Problem set 17 for Cosmology (ns-tp430m)

Problems are due at Thu July 3. In total 10 points

32. Spectrum of a massless minimally coupled scalar field. (*10 points*)

Consider a massless minimally coupled real scalar field with the action,

$$S = \int d^4x\sqrt{-g}\Big(\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)\Big)\,, \tag{1}$$

where $g = det(g_{\mu\nu})$ is the determinant of the metric. In a FLRW cosmology the metric tensor, when expressed in conformal coordinates, is of the form,

$$g_{\mu\nu} = a^2\eta_{\mu\nu}, \qquad \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)\,, \tag{2}$$

where $a = a(\eta)$ denotes the scale factor, and η is conformal time.

Consider the evolution of vacuum fluctuations of a scalar field given by the action (1) during de Sitter inflationary epoch, and then match it onto radiation era. Take the scale factor during de Sitter inflation to be of the form,

$$a = -\frac{1}{H_I\eta} \qquad (-\infty < \eta \leq -1/H_I)\,, \tag{3}$$

such that $a = 1$ at the end of inflation. Here H_I denotes the Hubble parameter during inflation.

(a) (*1 point*) By varying the action (1) in the space-time (2), show that the equation of motion for the scalar field reads,

$$\partial_\eta^2\phi + 2\frac{a'}{a}\partial_\eta\phi - \nabla^2\phi = 0\,, \tag{4}$$

where $\nabla^2 = \sum_{i=1}^3 \partial^2/\partial x_i^2$, $a' = da/d\eta$ and $\partial_\eta = \partial/\partial\eta$.

The scalar field ϕ can be promoted to an operator, $\phi \to \hat{\phi}$, by the canonical quantisation,

$$[\hat{\phi}(\vec{x},\eta), \hat{\pi}_\phi(\vec{x}\,',\eta)] = i\hbar\delta^3(\vec{x}-\vec{x}\,') \tag{5}$$

where $\hat{\pi}_\phi = a^2 d\hat{\phi}/d\eta$ denotes the canonical momentum of $\hat{\phi}$. This can be achieved by the following decomposition,

$$\hat{\phi}(\vec{x},\eta) = \frac{1}{a}\int \frac{d^3k}{(2\pi)^3} \mathrm{e}^{i\vec{k}\cdot\vec{x}}\Big[\varphi_k(\eta)\hat{a}_{\vec{k}} + \varphi_k^*(\eta)\hat{a}^\dagger_{-\vec{k}}\Big]\,, \tag{6}$$

where $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ denote the annihilation and creation operators, and φ_k and φ_k^* are the mode functions, which due to the homogeneity of the background space (one assumes a spatial translation invariance of the state), depend only on the modulus $k = \|\vec{k}\,\|$. The annihilation and creation operators satisfy the commutation relations,

$$[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = \hbar(2\pi)^3\delta^3(\vec{k}-\vec{k}\,')\,, \qquad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0\,, \qquad [\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = 0\,, \tag{7}$$

and the mode functions $\varphi_{\vec{k}} =$ and $\varphi_{\vec{k}}^{\dagger}$ obey the following Wronskian normalisation condition,

$$\mathrm{W}\Big[\varphi_{\mathrm{k}}, \varphi_{\mathrm{k}}^{*}\Big] = \varphi_{\mathrm{k}}\Big(\frac{d}{d\eta}\varphi_{\mathrm{k}}^{*}\Big) - \Big(\frac{d}{\eta}\varphi_{\mathrm{k}}\Big)\varphi_{\mathrm{k}}^{*} = i\,. \tag{8}$$

In particular, the annihilation operator $\hat{a}_{\vec{k}}$ annihilates the vacuum, $\hat{a}_{\vec{k}}|0\rangle = 0$, while the creation operator $\hat{a}_{\vec{k}}^{\dagger}$ creates one particle excitation of momentum $\vec{k}$ out of the vacuum, $\hat{a}_{\vec{k}}^{\dagger}|0\rangle = |1_{\vec{k}}\rangle$.

(b) (*1 point*) Show that the mode functions φ_k in a FLRW space-time (2) obey the following equation of motion,

$$\varphi_k'' + \Big(k^2 - \frac{a''}{a}\Big)\varphi_k = 0\,, \quad (k = \|\vec{k}\|)\,. \tag{9}$$

(c) (*2 points*) Show that during de Sitter inflation Eq. (9) reduces to

$$\varphi_k'' + \Big(k^2 - \frac{2}{\eta^2}\Big)\varphi_k = 0\,, \tag{10}$$

and show further that the fundamental positive and negative frequency solutions are (the Bunch-Davies vacuum) are

$$\varphi_k = \frac{1}{\sqrt{2k}}\Big(1 - \frac{i}{k\eta}\Big)\mathrm{e}^{-ik\eta}\,, \qquad \varphi_k^{*} = \frac{1}{\sqrt{2k}}\Big(1 + \frac{i}{k\eta}\Big)\mathrm{e}^{ik\eta}\,. \tag{11}$$

Note that the mode normalisation, $1/\sqrt{2k}$, follows from the Wronskian (8).

