Problem set 16 for Cosmology (ns-tp430m)

Problems are due at Thu June 19. In total 18 points.

30. De Sitter space. (*10 points*)

Perhaps the simplest representation of de Sitter space is the following hyperbolic embedding

$$X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = -\frac{1}{H^2} \tag{1}$$

into a flat five dimensional space, with the line element given by,

$$ds_5^2 = \sum_{A,B=0}^{4} \eta^{AB} dX_A dX_B = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 - dX_4^2 \tag{2}$$

where $\eta^{AB} = \mathrm{diag}(1, -1, -1, -1, -1)$. From this embedding, it trivially follows that the de Sitter space line element is invariant under SO(1,4) transformations, which are the transformations that leave invariant the distance between the two points,

$$Z(X; X') = -\frac{H^2}{2} \sum_{A,B=0}^{4} \eta^{AB} (X - X')_A (X - X')_B - 1 = H^2 \sum_{A,B=0}^{4} \eta^{AB} X_A X'_B \,, \tag{3}$$

such that two points X and X' are space-like (time-like) separated when $Z(X; X') > -1$ ($Z(X; X') < -1$). X and X' are said to be light-like separated when $Z(X; X') = -1$.

(a) (*4 points*) Show that under the coordinate transformation,

$$\begin{aligned}
X_0 &= \frac{1}{H} \sinh(Ht) \cosh(\lambda) \\
X_1 &= \frac{1}{H} \cosh(Ht) \\
X_2 &= \frac{1}{H} \sinh(Ht) \sinh(\lambda) \cos(\theta) \\
X_3 &= \frac{1}{H} \sinh(Ht) \sinh(\lambda) \sin(\theta) \cos(\phi) \\
X_4 &= \frac{1}{H} \sinh(Ht) \sinh(\lambda) \sin(\theta) \sin(\phi)
\end{aligned} \tag{4}$$

where H denotes the Hubble parameter, the line element (2) transforms into,

$$ds^2 = dt^2 - \frac{1}{H^2} \sinh^2(Ht) \Big(d\lambda^2 + \sinh^2(\lambda) d\Omega_2^2 \Big) \,, \qquad (-\infty < t < \infty, 0 \le \lambda < \infty) \,, \tag{5}$$

such that the scale factor, $a = \sinh(Ht)$ for $t > 0$. Here $d\Omega_2^2 = d\theta^2 + \sin^2(\theta) d\phi^2$ is the line element of the two dimensional unit sphere S^2 ($\theta \in [0, \pi]$, $\phi \in [0, 2\pi\rangle$).

(b) (*2 points*) Show that, when expressed in terms of conformal time η, the scale factor a is given by,

$$a(\eta) = -\frac{1}{\sinh(H\eta)} \,, \qquad (-\infty < \eta < 0) \tag{6}$$

and that the line element in conformal coordinates (defined by $ad\eta = dt$) reads,

$$ds^2 = \frac{1}{\sinh^2(H\eta)}\Big[d\eta^2 - \frac{1}{H^2}\Big(d\lambda^2 + \sinh^2(\lambda)d\Omega_2^2\Big)\Big]\,. \tag{7}$$

(c) (*2 points*) Show that upon the following coordinate transformation

$$\begin{aligned} X_0 &= \sqrt{\frac{1}{H^2} - r^2}\,\sinh(Ht)\,, \qquad X_1 = \sqrt{\frac{1}{H^2} - r^2}\,\cosh(Ht)\,, \qquad (0 \leq r < 1/H) \\ X_2 &= r\sin(\theta)\cos(\phi)\,, \qquad X_3 = r\sin(\theta)\sin(\phi)\,, \qquad X_4 = r\cos(\theta)\,, \end{aligned} \tag{8}$$

the line element (2) transforms into the following static metric,

$$ds^2 = (1 - H^2r^2)dt^2 - \frac{dr^2}{1 - H^2r^2} - r^2 d\Omega_2^2\,, \qquad (0 \leq r < \infty, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi)\,. \tag{9}$$

The price to pay to get a static slicing of de Sitter space is breakdown of spatial homogeneity (expressed through the dependence on the radial coordinate r). Similarly to the Schwarzschild metric, this metric exhibits an event horizon. For an observer at the spatial origin, $r = 0$, the event horizon corresponds to $r = 1/H$. Can you give a simple argument supporting the statement that the metric (8) covers at most 1/2 of the de Sitter manifold? In fact, when viewed on the Carter-Penrose diagram, the static coordinates cover 1/4 of de Sitter space.

