Problem set 4 for Cosmology (ns-tp430m)

Problems are due at Thu Mar 6 (in total 17 points).

8. Fermions in curved space-times. (*7 points*)

Consider the following covariant Dirac action for fermions in curved space times,

$$S_{\text{fermion}} = \int d^4x\sqrt{-g}\Big(\bar{\psi}i\gamma^\mu\nabla_\mu\psi - m_\psi\bar{\psi}\psi\Big)\,, \tag{1}$$

where $\bar{\psi} = \psi^\dagger\gamma^0(x)$ and m_ψ denotes the fermion mass. The covariant derivative acting on a fermion field is given in terms of the spin connection Γ_μ as,

$$\nabla_\mu\psi = (\partial_\mu - \Gamma_\mu)\psi\,, \tag{2}$$

which is in turn defined by

$$\nabla_\mu\gamma_\nu \equiv \partial_\mu\gamma_\nu - \Gamma^\alpha_{\mu\nu}\gamma_\alpha - \Gamma_\mu\gamma_\nu + \gamma_\nu\Gamma_\mu = 0\,. \tag{3}$$

By recalling that,

$$\gamma_\mu = e^b_\mu\gamma_b\,, \tag{4}$$

one can show that a solution of (3) is

$$\Gamma_\mu = -\frac{1}{8}e^\nu_c\Big(\partial_\mu e_{\nu d} - \Gamma^\alpha_{\mu\nu}e_{\alpha d}\Big)[\gamma^c,\gamma^d]\,, \tag{5}$$

where $[\gamma^c,\gamma^d] = \gamma^c\gamma^d - \gamma^d\gamma^c$ denotes the commutator. Here we are using $a, b, c, d, ..$ for the tangent space indices, on which γ^a and η_{ab} are space-time independent, and $\mu,\nu,\rho,\sigma,..$ for the space-time indices.

We consider homogeneous conformal space-times, with the conformally flat metric tensor,

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \equiv e^b_\mu(\eta)e^c_\nu(\eta)\eta_{bc}\,, \qquad \eta_{bc} = \text{diag}(1,-1,-1,-1)\,, \quad (b,c = 0,1,2,3)\,, \tag{6}$$

where the scale factor $a = a(\eta)$ is a function of conformal time η (*e.g.* in de Sitter space-time, $a = -1/(H_I\eta)$ $(\eta < 0)$, where H_I denotes the Hubble parameter of de Sitter space).

(a) (*1 point*)

Check that in conformal space-times the vierbein has the form

$$e^c_\mu(\eta) = \delta^c_\mu a(\eta)\,, \qquad e^\mu_c(\eta) = \delta^\mu_c a(\eta)^{-1}\,, \qquad (\mu = 0,1,2,3;\ c = 0,1,2,3)\,. \tag{7}$$

and that the Dirac matrices obeying the correct anticommutation relation are,

$$\gamma^\mu(\eta) = e^\mu_b\gamma^b = a(\eta)^{-1}\delta^\mu_b\gamma^b\,. \tag{8}$$

(b) *(3 points)*

Show that the Levi-Cività connection $\Gamma^{\alpha}_{\mu\nu}$ and the spin connection Γ_{μ} are of the form,

$$\begin{aligned}\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{c}\frac{a'}{a}\Big(\delta^0_\mu\delta^\alpha_\nu + \delta^0_\nu\delta^\alpha_\mu - \delta^\alpha_0\eta_{\mu\nu}\Big)\\ \Gamma_\mu &= \frac{1}{4}\frac{a'}{a}\Big[\gamma^0,\gamma^b\Big]\eta_{\mu b}\,,\end{aligned} \tag{9}$$

where $a' = da/d\eta$ and $\eta_{\mu b} = \mathrm{diag}(1,-1,-1,-1)$.

