Problemset 2 for Cosmology (ns-tp430m)

Problems are due at Thu Feb 20. (9 points in total)

4. Thermal sphere. *(5 points)*

Newtonian spherically symmetric gravitating systems of many particles satisfy the Poisson equation for the gravitational Newton potential ϕ_N,

$$\nabla^2\phi_N = 4\pi G_N \rho_N \,, \tag{1}$$

where ρ_N denotes the mass density, which in a spherically symmetric system is a function of the distance r from the center of mass, $\phi_N = \phi_N(r)$. In an equilibrated system, the distribution of particles can be approximated by the thermal distribution function $f = f(r, v)$, which is a function of $v = \|\vec{v}\,\|$ and $r = \|\vec{r}\,\|$ only,

$$f = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\Big(-\frac{v^2/2 + \phi_N}{\sigma^2}\Big) \,, \tag{2}$$

where v is particle's velocity, $\phi_N = \phi_N(r)$ Newton's gravitational potential, $\sigma^2 = \langle\vec{v}^2\rangle/3 \equiv k_B T/m$ and

$$\rho_N(r) = \int d^3v f \,. \tag{3}$$

(a) *(2 points)*

Show that ϕ_N satisfies the following (mean field) equation of motion,

$$\frac{d^2}{dr^2}\phi_N + \frac{2}{r}\frac{d}{dr}\phi_N = 4\pi G_N \rho_1 \exp\Big(-\frac{\phi_N(r)}{\sigma^2}\Big) \,. \tag{4}$$

(b) *(3 points)*

Show that the following *Ansatz* solves equation (1)

$$\rho_N(r) = \frac{\sigma^2}{2\pi G_N r^2} \,; \qquad \phi_N(r) = -\sigma^2 \ln\Big(\frac{\sigma^2}{2\pi G_N \rho_1 r^2}\Big) \,. \tag{5}$$

This solution is known as the *thermal sphere*, and it is the only known analytic solution of Eq. (1). Next, show that the mass inside radius r reads,

$$M(r) = \frac{2k_B T}{m G_N}\, r \,. \tag{6}$$

Discuss the limits $r \to 0$ and $r \to \infty$.

5. Light deflection off a thermal sphere. *(4 points)*

Calculate the deflection angle of light in the presence of a mass distribution of a thermal sphere, given by the thermal sphere potential $\phi_N = -\sigma^2 \ln[\sigma^2/(2\pi G_N \rho_1 r^2)]$, $\sigma^2 = k_B T/m$ (see problem 4 above), by making use of the formula,

$$\vec{\alpha}(d) = -\frac{2}{c^2}\int d\ell\, \nabla_\perp \phi_N(\vec{x}) \tag{7}$$

where $\nabla_\perp$ is the gradient operator in the lens plane, whose two components are transversal (perpendicular) to the photon path, ℓ is the distance along the light geodesic, and d is the impact parameter, which is the shortest distance from the center of mass ($\vec{x} = 0$) to the geodesic. Assume that the photon path can be approximated by an almost straight line, $\vec{x} \simeq (d, 0, z)$, ($z \in \{-\infty, \infty\}$). In this case Eq. (7) reduces to,

$$\alpha(d) = -\frac{2}{c^2}\int_{-\infty}^{\infty} dz \partial_x \phi_N(\vec{x})\,. \tag{8}$$

Assume that the mass distribution of an elliptical galaxy can be well approximated by a thermal sphere, with a typical dispersion of a velocity component $\sigma = 300$ km/s. Calculate the light deflection angle originating at a distant point source (quasar or galaxy). Express your answer in arc-seconds. Comment on the dependence of the deflection angle α on d!