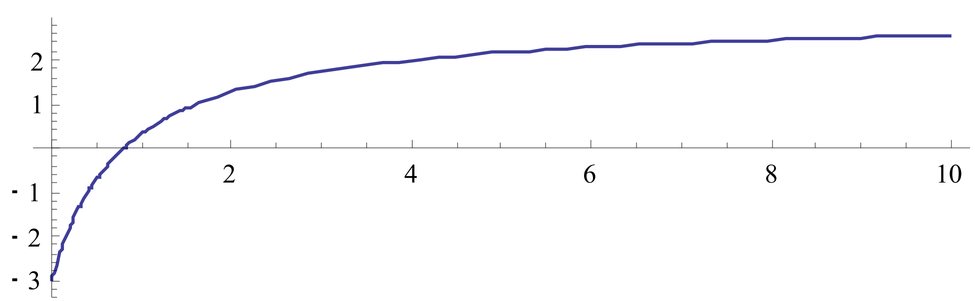

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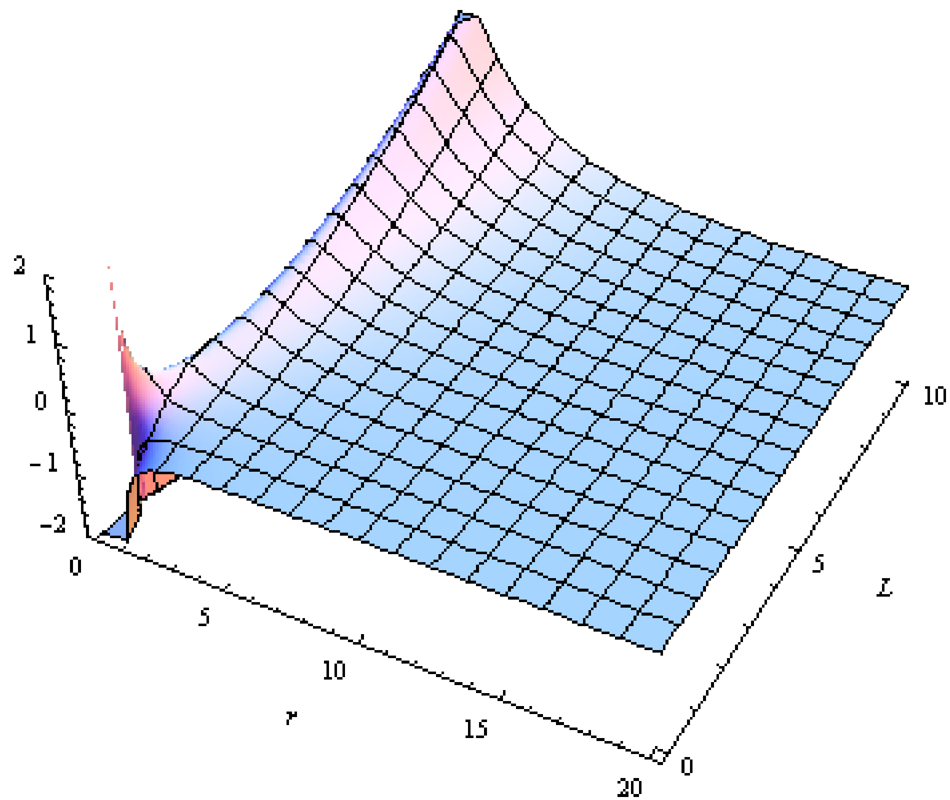

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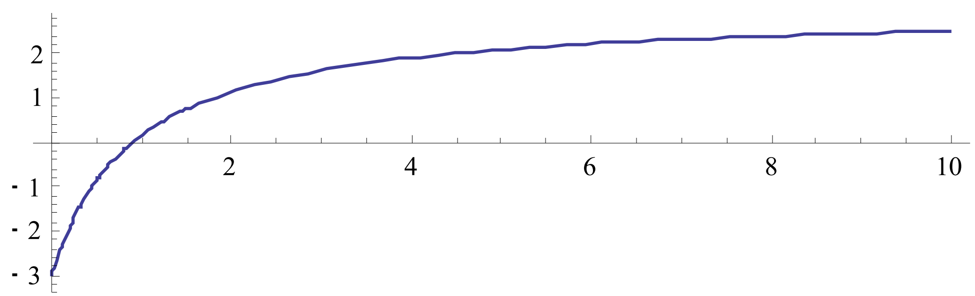

