# Problem set 11 for Cosmology (ns-tp430m)

*Problems are due at Thu May 8. In total 14 points plus 2 bonus points*

**21.** *Baryon-to-photon ratio.* (*4 points*)
According to the Planck satellite measurement, the baryon-to-photon ratio today is,

$$\eta_{\rm b} \equiv \frac{n_{\rm b}}{n_\gamma} = 6.4 \pm 0.1 \times 10^{-10}\,, \tag{1}$$

where $n_{\rm b}$ denotes the baryon density, and $n_\gamma$ is the photon density.

**(a)** (*2 points*)

What is the baryon-to-entropy ratio $n_b/s$ today? Here $s$ denotes the entropy density. Recall that the effective number of degrees of freedom today is, $g_*(T_0) \simeq 3.36$ ($T_0 \simeq 2.73$ K), $g_{*S}(T_0) \simeq 3.91$ (the difference is due to a lower temperature of the neutrino fluid today, $T_\nu \simeq 1.96$ K), while above the electroweak transition, $g_*(T \gg T_{\rm ew}) \simeq 106.75$ ($k_B T_{\rm ew} \sim 100$ GeV).

**(b)** (*2 points*)

What is $n_b/s$ and $\eta_b = n_b/n_\gamma$ at temperatures above the neutrino decoupling temperature, $100\ {\rm MeV} \gg k_B T \gg 1$ MeV, where the only additional relativistic particles are electrons and positrons? Assume that baryons are generated at a temperature much above $100\ {\rm MeV}/k_B$.

**22.** *Thermal effective potential.* (*10 points + 2* bonus points*)
Thermal effects may cause phase transitions in the early Universe (Kirzhnits, Linde, 1972), most notable being the electroweak (phase) transition and the strong phase transition. While the full account of phase transition dynamics usually requires a nonperturbative treatment, some of the important features of the transition are captured by the effective potential calculated by truncating the effective action at a certain order in the loop expansion (an expansion in powers of $\hbar$). In this problem we set $\hbar = 1$ and $c = 1$.

A simple thermal effective potential is the one-loop approximation to the effective action of a (self-interacting) real scalar field $\phi$, whose tree-level Lagrangian reads,

$$\mathcal{L}_0 = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi) \qquad \eta^{\mu\nu} = {\rm diag}(1,-1,-1,-1)\,, \tag{2}$$

where the potential reads,

$$V(\phi) = \bar{V}_0 + \frac{1}{2}m_0^2\phi^2 + \frac{\lambda}{4!}\phi^4\,, \tag{3}$$

and the tree-level action is $S_0[\phi] = \int d^4x \mathcal{L}_0$.

When $m_0^2 \geq 0$, the field expectation value, $\langle\phi\rangle = 0$. When treated in a tree-level approximation, the field is in its symmetric minimum, $\langle\phi\rangle = 0$. When, on the other hand, $m_0^2 = (d^2V/d\phi)^2(\phi = 0) < 0$, the $Z_2$-symmetry $\phi \to -\phi$ is spontaneously broken by the vacuum, such that the scalar vacuum corresponds to one of the two minima, $\phi = \pm\mu$, $\mu^2 = -6m_0^2/\lambda$. In this case it is convenient to consider the action for a shifted scalar field,

$$\phi = \phi_0 + \varphi\,, \qquad \phi_0^2 = -\frac{6m_0^2}{\lambda} > 0\,, \tag{4}$$

such that the quadratic part of the shifted action in an adiabatically expanding universe (the scale factor can be chosen $a = 1$) reads,

$$\begin{aligned} \int d^4x\mathcal{L}_0 &= \int d^4x\int d^4y \frac{1}{2}\varphi(x)\mathcal{D}(x;y;\phi_0)\varphi(y)\,, \qquad (5)\\ \mathcal{D}(x;y;\phi_0) &= -\sqrt{-g}(\Box_x + m^2)\delta^4(x-y)\,, \quad m^2 = m^2(\phi_0) = m_0^2 + \frac{\lambda}{2}\phi_0^2 = -2m_0^2\,, \quad \Box_x = \partial_x^2\,,\ \sqrt{-g} = 1\,. \end{aligned}$$

Figure 1: The one-loop Feynman diagram contributing at one-loop order ($\hbar$) to the effective action of a real scalar theory.

