Classical field theory (NS-364B) – Hand in problems, set 4

Tue June 11 2013, 13:15-17:00 BBG065
To be handed in at the latest at the exercise class of June 25.

Problem 1: A real scalar field and its Green's and two point functions *(12 points)*

Consider a real massless scalar field ϕ, whose action is given by,

$$S[\phi] = \int d^4x \Big(\frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)\eta^{\mu\nu} - j_\phi\phi\Big)\,, \tag{1}$$

where j_ϕ is some scalar source current. The canonical momentum of ϕ is $\pi = \partial_t\phi$, and the canonical Poisson bracket and the corresponding the canonical quantization rule is hence,

$$\{\phi(\vec{x},t), \pi(\vec{x}',t)\} = \delta^3(\vec{x}-\vec{x}') \implies [\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{x}',t)] = \imath\hbar\delta^3(\vec{x}-\vec{x}')\,, \tag{2}$$

where by a *hat* we denote operators. Since the source current in the action (1) couples linearly to the field, the corresponding Green functions that should be used to solve the quantum equations for the field are identical to those of a classical scalar field.
This exercise is to some extent inspired by section 2.7 of Birrell & Davies, "Quantum Fields in Curved Space" (Cambridge University Press). From now on we shall set $\hbar = 1$.

(A) (*1 point*)

Show that, when quantised, the scalar field obeys the following equation of motion,

$$-\partial^2\hat{\phi}(x) = \hat{j}_\phi\,, \tag{3}$$

where $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembertian operator, and $\hat{j}_\phi$ is a quantised current.

(B) (*2 points*)

If the current $\hat{j}_\phi$ does not depend on $\hat{\phi}$, then the problem becomes linear and can be solved by the method of Green's functions. Show that the general solution in this case can be written as,

$$\begin{aligned}\hat{\phi}(x) &= \hat{\phi}_{0r}(x) + \int d^4x' G_r(x;x')\hat{j}_\phi(x')\,, & (4)\\ &= \hat{\phi}_{0a}(x) + \int d^4x' G_a(x;x')\hat{j}_\phi(x')\,, & (5)\end{aligned}$$

where G_r and G_a denote the retarded and advanced Green functions, respectively, and $\hat{\phi}_{0r}(x)$ and $\hat{\phi}_{0a}(x)$ are (in principle different) homogeneous free solutions that obey wave equations, $-\partial^2\hat{\phi}_{0r}(x) = 0 = -\partial^2\hat{\phi}_{0a}(x)$. $\hat{\phi}_{0r}(x)$ and $\hat{\phi}_{0a}(x)$ are determined by the homogeneous wave amplitudes existing on an intial (t_{in}) and final spatial surface (t_{fin}), respectively. At the lecture we have constructed explicitly the retarded Green function G_r, and shown that

$$G_r(x;x') = -\frac{\theta(t-t')}{2\pi}\delta(\Delta x^2)\,, \qquad \Delta x^2 = c^2(t-t')^2 - \|\vec{x}-\vec{x}'\|^2 \equiv c^2\Delta t^2 - r^2\,. \tag{6}$$

By making use of the same method (going to momentum space, and making a judicious choice of the contour of integration over k^0 which was shown at the lecture), construct G_a, and show that it can be written as,

$$G_a(x;x') = \frac{\theta(t'-t)}{2\pi}\delta(\Delta x^2)\,. \tag{7}$$

Pay, in particular, attention to making the correct choice of the integration contour in the complex k^0-plane. Explain what physical consideration motivated your choice of the integration contour that lead to G_a, and explain further how is the integration contour related to the epsilon prescription of the poles in the complex k^0-plane.

(C) (*4 points*)

The positive and negative frequency Wightman functions are defined as homogeneous solutions of the wave equation,

$$-\partial_x^2 G^{\pm}(x;x') = 0 = -\partial_{x'}^2 G^{\pm}(x;x')\,, \tag{8}$$

whereby (in the vacuum) $G^+(x;x')$ is determined by the contribution by integrating counterclockwise around the positive frequency pole $k^0 = \omega = c\|\vec{k}\,\|$, and $G^-(x;x')$ picks up the contribution by integrating clockwise around the negative frequency pole $k^0 = -\omega = -c\|\vec{k}\,\|$. Calculate $G^{\pm}$ in position space by performing suitable 4-momentum integrations and show that

$$G^+(x;x') = \frac{-\imath}{4\pi^2}\frac{1}{\Delta x_+^2}\,, \qquad G^-(x;x') = \frac{-\imath}{4\pi^2}\frac{1}{\Delta x_-^2} \tag{9}$$

where $\imath^2 = -1$ and

$$\Delta x_+^2 = (ct - ct' - \imath\epsilon)^2 - \|\vec{x} - \vec{x}'\,\|^2\,, \qquad \Delta x_-^2 = (ct - ct' + \imath\epsilon)^2 - \|\vec{x} - \vec{x}'\,\|^2\,. \tag{10}$$

Explain the origin of the infinitesimal parameter $\epsilon > 0$ in Eq. (10).

