Problem set 5 for Cosmology (ns-tp430m)

Problems are due at Thu Mar 20 (in total 12 points).

10. Expanding space-times with constant curvature. *(4 points)*

The line element of space-times with constant curvature can be written as,

$$ds^2 = c^2 dt^2 - a^2 \left[\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega_2^2 \right] , \tag{1}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2(\theta) d\phi^2$ is the surface element of the two-sphere S^2 $(\theta \in [0, \pi], \phi \in [0, 2\pi))$, $a = a(t)$ denotes the scale factor, and $\kappa > 0, \kappa = 0, \kappa < 0$ for a positively curved, flat or negatively curved geometry, respectively.

(a) *(2 points)* Show that the line element of an expanding space-time with a constant negative spatial curvature can be also written as,

$$ds^2 = c^2 dt^2 - a^2 \left[d\chi^2 + \frac{\sinh^2(\sqrt{-\kappa}\chi)}{-\kappa} d\Omega_2^2 \right] , \tag{2}$$

$\chi \in [0, \infty)$ is a radial coordinate. What is the analogous form for the line element in positively curved (spherical) geometry and what in flat geometry?

(b) *(2 points)* Show that the spatial part of the metric tensor (2) can be written as a 3-dimensional hyperboloid embedded into a 3+1 dimensional Minkowski space.

11. Evolution of (relative) densities. *(8 points)*

(a) *(2 points)*

By making use of the FLRW equations,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3c^2}(\rho + \rho_Q) + \frac{\Lambda}{3} - \frac{\kappa c^2}{a^2} \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3c^2}(\rho + \rho_Q + 3(p + p_Q)) + \frac{\Lambda}{3} , \tag{4}$$

derive the following equation,

$$1 = \Omega_m(t) + \Omega_Q(t) + \Omega_\Lambda(t) + \Omega_\kappa(t) \tag{5}$$

$$q(t) = \frac{1}{2}\Omega_m(t) + \frac{1}{2}(1 + 3w_Q)\Omega_Q(t) - \Omega_\Lambda(t) , \tag{6}$$

where $\Omega_m(t) = \rho(t)/\rho_{\rm cr}(t)$ is the energy density of non-relativistic matter relative to the critical density, $\rho_{\rm cr}(t) = [3c^2/(8\pi G_N)]H^2(t)$ (assume that ρ is dominated by non-relativistic matter), $\Omega_Q(t) = \rho_Q(t)/\rho_{\rm cr}(t)$, $\rho_Q(t)$ is the energy density in Q-matter (quintessence), whose equation of state is, $p_Q = w_Q \rho_Q$ ($w_Q < -1/3$), and $q(t) = -(\ddot{a}(t)/a)/H^2(t)$ is the deceleration parameter. $\Omega_\kappa(t) = -c^2\kappa/H^2(t)$ is the rescaled 'energy density' in the curvature term, $H(t)$ is the Hubble parameter, $\Omega_\Lambda(t) = \Lambda/[3H^2(t)]$.

(b) *(4 points)*

Of course, Eqs. (5–6) are also valid today, when $a(t_0) = a_0 = 1$, $H(t_0) = H_0 \simeq 68$ km/s/Mpc, $\Omega_m(t_0) = \Omega_{m0} \simeq 0.31$, $\Omega_\Lambda(t_0) = \Omega_{\Lambda 0} \simeq 0.69$, $\Omega_\kappa(t_0) \simeq 0 \simeq \Omega_Q(t_0)$. Calculate $q_0 = q(t_0)$ today. Calculate further $\Omega_\Lambda(t)$ and $q(t)$ and sketch the results as a function of time. What are the asymptotic values (at very early and at very late times) of $q(t)$ and $\Omega_\Lambda(t)$? Do those agree with your naive expectation?

(c) *(2 points)* The cosmological redshift z is defined as $z = a_0/a - 1$. Estimate the redshift at which $q(z) = q(t(z)) = 0$.

Hint: The following integral you may find useful, $\int dx/\sqrt{x+x^2} = 2\mathrm{Arcsin}[\sqrt{x}\,]$.