To calculate the one-loop thermal contribution to the effective action, whose Feynman graph representation is shown in figure 1, observe that the scalar thermal Feynman (time ordered) propagator is defined as,

$$i\Delta_F(x-x') = \frac{\mathrm{Tr}[\mathrm{e}^{-\beta\hat{H}}\mathcal{T}(\hat{\phi}(x)\hat{\phi}(x'))]}{\mathrm{Tr}[\mathrm{e}^{-\beta\hat{H}}]}\,, \tag{6}$$

where $\mathcal{T}$ denotes time ordering,

$$\mathcal{T}(\hat{\phi}(x)\hat{\phi}(x')) = \theta(t-t')\hat{\phi}(x)\hat{\phi}(x') + \theta(t'-t)(\hat{\phi}(x')\hat{\phi}(x)) \tag{7}$$

$$\hat{H} = \int d^3x\Big[\frac{1}{2}(\hat{\pi}_\phi)^2 + \frac{1}{2}(\nabla\hat{\phi})^2 + V(\hat{\phi})\Big]\,, \qquad \hat{\pi}_\phi = \frac{d\hat{\phi}}{dt} \tag{8}$$

is the Hamiltonian operator, and $\beta = 1/(k_BT)$.

**(a)** (*4 points*)

By making use of the definition (6), show that the scalar propagator (6) obeys the boundary condition ($\hbar = 1$),

$$i\Delta_F(z^0 = 0, \vec{z}) = i\Delta_{\bar{F}}(z^0 = -i\beta, \vec{z}), \qquad \text{where } z = x - y\,, \tag{9}$$

where $i\Delta_{\bar{F}}$ is the anti-time ordered propagator. In this *imaginary time formalism* the scalar Feynman propagator obeys the equation of motion,

$$-\sqrt{-g}(\Box_x + m^2)i\Delta_F(x-x') = i\delta^4(x-x')\,. \tag{10}$$

Show that the Feynman propagator $i\Delta_F(x-x')$, which obeys the boundary condition (9), is given in the imaginary (Matsubara) formalism by (see *e.g.* Le Bellac, Thermal Field Theory (1996)),

$$i\Delta_F(x-x') \to i\Delta_E(x-x') = k_BT\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\mathrm{e}^{-[2\pi nk_BT(t-t')-i(\vec{x}-\vec{x}\,')\cdot\vec{k}]}i\tilde{\Delta}_E(k)\,, \tag{11}$$

and

$$i\tilde{\Delta}_E(k) = \frac{i}{\omega_n^2 - \omega^2(k)}\,, \qquad \omega(k) = \sqrt{\vec{k}^2 + m^2}\,, \tag{12}$$

where $\omega_n = 2\pi ink_BT$ ($n = 0, \pm1, \pm2, ..$) denote the imaginary Matsubara frequencies and $i\Delta_E$ is the Euclidean (Matsubara) propagator.

Alternatively, one can use the real time formalism, to obtain (you do not need to show this),

$$i\Delta_F(x-x') = \int\frac{d^4k}{(2\pi)^4}\mathrm{e}^{-ik\cdot(x-x')}i\tilde{\Delta}_F(k)\,, \tag{13}$$

with

$$i\tilde{\Delta}_F(k) = \frac{i}{k^2 - m^2 + i\epsilon} + 2\pi n_{\mathrm{BE}}(\omega)\delta(k^2 - m^2)\,, \qquad n_{\mathrm{BE}}(\omega) = \frac{1}{\mathrm{e}^{\beta\omega(k)} - 1}\,, \tag{14}$$

where $\epsilon > 0$ is an infinitesimal parameter used to move the propagator poles away from the real axis (needed for the $k^0$-integration prescription for the Feynman propagator).

The one-loop contribution to the effective potential for a real scalar field shown in figure 1 reads (up to a constant),

$$V_{T1}(\phi) = -\frac{i}{2}\int_k \ln[\tilde{\mathcal{D}}(k;\phi_0)] \tag{15}$$

where

$$\tilde{\mathcal{D}} = k^2 - m^2\,, \qquad m^2 = m_0^2 + \frac{\lambda}{2}\phi_0^2 \tag{16}$$

denotes the inverse propagator (5) in momentum space, $\int_k = ik_BT\sum_{n=-\infty}^{\infty}\int[d^3k/(2\pi)^3]$, $k^2 = \omega_n^2 - \vec{k}^2$ in the imaginary time formalism, and $\int_k = \int[d^4k/(2\pi)^4]$, $k^2 = k_0^2 - \vec{k}^2$ in the real time formalism.