(d) (*2 points*) Show that the de Sitter invariant function (3) in the static coordinate system (8) is $(0 \leq r \leq H^{-1})$,

$$Z(x, x') = -(1 - H^2r^2)^{\frac{1}{2}}(1 - H^2r'^2)^{\frac{1}{2}}\cosh\left[H(t - t')\right] - H^2rr'\cos(\angle(\vec{x}, \vec{x}'))\,. \tag{10}$$

where $\cos(\angle(\vec{x}, \vec{x}')) = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi')$. At the origin $r = 0 = r'$, this is simply, $Z(x, x') = -\cosh\left[H(t - t')\right]$.

31. Power law inflation with an exponential potential. (*8 points*)

Consider inflation realised by a scalar field in the theory with the action of the form,

$$S = \int d^4x\sqrt{-g}\Big(\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V\Big) \tag{11}$$

with the metric, $g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2)$, where $a = a(t)$ denotes the scale factor, and $-g = -\det(g_{\mu\nu}) = a^6$ is the determinant of the metric. Assume that the potential is of the exponential form,

$$V(\phi) = M^4 \mathrm{e}^{-\lambda\phi/M}\,, \tag{12}$$

where M is an energy scale, and λ is a coupling constant. The energy density and the pressure of a homogeneous scalar field, $\phi = \phi(t)$ are,

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V \\ \mathcal{P}_\phi &= \frac{1}{2}\dot{\phi}^2 - V\,. \end{aligned} \tag{13}$$

Assume further that the homogeneous field ϕ obeys the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{\rho_\phi}{3M_P^2}\,, \tag{14}$$

where $M_P \equiv (8\pi G_N)^{-1/2} \simeq 2.4 \times 10^{18}$ GeV denotes the reduced Planck mass.

(a) (*1 point*) Show that the equation of motion for the homogeneous scalar field reads,

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0\,, \qquad V' = -\lambda M^3 \mathrm{e}^{-\lambda\phi/M}\,, \tag{15}$$

where $V' = \frac{d}{d\phi}V(\phi)$ and $\dot{\phi} = d\phi/dt$.

(b) (*2 points*) Show that there is a solution of the form,

$$\phi = \phi_0 \ln\left(\frac{t}{\tau}\right) \tag{16}$$

where ϕ_0 and τ are the following constants,

$$\begin{aligned} \phi_0 &= \frac{2M}{\lambda} \\ \tau &= \frac{\sqrt{2}}{\lambda M}\left[\frac{6}{\lambda^2}\left(\frac{M}{M_P}\right)^2 - 1\right]^{\frac{1}{2}}\,, \qquad (\lambda < \sqrt{6}\,M/M_P)\,. \end{aligned} \tag{17}$$

(c) (*1 point*) Show that the energy density in this homogeneous mode is,

$$\rho_\phi = \frac{12M^4}{\lambda^4 M_P^2 t^2}\,. \tag{18}$$

(d) (*1 point*) Show that the scale factor as a function of time,

$$a = a_0\left(\frac{t}{t_0}\right)^{\frac{2M^2}{\lambda^2 M_P^2}}\,, \tag{19}$$

such that this solution corresponds to a power law expansion, $a \propto t^\alpha$, with $\alpha > 1/3$. The limiting case, $\alpha = 1/3$ corresponds to kination, and it is realised in the case when the potential $V = V(\phi)$ can be neglected, when compared to the kinetic term, $\dot{\phi}^2/2$. When, on the other hand, $\alpha > 1$, the Universe inflates.

(e) (*1 point)* Show that the Universe undergoes an accelerated expansion when,

$$\lambda < \frac{\sqrt{2}M}{M_P}\,. \tag{20}$$

(f) (*1 point)* Show that the equation of state, $\mathcal{P}_\phi = w_\phi \rho_\phi$ is determined by,

$$w_\phi = -1 + \frac{\lambda^2 M_P^2}{3M^2} \tag{21}$$

such that the condition for inflation,

$$\rho_{\text{active}} = \rho + 3\mathcal{P} < 0 \tag{22}$$

is fulfilled when the condition (20) for λ is satisfied.

(g) (*1 point*) Show that during N e-foldings of inflation, the field changes by, $\Delta\phi = \lambda N M_P^2/M < \sqrt{2}\, N M_P$.

The solution discussed in this problem is an *attractor*, in the sense that if the initial condition is such that $\dot{\phi}^2/2 \gg V$, or $\dot{\phi}^2/2 \ll V$, after a sufficiently long time, the solution eventually approaches, $\phi = (2M/\lambda)\ln(t/\tau)$, with τ given in (17).