Classical field theory 2013 (NS-364B) – Hand in problems, set 2 (in total *15 points*)

Tue May 28 2012, 13:15-17:00 BBG065. To be handed in at the exercise class of Jun 12.

Problem 1: Energy conservation in EM *(3 points)*

Show that

$$\partial_t \rho + \nabla \cdot \vec{S} = -\vec{E} \cdot \vec{j} \tag{1}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, $\vec{j}$ the electromagnetic current density, u the electromagnetic energy density and $\vec{S}$ the Poynting vector defined as

$$\rho = \frac{1}{8\pi}(\vec{E}^{\,2} + \vec{B}^{\,2}) \, ; \qquad \vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B} \, . \tag{2}$$

Eq. (2) has the form of an inhomogeneous continuity equation and it can be interpreted as follows. A change in the electromagentic energy density in some small region can be caused either by a flow of energy out of that region (described by $\vec{S}$) or by a dissipation (conversion of energy into heat), characterised by the $-\vec{E} \cdot \vec{j}$ term.
Hint: Use the Maxwell's equations.

Problem 2: Lorentz transformations of the electric and magnetic fields *(3 points)*

A general Lorentz transformation can be written as a 4×4 matrix,

$$\Lambda = \begin{pmatrix} \gamma & \gamma \vec{v}^{T}/c \\ \gamma \vec{v}/c & I + (\gamma - 1)\vec{v} \otimes \vec{v}^{T}/v^2 \end{pmatrix} \tag{3}$$

where $\gamma = [1 - (\vec{v}/c)^2]^{-1/2}$ and $v = \|\vec{v}\,\|$, such that

$$\Lambda^0_0 = \gamma \, , \qquad \Lambda^0_i = \Lambda^i_0 = \gamma v^i/c \, , \qquad \Lambda^i_j = \delta_{ij} + (\gamma - 1)v^i v^j/v^2 \tag{4}$$

By making use of the tensorial character of the electromagnetic field sthrength $F^{\mu\nu}$ and its decomposition in terms of the electric ($\vec{E}$) and magnetic fields ($\vec{B}$), derive the general transformation laws for the fields ($\vec{E}, \vec{B}$).

Problem 3: A particle in constant fields *(4 points)*

(A) A (relativistic) particle is moving in a constant electric $\vec{E}$. Find a general solution for the trajectory $\vec{x}(t)$ of a test particle.

(B) *Remark:* Provided the transformed fields satisfy $\|\vec{B}'\| < \|\vec{E}'\|$, one can use the Lorentz transformations found in Problem 2 to find a trajectory of a particle moving in a constant electric field $\vec{E}'$ and magnetic field $\vec{B}'$ (you do not need to do anything here!).

Question for you: Explain why in this case $\|\vec{B}'\| < \|\vec{E}'\|$ must be satisfied?

(C) Solve for a general particle's trajectory in a constant magnetic field $\vec{B}$.

Hint: Start with the equation $d(m\gamma\vec{v})/dt = (\vec{v}/c) \times \vec{B}$, and choose for simplicity $\vec{B}$ to point in the z-direction, $\vec{B} = (0, 0, B)$. Show next that $\|\vec{v}\| = \text{const}$ and $\gamma = (1 - (\vec{v}/c))^{-1/2} = \text{const}$. After this you can quite easily integrate the equation of motion.

If you combine the solution found in this problem with that in Problem 2, you can get a particle trajectory in a general constant electromagnetic field.

Problem 4: Quantization of Electromagnetism *(5 points)*

The action for electromagnetism is of the form (here we use the units in which $c = 1$),

$$S_{\rm EM} = \int d^4x \mathcal{L}_{\rm EM}\,, \qquad \mathcal{L}_{\rm EM} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu\,, \tag{5}$$

where $F = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv 2\partial_{[\mu}A_{\nu]}$ is the field strength. and $j^\mu = (c\rho, \vec{j})$ is a charge current and $A^\mu = (\varphi, \vec{A})$ is electromagnetic potential.

(A) *(1 point)*

Calculate the canonical momenta $\pi_\mu(x) = \delta S_{\rm EM}/\delta\partial_0 A^\mu(x)$ and the corresponding Hamiltonian density

$$\mathcal{H} = \pi_\mu \partial_0 A^\mu - \mathcal{L} \tag{6}$$

Show that, up to boundary terms, the Hamiltonian density can be written as,

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + C\varphi\,, \qquad \mathcal{H}_0 = \frac{1}{8\pi}\left(\vec{E}^2 + \vec{B}^2\right) - \vec{A}\cdot\vec{j} \\ C &= -\frac{1}{4\pi}\nabla\cdot\vec{E} + \rho\,, \end{aligned} \tag{7}$$

where $\vec{E} = 4\pi\vec{\pi}$.

(B) *(1 point)*

Calculate the (naive) Poisson brackets for the canonical pair (π_μ, A^ν). Show that

$$\{E_i(\vec{x},t), A^j(\vec{y},t)\} = 4\pi\delta_i^j\delta^3(\vec{x}-\vec{y})\,, \qquad \{E_i(\vec{x},t), \varphi(\vec{y},t\} = 0\,. \tag{8}$$

(C) *(2 points)*

You have learned in the lecture that it would be incorrect to quantize the Poisson brackets (8), since the true dynamical degrees of freedom are the tranverse fields $\vec{E}^T$, $\vec{A}^T$. Act with the transverse projector, $\delta_l^i + \nabla^{-2}\partial^i\partial_l$ on the naive Poisson bracket (8), and show that the result is the following *Dirac bracket*,

$$\begin{aligned} \{E_l^T(\vec{x},t), A^{T^j}(\vec{y},t)\}_D &= 4\pi\delta_l^j\delta^3(\vec{x}-\vec{y}) - \partial_l\partial^j\frac{1}{\|\vec{x}-\vec{y}\|} \\ \{E_i(\vec{x},t), \varphi(\vec{y},t\}_D &= 0\,, \end{aligned} \tag{9}$$

where ∇^{-2} is the inverse Laplacian, defined by,

$$[\nabla_x^2\delta^3(\vec{x}-\vec{y})]^{-1} = G(\vec{x}-\vec{y}) = -\frac{1}{4\pi}\frac{1}{\|\vec{x}-\vec{y}\|}\,,$$

and $G(\vec{x}-\vec{y})$ is the Green function of the operator ∇^2. Show by explicit calculation that the Dirac bracket (9) is tranverse on both legs $\vec{x}$ and $\vec{y}$.

(D) *(1 point)*

Perform a canonical quantization of electromagentism by promoting the Dirac brackets (9) to commutators according to the rule,

$$\{A, B\}_D \to \frac{[\hat{A}, \hat{B}]}{i\hbar}\,,$$

where $\hat{A}$ and $\hat{B}$ are operators. Write down explicitly

Note that promoting $\{E_i(\vec{x},t), \varphi(\vec{y},t)\}_D = 0$ to a commutator leads a trivial result, $[\hat{E}_i(\vec{x},t), \hat{\varphi}(\vec{y},t)] = 0$. What does this trivial quantization rule imply physically?