(D) (*2 points*)

The Pauli-Jordan, or spectral, two point function can be defined as

$$G_{PJ} \equiv \rho = G^- - G^+\,. \tag{11}$$

Show that

$$G_{PJ} \equiv \rho = -\frac{\mathrm{sign}(t-t')}{2\pi}\delta(\Delta x^2 - \epsilon^2)\,, \tag{12}$$

where Δx^2 is defined in (6) and $\mathrm{sign}(t-t') = \Theta(t-t') - \Theta(t'-t)$ and $\Theta(t-t')$ denotes the Heaviside function. Show further that

$$G_r(x;x') = \theta(t-t')G_{PJ}(x;x')\,, \qquad G_a(x;x') = \theta(t'-t)G_{PJ}(x;x')\,. \tag{13}$$

Hint: Make use of the Sokhotsky-Plemelj theorem,

$$\frac{1}{x \mp \imath\epsilon} = \mathcal{P}\frac{1}{x} \pm \imath\pi\delta(x)\,,$$

where $\mathcal{P}$ denotes a princial value (when integrating) and $\epsilon > 0$ is an infinitezimal parameter.

(E) (*2 points*)

The Wightman functions of a quantum theory can be written as expectation values of products of two field operators,

$$G^{+}(x;x') = -\imath\langle\Psi|\hat{\phi}(x)\hat{\phi}(x')|\Psi\rangle\,, \qquad G^{-}(x;x') = -\imath\langle\Psi|\hat{\phi}(x')\hat{\phi}(x)|\Psi\rangle\,, \tag{14}$$

where $|\Psi\rangle$ is the vacuum ket state of the scalar field theory and $\hat{\phi}(x)$ is the field operator in Heisenberg picture.

Show that the definitions (14) are consistent with Eqs. (12–13) and the canonical commutation relation (2), *i.e.* show that, when (14) is inserted into the equation of motion for $G_{r,a}$, one gets

$$-\,\partial_x^2 G_{r,a}(x;x') = \delta^4(x-x') = -\partial_{x'}^2 G_{r,a}(x;x')\,. \tag{15}$$

This analysis shows that the Green function of a classical and quantum theory are related by a simple relation,

$$[G(x;x')]_{\rm quantum} = \hbar[G(x;x')]_{\rm classical}\,. \tag{16}$$

This can be easily seen on the example of the retarded Green function of the quantum theory, which can be written as,

$$G_r(x;x') = -\imath\theta(t-t')\langle\Psi|[\hat{\phi}(x),\hat{\phi}(x')]|\Psi\rangle\,.$$

Now, by acting with $-\partial^2$ on both sides of this equation and making use of (2) one gets $-\partial^2 G_r(x;x') = \hbar\delta^4(x-x')$. The significance of $\hbar$ in (16) is profound. Namely, it shows that Green functions of a quantum theory $G\to 0$ when $\hbar\to 0$. The physical interpretation is that vacuum Green functions provide a measure of quantum fluctuations of the vacuum, which are present only when $\hbar\neq 0$. Examples can be found in sec 3.3 of Birrell & Davies, "Quantum Fields in Curved Space" (Cambridge University Press).

(F) (*1 point*)

Construct the Hadamard (or statistical) two-point function for the problem at hand, which is defined by,

$$G_H(x,x') \equiv F(x,x') \;=\; \frac{1}{2}\Big(\imath G^{+}(x;x') + \imath G^{-}(x;x')\Big)\,. \tag{17}$$

Notice that G_H and G_{PJ} are in fact the real and imaginary parts of the Wightman functions, which are also related as $(G^+)^* = -G^-$. While the spectral function contains information about what are the available states in the system, the statistical two-point function provides information on how these states are populated. For this Pauli blocking principle and Bose enhancement play an important role, which say that each fermionic state can be populated by at most one particle, while bosonic states can be populated by arbitrary number of particles. But, how that exactly affects what are the physically reasonable choices of these two point functions, is to be explained some other time. For now we just comment that in this exercise you have constructed vacuum two-point functions, for which all states are empty, *i.e.* they are occupied by zero real particles. The vacuum two-point functions are non-vanishing due to virtual particle fluctuations, which exist also in the vacuum. Under certain circumstances, these virtual vacuum fluctuations can be detected.