Problemset 3 for Cosmology (ns-tp430m)

Problems are due at Thu Feb 27. (15 plus 3 bonus points in total)

6. A particle in an expanding universe *(7 points plus 3 bonus points)*

Consider a massive test particle in a (spatially) flat FLRW space-time in conformal coordinates $x_c^\mu = (c\eta, x_c^i)$ (recall that conformal time $\eta = x_c^0/c$ is related to physical time by $ad\eta = dt$),

$$ds^2 = a^2(\eta)\eta_{\mu\nu}dx_c^\mu dx_c^\nu \tag{1}$$

(a) *(1 point)* Show that the Christoffel connection in the space-time (1) is given by,

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{c}\mathcal{H}\left[\delta^\mu_\alpha\delta^0_\beta + \delta^\mu_\beta\delta^0_\alpha - \delta^\mu_0\eta_{\alpha\beta}\right] , \qquad \mathcal{H} = \frac{d\ln(a)}{d\eta} \equiv \frac{a'}{a} , \tag{2}$$

where $\mathcal{H}$ denotes a conformal Hubble rate, which is related to the physical Hubble (expansion) rate H as, $H = a\mathcal{H}$.

(b) *(2 points)* Show that the geodesic equation for the four velocity $u_c^\mu = cdx_c^\mu/ds = dx_c^\mu/d\tau$ can be written as

$$\frac{du_c^\mu}{d\tau} + \frac{a'}{a}\left(\frac{2u_c^0u_c^\mu}{c} - \frac{c\delta_0^\mu}{a^2}\right) = 0 , \qquad \eta_{\mu\nu}u_c^\mu u_c^\nu = \frac{c^2}{a^2} . \tag{3}$$

(c) *(2 points)* Solve the spatial geodesic equation (3) and show that

$$\frac{d(a^2u_c^i)}{d\eta} = 0 , \tag{4}$$

such that $u_{c\,i} = -a^2\delta_{ij}u_c^j$ is conserved, *i.e.* that u_c^i scales as $1/a^2$. Is there is a Killing vector associated to the conservation of $u_{c\,i}$? If yes, which one?

(d) *(2 points)* Show that the equation for u_c^0 in (3) can be recast as,

$$\frac{d}{d\eta}\left[a^2\left(a^2(u_c^0)^2 - c^2\right)\right] = 0 , \tag{5}$$

Solve this equation for u_c^0. Make sure that you properly fix the integration constant. *Hint:* Use the line element in (3).

(e) *(1 point)* Introduce a 4-momentum, $p_c^\mu = mu_c^\mu$, and show that the solution for p_c^0 can be written as,

$$\frac{E_c^2}{a^4c^2} \equiv (p_c^0)^2 = (p_c^i)^2 + \frac{m^2c^2}{a^2} , \quad (E_c/c = a^2p_c^0 = p_{c\,0}) . \tag{6}$$

The Einstein's relation for the physical energy E and momentum $\vec{p}_{\rm phys}$ is then obtained by simply multiplying (7) by a^2,

$$\frac{E^2}{c^2} = \|\vec{p}_{\rm phys}\|^2 + m^2c^2 , \tag{7}$$

from where we conclude,

$$\|\vec{p}_{\rm phys}\|^2 = a^2\eta_{ij}p_c^i p_c^j = \frac{\eta^{ij}}{a^2}p_{c\,i}p_{c\,j}\,, \tag{8}$$

such that in an expanding universe the physical momentum $\|\vec{p}_{\rm phys}\|$ of particles scales as $\propto 1/a$. When taken in the nonrelativistic limit, this relation then implies that physical velocities of particles scale due to the Universe's expansion as $\propto 1/a$. This phenomenon of the redshift of physical momenta and velocities of particles is also known as the *Hubble damping*.

(f) *(3* bonus points)* Repeat the above analysis for a photon in an expanding universe.

7. Kination. *(8 points)*

Consider a real scalar massless field minimally coupled to gravitation, whose action reads,

$$S_\phi = \int d^4x\sqrt{-g}\,\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) \tag{9}$$

in a homogeneous FLRW background,

$$g_{\mu\nu} = {\rm diag}\Big(1, -a^2, -a^2, -a^2\Big)\,. \tag{10}$$

(a) *(2 points)* Derive the equation of motion for the homogeneous mode of the scalar field, $\phi = \phi(t)$, and discuss its dependence on the scale factor, $\phi = \phi(a)$.

(b) *(2 points)* How does the energy density of ϕ scales with the scale factor a, and what is the equation of state, $\mathcal{P}_\phi = w_\phi\rho_\phi$, *i.e.* what is w_ϕ?

Recall that the stress energy is defined as,

$$T^\phi_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - g_{\mu\nu}\mathcal{L}_\phi\,. \tag{11}$$

Assume an ideal fluid form,

$$T^\phi_{\mu\nu} = (\rho_\phi + \mathcal{P}_\phi)\frac{u_\mu u_\nu}{c^2} - g_{\mu\nu}\mathcal{P}_\phi\,, \tag{12}$$

and take the fluid to be in its rest-frame, in which $u_\mu = c\delta^0_\mu$.

(c) *(4 points)* Assume that the Universe's energy density is dominated by the kinetic energy of a scalar field, $\rho \simeq \rho_\phi$. The Friedmann equation (with $\Lambda = 0$) reads

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3c^2}\rho_\phi \tag{13}$$

where $\dot{a} = da/dt$ is a time derivative of the scale factor. Solve for $\phi = \phi(t)$ and determine how the scale factor depends on time t.

An epoch of the Universe, in which the kinetic energy of a scalar field dominates, is called *kination*, and it is not known whether such an epoch existed.