Problem set 10 for Cosmology (ns-tp430m)

Problems are due at Thu May 1. In total 12 points

19. The stress-energy tensor and entropy. (*8 points*)

The general relativistic expression for the stress-energy tensor in terms of the distribution function for a species a, $n_a = n_a(\vec{x}, \vec{p}, t)$, is given by,

$$T^{\mu}_{\nu}(x^{\alpha}) = \sum_a g_a \int \frac{d^3P}{(2\pi\hbar)^3}(-g)^{-1/2}\frac{cP^{\mu}P_{\nu}}{P_0}n_a\,, \tag{1}$$

where $P^{\mu} = mdx^{\mu}/d\lambda$ is the comoving momentum, $g_{\mu\nu}P^{\mu}P^{\nu} = m^2c^2$, $d/d\lambda = (dx^0/d\lambda)(d/dx^0) = [E/(mc^2)]d/dt$, g_a is the number of spin states for species a, and $g = \det[g_{\mu\nu}]$. For bosons in chemical equilibrium, n_a reduces to the Bose-Einstein distribution function,

$$n_a \to n_{aBE} = \frac{1}{\mathrm{e}^{(E_a-\mu_a)/k_BT} - 1}\,, \qquad E_a = E_a(p, m_a) \tag{2}$$

and for fermions we have the Fermi-Dirac distribution function,

$$n_a \to n_{aFD} = \frac{1}{\mathrm{e}^{(E_a-\mu_a)/k_BT} + 1}\,, \qquad E_a = E_a(p, m_a)\,, \tag{3}$$

where k_B denotes the Boltzmann constant.

Consider a FLRW expanding universe, with the metric tensor, $g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2)$.

(a) (*2 points)*

Eliminate the comoving momenta P_μ in (1) in favour of the proper (physical) momentum, whose magnitude is defined as,

$$(\vec{p})^2 = -g^{ij}P_iP_j, \tag{4}$$

and the direction vector, $\vec{p} \propto \vec{P}$. Determine the dispersion relation (the energy as a function of the momentum), $E = E(p, m)$.

(b) (*1 point*)

In a FLRW universe the comoving momenta P_i remain constant. Show that the physical momenta $(\vec{p})^2 = -g_{ij}P^iP^j$ scale inversely with the scale factor as $p_i \propto 1/a$.

(c) (*2 points)*

Consider next an ideal fluid, whose stress-energy tensor in the fluid rest frame acquires the form,

$$T^{\mu}_{\nu} = \mathrm{diag}(\rho, -\mathcal{P}, -\mathcal{P}, -\mathcal{P})\,. \tag{5}$$

Show from (1) that the energy density and pressure are

$$\begin{aligned} \rho &= \sum_a g_a \int \frac{d^3p}{(2\pi\hbar)^3} E(p) n_a \\ \mathcal{P} &= \sum_a g_a \int \frac{d^3p}{(2\pi\hbar)^3} \frac{c^2p^2}{3E(p)} n_a \,. \end{aligned} \tag{6}$$

(d) (*3 points*)

From the covariant stress-energy conservation in a FLRW universe, we know that

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}(\rho + \mathcal{P}) = 0\,. \tag{7}$$

Show that in a universe, in which $\mu_a \ll k_B T$,

$$\frac{\partial \mathcal{P}}{\partial T} = \frac{\rho + \mathcal{P}}{T}\,. \tag{8}$$

Show further that the entropy in a comoving volume, $S = sa^3$ is conserved in an expanding universe (in which the expansion is driven by a single perfect fluid),

$$\frac{dS}{dt} = 0\,, \tag{9}$$

where s denotes the entropy density, defined as

$$s = \frac{\rho + \mathcal{P}}{T}\,. \tag{10}$$

20. Relativistic degrees of freedom. (*4 points*)

(a) (*2 points*)

Calculate the effective number of relativistic degrees of freedom g_* in the Minimal Standard Model (MSM) of elementary particles. Assume that all particles are in thermal equilibrium at the same temperature, and that the temperature is much higher than the electroweak scale, $k_B T \gg E_{\rm ew} = 100$ GeV.

(b) (*2 points*)

Calculate the effective number of relativistic degrees of freedom g_* in the Minimal Supersymmetric Standard Model (MSSM), where to each bosonic degree of freedom a fermionic degree of freedom is associated, and *vice versa.* Recall that, unlike to the MSM, the MSSM has two Higgs doublet fields.