Problem set 12 for Cosmology (ns-tp430m)

Problems are due at Thu May 15. In total 14 points plus 2 bonus points

23. Nucleation of bubbles at a first order phase transition: Thin wall approximation. (*10 points + 2* bonus points*)

Phase transitions in the early Universe are thermally induced, which means that one can infer a phase transition dynamics from the appropriate thermal effective potential. As regards the electroweak phase transition, the following form of the thermal effective potential can be used as a reasonably good approximation for the dynamics at a first order electroweak phase transition (in this exercise we set $c = 1$, $k_B = 1$ and $\hbar = 1$),

$$V_T(\phi) = \frac{1}{2}m_T^2\phi^2 - ET\phi^3 + \frac{\lambda_T}{4}\phi^4 \,, \qquad m_T^2 = 2D(T^2 - T_0^2)\,, \tag{1}$$

where ϕ is a real scalar field, which can be thought of as a thermaly induced Higgs field expectation value, and m_T^2, E and λ_T the effective parameters of the potential, which are functions of the Higgs mass, quartic coupling, and gauge and Yukawa couplings of the Minimal Standard Model, and of course of the temperature T.

At very high temperatures, $T \gg T_c$, where T_c denotes the critical temperature, the potential has one minimum at $\phi = 0$.

(a) (*2 points*)

When $m_T^2 > 0$ and $E > 0$, the effective potential (1) may have two minima. Show that a second minimum starts developing at the temperature T_1 given by,

$$T_1^2 = \frac{T_0^2}{1 - 9E^2/(8\lambda_{T_1}D)}\,, \tag{2}$$

and that the field value at this minimum is

$$\phi_1 = \frac{3ET_1}{2\lambda_{T_1}}\,. \tag{3}$$

(b) (*2 points*)

Show that the critical temperature, T_c (at which there are two degenerate minima) is given by the following relation,

$$T_c^2 = \frac{T_0^2}{1 - E^2/(\lambda_{T_c}D)},, \tag{4}$$

and that the corresponding field values are,

$$\phi = 0\,, \qquad \phi_c = \frac{2ET_c}{\lambda_{T_c}}\,. \tag{5}$$

(c) (*2 points*)

Show that at the temperature $T = T_0$, the minimum at the origin disappears. Show that at this temperature the field value of the second minimum is given by

$$\phi_0 = \frac{3ET_0}{\lambda_{T_0}}\,. \tag{6}$$

When a second minimum is formed and becomes lower than the minimum at the origin, it is energetically favourable that the Universe be in the broken phase with $\phi > 0$. At high temperatures, the tunneling rate per unit volume into the broken ("true vacuum") phase is of the order,

$$\mathcal{P} \sim T^4 \mathrm{e}^{-S_3/T} \tag{7}$$

where S_3 is the three dimensional instanton action (the critical bubble action),

$$S_3 = 4\pi \int r^2 dr \Big[\frac{1}{2}\Big(\frac{d\phi}{dr}\Big)^2 + V_T(\phi)\Big] . \tag{8}$$

If a bubble of the true vacuum begins nucleating at a temperature, which is just slightly below the critical temperature, the difference between the free energy (effective action) of the two minima, ΔV (chosen by convention to be positive) is much smaller that the size of the energy barrier between them, then the radius of the bubble r is much larger than the thickness of the bubble wall δr (defined as the region over which $\phi = \phi(r)$ changes significantly).

(d) (*2 points*)

In this *thin wall approximation* one can show that (8) can be approximated by,

$$S_3 \approx -\frac{4\pi}{3} r^3 \Delta V + 4\pi r^2 S_1 , \tag{9}$$

where

$$S_1 \simeq \int d\phi \sqrt{2V_T(\phi)} \simeq \sigma \tag{10}$$

and σ denotes the surface tension. In deriving (9) and (10) we assumed the equipartition of gradient and potential energy, $(1/2)(d\phi/dr)^2 \simeq V_T(\phi)$ on the boundary of the bubble.

(e) (*2 points*)

Calculate the radius of the critical bubble by finding the extremum of the action (9), and show that it equals,

$$r_c = \frac{2S_1}{\Delta V} . \tag{11}$$

Show that the critical action (9) is then,

$$S_{3c} \simeq \frac{16\pi}{3} \frac{S_1^3}{(\Delta V)^2} , \tag{12}$$

which is a maximum of the action. This is because critical bubbles are unstable, which means that, once they form, critical bubbles either expand or contract.

The bubbles begin nucleating when the probability for bubble nucleation per unit time becomes of the order of the expansion rate of the Universe H.

One can then show that the fraction of the Universe filled with bubbles at a time t is given by,

$$f = 1 - \mathrm{e}^{-\Delta} , \tag{13}$$

where

$$\Delta(t) = \int_0^t dt' \frac{4\pi}{3} v_b^3 (t - t')^3 \mathcal{P} , \tag{14}$$

v_b is the (radial) velocity of the bubble expansion, and $\mathcal{P} = \mathcal{P}(t')$ denotes the tunneling rate per unit volume (7). This rate becomes significant only below the critical temperature T_c, such that $t = 0$ in (14) can be chosen as $T(0) = T_c$. In general, $\mathcal{P}$ is a complicated function of t' that can be read off from Eq. (7), which is somtimes approximated by a step function $\mathcal{P}(t') = \mathcal{P}_0\, \theta(t')$. A better approximation for $f = f(T)$ is obtained by expanding the action S_3 in (7) around $T = T_c$.

(f) (*2* bonus points*) Construct an argument for the correctness of the expressions (13–14). (You do not need to calculate $f = f(T)$.)

24. Particle density. (*4 points*)
What is the relation between the density of particles (minus antiparticles)

$$N_{b,f} \equiv (N_p - N_{\bar{p}})_{b,f} \tag{15}$$

and the chemical potential for (a) bosons and (b) fermions, where N_p denotes the number density of particles and $N_{\bar{p}}$ is the number density of antiparticles.

Assume that the phase space densities of bosons n_b and fermions n_f are of the form,

$$n_{b\pm} = \frac{1}{\mathrm{e}^{\beta(cp\mp\mu_b)} - 1}, \qquad n_{f\pm} = \frac{1}{\mathrm{e}^{\beta(cp\mp\mu_f)} + 1} \tag{16}$$

where $\beta = 1/(k_B T)$, T denotes the temperature and k_B the Štefan-Boltzmann constant, $p = \|\vec{p}\|$ the momentum of particles (we are working in the super-relativistic limit in which $E = cp$ and $\pm\mu_{f,b}$ denotes the chemical potential for particles (the + sign) and antiparticles (the − sign), respectively. Express your answer in terms of the temperature $k_B T$, chemical potentials $\mu_{b,f}$, and the Riemann zeta-function, defined as, $\zeta(z) = \sum_{n=1}^{\infty}(1/n^z)$.

Hint: Assume that the chemical potentials are small, and then Taylor-expand to linear order in $\mu_{b,f}$.