(c) *(3 points)*

Show that, when contracted with gamma matrices, the covariant derivative acting on a spinor acquires the form,

$$i\nabla\!\!\!\!/ \equiv e^\mu_c i\gamma^c\nabla_\mu = a^{-\frac{5}{2}} i\gamma^c\delta^\mu_c\partial_\mu a^{\frac{3}{2}}\,. \tag{10}$$

Show that from here it then follows that the Dirac equation for fermions in homogeneous conformal space-times can be written as

$$\Big(i\gamma^c\delta^\mu_c\partial_\mu - am_\psi\Big)\psi_{cf} = 0\,,\qquad \psi_c = a^{3/2}\psi\,, \tag{11}$$

where ψ_c denotes a conformally rescaled spinor. Comment on the physical implications of this result.

9. The Friedmann equation. *(10 points)*

Consider the spatially flat metric of an expanding universe,

$$g_{\mu\nu} = \mathrm{diag}(1,-a^2,-a^2,-a^2)\,, \tag{12}$$

where $a = a(t)$ is the scale factor (of the Universe), and it is a function of time.

(a) *(2 points)*

Calculate the corresponding Levi-Cività connection,

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\Big(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\Big)\,. \tag{13}$$

(b) *(3 points)*

Calculate the Ricci tensor, by making use of the expressions,

$$\mathcal{R}_{\alpha\beta} = \mathcal{R}^\mu{}_{\alpha\mu\beta}\,,\qquad \mathcal{R}^\mu{}_{\alpha\beta\gamma} = \partial_\beta\Gamma^\mu_{\alpha\gamma} - \partial_\gamma\Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha} - \Gamma^\mu_{\sigma\gamma}\Gamma^\sigma_{\beta\alpha}\,, \tag{14}$$

and show that the Ricci tensor is of the form,

$$\mathcal{R}_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a}\,, \qquad \mathcal{R}_{ij} = -\frac{1}{c^2}\Big(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\Big)g_{ij}\,, \qquad (g_{ij} = -a^2\delta_{ij})\,, \tag{15}$$

while the Ricci scalar reads,

$$\mathcal{R} \equiv g^{\mu\nu}\mathcal{R}_{\mu\nu} = -\frac{6}{c^2}\Big(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\Big)\,. \tag{16}$$

(c) *(2 points)*

Make use of the Einstein equation

$$G_{\mu\nu} - \frac{1}{c^2}\Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}\,, \tag{17}$$

where

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}\,, \tag{18}$$

denotes the Einstein curvature tensor, Λ is the cosmological term, and the stress-energy tensor of an ideal fluid equals in the fluid rest frame, in which $u^\mu = c(1,\vec{0}\,)$,

$$T_{\mu\nu} = (\rho + \mathcal{P})\frac{u_\mu u_\nu}{c^2} - g_{\mu\nu}\mathcal{P}\,, \tag{19}$$

and derive the Friedmann (Friedmann-Lemaître-Robertson-Walker, `FLRW`) equations,

$$\begin{aligned} H^2 \equiv \frac{\dot{a}^2}{a^2} &= \frac{8\pi G_N}{3c^2}\rho + \frac{\Lambda}{3} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G_N}{3c^2}(\rho + 3\mathcal{P}) + \frac{\Lambda}{3}\,, \end{aligned} \tag{20}$$

where $\dot{a} = da/dt$ and $\ddot{a} = d^2a/dt^2$, and $H = H(t)$ is the Hubble parameter.

(d) *(3 points)*

Show that the covariant stress-energy conservation, $\nabla^\mu T_{\mu\nu} = 0$, implies,

$$\dot{\rho} + 3H(\rho + \mathcal{P}) = 0\,, \tag{21}$$

where $\rho = \rho(t)$ and $\mathcal{P} = \mathcal{P}(t)$. Show that this is not an independent constraint, and that it can be derived from Eqs. (20).

Discuss the solutions of equations (20–21) for the cases (1) $\rho = \mathcal{P} = 0$, $\Lambda = \Lambda_0 = \text{const.}$ (de Sitter space), (2) $\mathcal{P} = w\rho \propto 1/a^4$, $\Lambda = 0$ (what is the value of w in this case?), and (3) $\rho \propto 1/a^3$, $\Lambda = 0$ (what is the value of w in this case?).