Classical field theory 2013 (NS-364B) – Class exercises + Homework 1

Tue Apr 23 2013, 13:15-17:00 BBL 065

CLASS EXERCISES

Problems 1-3 to section 4 on page 149 of Lecture Notes.

HOMEWORK 1 *(due at Tue May 7 2013)*

1. Consider two vector fields, whose coordinates in the Cartesian coordinate system are $\vec{A} = (A^1, A^3, A^3)$ and $\vec{B} = (B^1, B^3, B^3)$.

(A) Find the coordinates of the vector field $\vec{A}$ in spherical coordinates. Express your answer in terms of A^1, A^2, A^3 and functions of spherical coordinates (r, θ, ϕ).

(B) Find all of the components of the metric tensor in spherical coordinates. *Hint:* Recall that the metric tensor in Cartesian coordinates is of the form $g_{ij} = \delta_{ij}$. To get its components in spherical coordinates use the transformation property of the metric tensor (its components transform as those of a rank (2,0) tensor).

(C) Show that the scalar product $\vec{A} \cdot \vec{B}$ is invariant under the coordinate transformation from parts (A-B).

2. Lie Groups and Algebras
Consider the Lie group $SU(2)$ of unitary complex matrices whose determinant equals unity and the corresponding Lie algebra $su(2)$.

(A) Show that the elements of $SU(2)$ can be parametrised as,

$$g = \cos(\theta) + \imath\hat{n} \cdot \vec{\sigma} \sin(\theta) \tag{1}$$

where $\imath^2 = -1$, $\hat{n}$ is a vector whose norm is unity ($\|\hat{n}\| = 1$) and θ is an angular parameter.

(B) Show that the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \qquad \sigma^2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} ; \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

are the generators of the Lie algebra $su(2)$ and that they obey the Pauli algebra,

$$[\sigma^i, \sigma^j] = 2\imath\epsilon^{ijl}\sigma^l \,. \tag{3}$$

Hint: To establish the connection between $su(2)$ and $SU(2)$ make use of the exponential map, $g = \exp(a)$, $g \in SU(2)$, $a \in su(2)$.

(C) Argue that the group $SU(2)$ is compact. You can show that e.g. (a) by finding an explicit homeomorphism between the $SU(2)$ and the three dimensional sphere S^3 and then by calculating the volume of S^3; (b) by computing the group volume Vol$[SU(2)]$ directly or (c) by showing that the volume of $SU(2)$ is limited from above by a finite number. Recall that, in addition, a compact group must be also closed.