(d) (*1 point*) Show by the appropriate matching that during radiation era, the scale factor is of the form,

$$a = H_I\eta \qquad (\eta \geq 1/H_I)\,, \tag{12}$$

where one assumes that the end time of inflation, $\eta = -1/H_I$, is identified with the beginning time of radiation era, $\eta = +1/H_I$.

(e) (*1 point*) Show that during radiation era Eq. (9) simplifies to

$$\varphi_k'' + k^2\varphi_k = 0\,, \tag{13}$$

and that its (most general, translationally invariant) solution can be written as,

$$\varphi_k^{\mathrm{rad}} = \frac{\alpha_k}{\sqrt{2k}}\mathrm{e}^{-ik\eta} + \frac{\beta_k}{\sqrt{2k}}\mathrm{e}^{ik\eta}\,, \tag{14}$$

where α_k and β_k are (time independent, but k-dependent) complex constants. Notice that, due to the assumed spatial homogeneity, α_k and β_k are functions of the modulus $k = \|\vec{k}\|$ only. Show that the Wronskian condition (8) implies,

$$|\alpha_k|^2 - |\beta_k|^2 = 1\,. \tag{15}$$

(f) (*2 points*) By making use of the matching conditions at the end of inflation/beginning of radiation,

$$\varphi_k\Big|_{\eta=-H_I^{-1}} = \varphi_k^{\rm rad}\Big|_{\eta=H_I^{-1}}\,; \qquad \frac{d\varphi_k}{d\eta}\Big|_{\eta=-H_I^{-1}} = \frac{d\varphi_k^{\rm rad}}{d\eta}\Big|_{\eta=H_I^{-1}}\,, \tag{16}$$

show that the coefficients α_k and β_k are given by,

$$\alpha_k = -\frac{1}{2}\frac{H_I^2}{k^2}\Big(1-2i\frac{k}{H_I}-2\frac{k^2}{H_I^2}\Big){\rm e}^{2ik/H_I}\,; \qquad \beta_k = \frac{1}{2}\frac{H_I^2}{k^2}\,. \tag{17}$$

(g) (*2 points*) The power spectrum $\mathcal{P}_\phi$ of the scalar field can be defined by

$$\langle 0|\hat\phi(\vec x,\eta)^2|0\rangle = \frac{1}{a^2}\int\frac{d^3k}{(2\pi)^3}|\varphi_k|^2 \equiv \int\frac{dk}{k}\mathcal{P}_\phi(k,\eta)\,. \tag{18}$$

Show that the power spectrum in radiation era, defined in (18), is given by,

$$\begin{aligned}\mathcal{P}_\phi &= \frac{H_I^4}{8\pi^2a^2k^2}\Big\{\Big(1+2\frac{k^4}{H_I^4}\Big)+2\frac{k}{H_I}\sin\Big(2k(\eta-1/H_I)\Big)-\Big(1-2\frac{k^2}{H_I^2}\Big)\cos\Big(2k(\eta-1/H_I)\Big)\Big\}\\ &= \frac{H_I^4}{4\pi^2a^2k^2}\sin^2\Big(k\eta-k/H_I\Big)\Big(1+\mathcal{O}(k/H_I)\Big)\,.\end{aligned} \tag{19}$$

This spectrum is scale invariant on super-Hubble scales, and exhibits Sakharov oscillations on sub-Hubble scales, with the maxima at, $\eta_{\rm max}(n) = (n+1/2)\pi/k$ $(n = 0, 1, 2, ..)$ and the minima (zeros) at, $\eta_{\rm min}(n) = n\pi/k$ $(n = 1, 2, ..)$, where we neglected the contribution, $1/H_I \ll \eta$.

In slow roll inflation the power spectrum of a massless minimally coupled scalar field can be used to calculate the power spectrum of the gauge invariant spatial curvature perturbation,

$$\mathcal{R}(x) = \psi(x) - \frac{H(t)}{\dot\phi(t)}\delta\phi(x)\,, \tag{20}$$

in the $\psi = 0$ gauge, where $\psi(x) = {\rm Tr}[\delta g_{ij}(x)]/(6a(t)^2)$ is the Newtonian potential perturbation and $\phi(x) = \phi(t) + \delta\phi(x)$. The result is

$$\mathcal{P}_{\mathcal{R}}(k,\eta) = \frac{H^2}{\dot\phi^2}\mathcal{P}_\phi(k,\eta)\,, \tag{21}$$

where, for a mode k, $H(t)$ and $\phi(t)$ are to be evaluated at the Hubble crossing of that mode during inflation, *i.e.* at the time t when $k = aH(t)$. The relation (21) can be then used to calculate the spectrum of scalar cosmological perturbations both in radiation and matter era, as it is explained in the last lecture. The perturbations in $\mathcal{R}$ then source the CMB temperature fluctuations and also provide the seeds for the large scale structure of the Universe.