Problem set 14 for Cosmology (ns-tp430m)

Problems are due at Thu Jun 5. In total 10 points plus 3 bonus points.

27. Kinetic equation for the electron-proton recombination. (*6 points + 3* bonus points*)

In an expanding Universe the following Boltzmann equation applies,

$$\frac{1}{a^3}\frac{d}{dt}\Big(a^3 n_1\Big) = -c n_{10} n_{20}\langle\sigma v\rangle\Big(\frac{n_1 n_2}{n_{10}n_{20}} - \frac{n_3 n_4}{n_{30}n_{40}}\Big)\,, \tag{1}$$

where

$$n_{a0} = g_a\int\frac{d^3p_a}{(2\pi\hbar)^3}\mathrm{e}^{-E_a/(k_BT)} \tag{2}$$

$$n_a = n_{a0}\mathrm{e}^{\mu_a/(k_BT)} \qquad (a=1,2,3,4)\,. \tag{3}$$

Here n_{a0} denotes the equilibrium density, $\langle\sigma v\rangle$ denotes the thermally averaged cross section, g_a is the number of degrees of freedom of a species a, μ_a is the chemical potential of a species a.

For the recombination of electrons (e) and protons (p), the following interaction is relevant,

$$e + p \leftrightarrow {}^1H + \gamma \qquad (B_H = 13.6\ \mathrm{eV})\,, \tag{4}$$

where ${}^1H$ denotes neutral hydrogen and γ a photon. The corresponding cross section is,

$$\langle\sigma v\rangle = 9.78\frac{\alpha_e^2\hbar^2}{m_e^2c^2}\Big(\frac{B_H}{k_BT}\Big)^{1/2}\ln\Big(\frac{B_H}{k_BT}\Big)[\mathtt{cm}^2]\,, \tag{5}$$

where $\alpha_e = 1/137$ denotes the (electromagnetic) fine structure constant, $m_ec^2 = 0.511$ MeV, and $B_H = (m_e + m_p - m_H)c^2 = 13.6$ eV.

(a) (*2 points*)

Show that integrating (2) results in the equilibrium density of particle species a,

$$n_{a0} = g_a\Big(\frac{m_ak_BT}{2\pi\hbar^2}\Big)^{\frac{3}{2}}\mathrm{e}^{-m_ac^2/(k_BT)}\,, \qquad (m_ac^2 \gg k_BT)\,. \tag{6}$$

(b) (*4 points*)

Show that, when applied to the reaction (4), Eq. (1) yields,

$$\frac{dX_e}{dt} = -c\langle\sigma v\rangle\Big(n_bX_e^2 - (1-X_e)\Big(\frac{m_ek_BT}{2\pi\hbar^2}\Big)^{3/2}\mathrm{e}^{-B_H/(k_BT)}\Big)\,, \tag{7}$$

where $X_e = n_e/n_b$. Assume that the Universe is electrically neutral, and recall that the chemical potential of photons is zero.

(b) (*3* bonus points*)

Solve Eq. (7) numerically, and estimate the redshift at which X_e freezes out and the value at which X_e freezes out. Recall that the baryon density is, $n_b = \eta_b n_\gamma$, where the photon density is, $n_\gamma = 2[\zeta(3)/\pi^2](k_B T/\hbar c)^3$, $\zeta(3) \simeq 1.202$, and $\eta_b = (6.025 \pm 0.073) \times 10^{-10}$ [Planck 2013].

28. Optical depth. (*4 points*)

The optical depth is defined as,

$$\tau(\eta) = c\sigma_T \int_{\eta_0}^{\eta} d\eta' a\,(n_e + n_p), \tag{8}$$

where η is conformal time ($ad\eta' = dt'$), $a = a(\eta')$ the scale factor, η_0 some initial conformal time, $n_e = n_e(\eta')$ and $n_p = n_p(\eta')$ the (free) electron and proton densities (per unit volume), and σ_T is the Thomson scattering cross section (which in general corresponds to the scattering cross section of an electromagnetic wave off a (heavy, nonrelativistic) charged particle),

$$\sigma_T = \frac{8\pi}{3} r_e^2 = 6.65 \times 10^{-29}\ \mathrm{m}^2\,, \tag{9}$$

where

$$r_e = \frac{\hbar e^2}{4\pi m_e c} = 2.8 \times 10^{-15}\ \mathrm{m}^2\,. \tag{10}$$

The Planck 2013 data imply the optical depth of the CMB photons (from the last scattering up to today),

$$\tau = 0.089 \pm 0.014\,. \tag{11}$$

Assume a simple model of reionisation, according to which the Universe gets completely and suddenly reionised at a certain redshift, z_{reion}. Calculate z_{reion}, which yields the observed optical depth (11)! What is the age of the Universe at the time z_{reion}?