**(b)** (*4 points*)

Show that the one-loop thermal contribution (15) to the effective potential can be written as,

$$V_{T1}(\phi) = \frac{k_B T}{2} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \ln\left[ -(2\pi n k_B T)^2 - \omega^2 \right] \tag{17}$$

where $\omega^2 = \vec{k}^2 + m^2$. By making use of the following sum,

$$\sum_{n=1}^{\infty} \frac{y}{y^2+n^2} = -\frac{1}{2y} + \frac{\pi}{2}\coth(\pi y)\,, \tag{18}$$

show that the sum over $n$ in (17) can be evaluated, and the result can be written in the form,

$$\begin{aligned} V_{T1}(\phi) &= \Delta V_{0,\text{vac}} + \Delta V_{T1} \\ \Delta V_{T1} &= \frac{(k_BT)^4}{2\pi^2}\int_0^\infty dx x^2 \ln\Big[1 - \exp\Big( -(x^2 + m^2/(k_BT)^2)^{1/2}\Big)\Big]\,, \end{aligned} \tag{19}$$

where

$$\Delta V_{0,\text{vac}} = \int \frac{d^3k}{(2\pi)^3}\frac{\omega}{2} \tag{20}$$

The divergent contribution, $\Delta V_{0,\text{vac}}$, is due to the zero temperature vacuum fluctuations. This contribution is usually combined with $\bar{V}_0$ in Eq. (3) to a finite constant potential term, $V_0 = \bar{V}_0 + \Delta V_{0,\text{vac}}$, and we do not consider it further.

The remaining contribution $\Delta V_{T1}$ in (19) is finite, and it is solely due to thermal excitations.

**(c)** (*2 points + 2* bonus points*)

Show that, in the high temperature limit,

$$k_BT \gg m\,, \tag{21}$$

$\Delta V_{T1}$ can be expanded in a Taylor series to yield,

$$\begin{aligned} \Delta V_{T1} &= -\frac{\pi^2}{90}(k_BT)^4 + \frac{1}{24}m^2(k_BT)^2 - \frac{1}{12\pi}m^3 k_BT - \frac{1}{64\pi^2}m^4\Big[\ln\Big(\frac{m^2}{(k_BT)^2}\Big) - c_0\Big] \\ &+ O(m^6/(k_BT)^2)\,, \end{aligned} \tag{22}$$

where $c_0 = (3/2) + 2\ln(4\pi) - 2\gamma_E \simeq 5.4076$, $\gamma_E \equiv -\psi(1) = 0.577215..$ denotes the Euler constant, and $\psi(z) = d[\ln\Gamma(z)]/dz$ is the di-gamma function. Note the negative cubic term,

$$-B_T k_BT\phi_0^3 \simeq -\frac{1}{12\pi}m^3k_BT \simeq -\frac{1}{12\pi}\Big(\frac{\lambda}{2}\Big)^{3/2}k_BT\phi_0^3\,, \tag{23}$$

$m^2 \approx (\lambda/2)\phi_0^2$, which can thermally induce a first order phase transition. When the same calculation is repeated for fermions, one finds no cubic term (23), implying that, when treated in the high-temperature approximation, fermions cannot induce a first order transition. This is because the cubic term arises due to the infrared singularity in the Bose-Einstein distribution, $n_{\rm BE} = 1/(e^{\beta\omega} - 1) \to k_BT/\omega$ when $E \to 0$, which is absent in the Fermi-Dirac distribution.

Strictly speaking, in Eq. (22) we calculated the one-loop thermal correction to the free energy (per unit volume), $\Delta V_{1T} \equiv F/V = \rho - Ts = -\mathcal{P}$, where we used the following relation for the entropy density, $s = (\mathcal{P}+\rho)/T$, such that $\Delta V_{1T}$ is also equal to minus the pressure.