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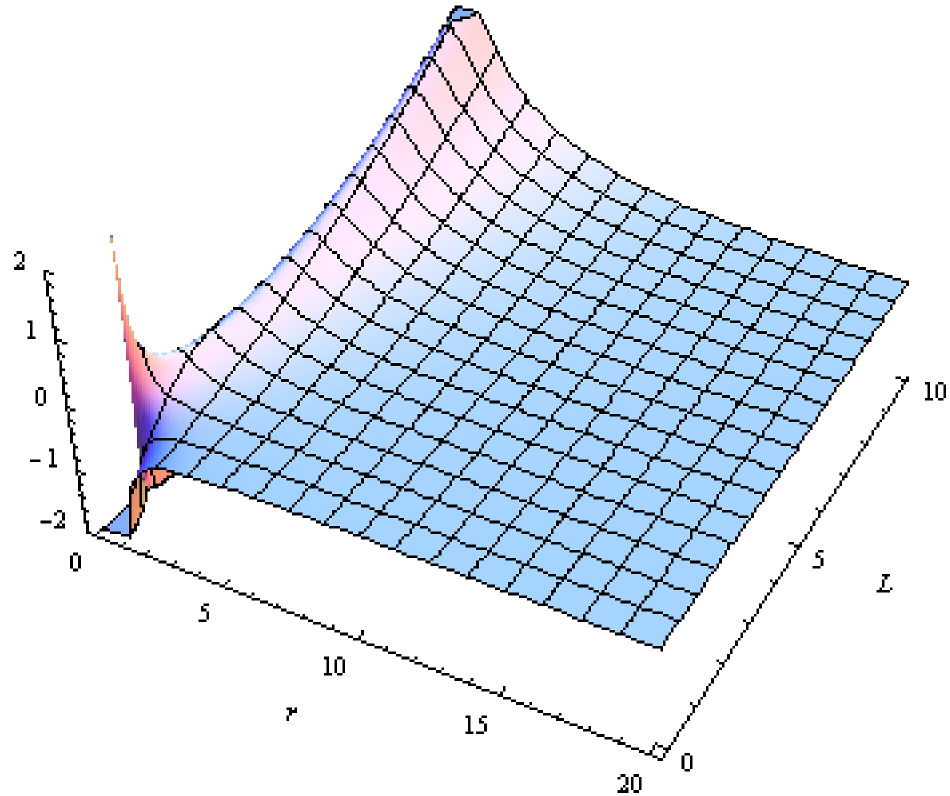
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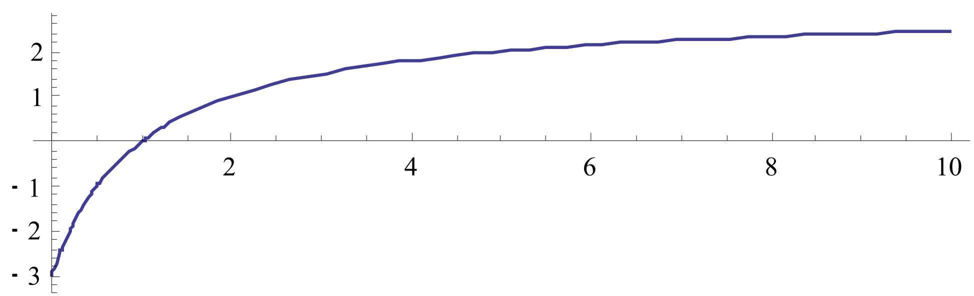
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The Penrose process

- As usual denote the conserved charge along geodesics associated with k:

$$E = -k_\mu p^\mu$$

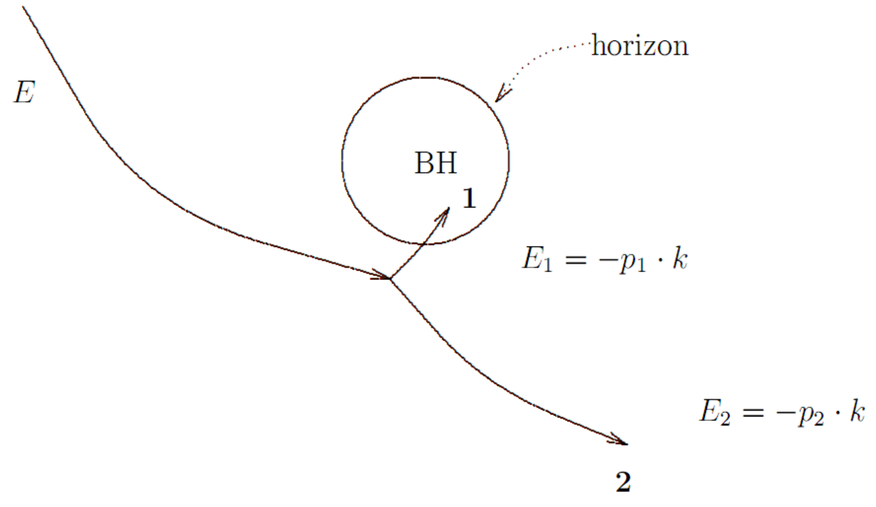
- Suppose we have a particle that decays to two others, one which falls into the black hole, and one which escapes to infinity:

- Normally $E_1>0$, and thus $E_2<E$

- However in the ergoregion k is spacelike and thus E_1 can be negative

- Hence for decay in the ergoregion we may have $E_2>E$

- Energy extraction from a black hole!



