Problem set 6 for Cosmology (ns-tp430m)

Problems are due at Thu Mar 27 (in total 12 points + 3 bonus points).

12. Luminosity distance. (*6 points*)

Luminosity distance d_L is defined in terms of the (absolute) luminosity $\mathcal{L}$ (in units of the energy per unit time per frequency interval produced by the source in its rest frame) and the measured flux $\mathcal{F}$ (in units of the energy per unit time per area per frequency interval measured by a detector) as follows,

$$d_L^2 = \frac{\mathcal{L}}{4\pi\mathcal{F}} \,. \tag{1}$$

Show that, in the absence of expansion and curvature (in flat Minkowski space-time), the luminosity distance corresponds to the actual (physical) distance.

Consider a star at the comoving distance r, which shines isotropically with an absolute luminosity $\mathcal{L}$ at a redshift $z = z(t)$. The light reaches the observer placed at $r = r_O = 0$ at $z = z_O = 0$. Show that, in an universe with spatially flat sections, the luminosity distance equals to

$$d_L = a_0 r[1 + z(t)] \,. \tag{2}$$

By performing a Taylor expansion of the scale factor around $t = t_0$,

$$\frac{a(t)}{a_0} = 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \dots \,, \tag{3}$$

where $a_0 = a(t_0)$, $H_0 = \dot{a}(t_0)/a_0$ is the Hubble radius today, $q_0 = -\ddot{a}(t_0)a_0/\dot{a}(t_0)^2$ is the deceleration parameter today, show that the following expression for the Hubble law holds,

$$\frac{H_0 d_L}{c} = z + \frac{1}{2}(1 - q_0) z^2 + \dots \,. \tag{4}$$

13. Open and closed universe. (*6 points + 3 bonus points*)

Consider a universe filled with matter with an equation of state $\mathcal{P} = w\rho$, where w is a constant, $w \neq -1/3$ and $w \geq -1$. Solve the corresponding conservation equation and the Friedmann equation.

(a) (*1 point*)

Show first that $\rho(t) = \rho_0 / a^{3(1+w)}$, where $\rho_0 = \rho(t_0)$, $a(t_0) = 1$.

(b) (*5 points*)

Solve the corresponding Friedmann equation for both cases: when $\kappa > 0$ and when $\kappa < 0$. Sketch the evolution of the Universe in both cases. Make a separate plot for $w > -1/3$ and for $-1 \leq w < -1/3$. Discuss in which cases the Universe begins with a vanishing scale factor

a (Big Bang singularity) and in which cases it ends with a vanishing scale factor (Big Crunch singularity).

Hint: Solve the Friedmann equation in conformal time, $dt = a d\eta$. Make use of the variable $x = (a/a_0)^{(1+3w)/2}$, $a_0 = [(8\pi G_N \rho_0)/(3|\kappa|c^4)]^{1/(1+3w)}$. The following integrals are useful,

$$\int \frac{dx}{\sqrt{1-x^2}} = \mathrm{Arcsin}(x)\,, \qquad \int \frac{dx}{\sqrt{1+x^2}} = \mathrm{Arcsinh}(x)\,. \tag{5}$$

(c) *(3 bonus points)* Use `Mathematica`, `Matlab` or `Maple` to find $t = t(\eta)$, and plot (numerically) $a(t)$ for 4 different cases ($w > -1/3, \kappa > 0$; $w > -1/3, \kappa < 0$; $w < -1/3, \kappa > 0$ and $w < -1/3, \kappa < 0$) analysed in part (b).