The Penrose process(2)

- Penrose has shown you can actually arrange the initial trajectory and the ‘decay’ such that afterwards you do follow a geodesic trajectory back outside the ergosphere and into the external universe

- The different possible particles/trajectories can be analyzed by looking at the potential found, and analysing the constraints:
  - The ‘parent’ particle has to come in from infinity
  - It has to reach the point of ‘decay’
  - The momenta have to be such that energy is extracted
  - One of the decay particles has to bounce out of the ergospere, and be able to escape to infinity

- The energy extracted this way of course has to come from somewhere: the angular momentum of the black hole has decreased

- What are the limits on this extraction?

Limits to energy extraction

- For particles passing through the outer event horizon we have: $-p_\mu \xi_+^\mu \geq 0$ since ξ_+ is future-directed null and p is future-directed timelike or null on the horizon
- Recall:

$$\xi_+ = k + \left(\frac{a}{r_+^2 + a^2}\right) m = k + \Omega_H m$$

- Hence

$$E - \Omega_H L \geq 0 \qquad (L = p_\mu m^\mu)$$

- This leads to:

$$\delta\left(M^2 + \sqrt{M^4 - J^2}\right) \geq 0$$

- As it turns out this (between brackets) is the area of the outer event horizon of the black hole

Limits to energy extraction(2)

- This is a special case of the second law of black hole thermodynamics.
- One can define the irreducible mass, M_{irr} of a black hole through this area:

$$M_{\text{irr}}^2 = \frac{A}{16\pi}$$

- Then the maximum amount of energy that can be extracted before slowing its rotation to zero is:

$$M - M_{\text{irr}} = M - \frac{1}{\sqrt{2}} \left(M^2 + \sqrt{M^4 - J^2} \right)^{1/2}$$

- It turns out (not unexpectedly) that the maximum amount can be extracted from an extreme Kerr black hole. In that case we can extract appr. 29% of its total energy.
- To put this in perspective: for an extreme Kerr black hole of solar mass, this would be *enough to power the earth for roughly* 10^{26} *years* at current consumption rates!

Super-Radiance

- The Penrose process actually has a close analog in the scattering of radiation by a Kerr black hole
- To analyze this, consider a scalar field, Φ, with energy-momentum tensor:

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\partial \Phi)^2$$

- Then by conservation of this tensor, we get the following:

$$\nabla_\mu \left(T^\mu{}_\nu k^\nu \right) = T^{\mu\nu} \nabla_\mu k_\nu = 0$$

- So we can consider the following current as the energy flux vector associated with Φ

$$j^\mu = -T^\mu{}_\nu k^\nu = -\partial^\mu \Phi k_\nu \partial^\nu \Phi + \frac{1}{2} k^\mu (\partial \Phi)^2$$

- Now what we want to do is look at a region of spacetime, with part of its boundary on the event horizon, and see what conservation of this current implies

Super-Radiance(2)

- The energy flux lost per unit of time (power) is,

$$P = \int dA \left(\frac{\partial}{\partial v} \Phi + \Omega_H \frac{\partial}{\partial \chi} \Phi \right) \left(\frac{\partial \Phi}{\partial v} \right)$$

- Working in Kerr coordinates, for a wavemode of angular frequency ω:

$$\Phi = \Phi_0 \cos(\omega v - \nu \chi), \quad \nu \in \mathbb{Z} \quad \text{(angular quantum number)}$$

- The time average power lost across the horizon is:

$$P = \frac{1}{2} \Phi_0^2 A \omega (\omega - \nu \Omega_H)$$

- While P is positive for most values of ω, it is in fact negative for the following range of values: $0 < \omega < \nu \Omega_H$

- A wave mode with parameters in this inequality is amplified by the black hole! (note it must have angular quantum number nonzero)

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  - Ellipsoidal coordinates image courtesy of Prof. S.M. Carroll, Lecture Notes on General Relativity, arXiv:gr-qc/9712019v1 3 Dec 1997
  - Singularity, ergosphere, and Penrose process image courtesy of Dr. P.K. Townsend; Black Holes, arXiv:gr-qc/9707012v1 4 Jul 1997