Problem set 7 for Cosmology (ns-tp430m)

Problems are due at Thu Apr 3 (in total 16 points + 3 bonus points).

14. *Units and the age of the Universe.* (*7 points)*

1 parsec (1pc) is defined as the distance from which 1 astronomical unit (1 AU) subtends an angle of 1 arc second ($1''$). 1 AU is defined as the mean distance between the Earth and the Sun, 1 AU $\simeq 1.5 \times 10^8$ km (more precisely 149 597 870.691 kilometers).

(a) (*1 point)* Express 1pc in terms of kilometers and light years.

(b) (*2 points)* The Hubble parameter today is $H_0 = 72 \pm 3$ km/s/Mpc. Express H_0 in inverse seconds, and in giga-electron volts (GeV). Show that $\hbar H_0 = 2.1332h \times 10^{-42}$ GeV, where h is defined as, $H_0 = 100h$ km/s/Mpc, such that $h = 0.72 \pm 0.03$. (Often we shall work in units in which the reduced Planck constant $\hbar = 1$.)

(c) (*4 points)* Estimate the age of the Universe as $t_0 \simeq 1/H_0$ (express it in giga-years, Gy).

In order to find out a more precise value of the age of the Universe, solve the Friedmann equation,

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G_N}{3c^2}\rho + \frac{\Lambda}{3} \tag{1}$$

and show by integrating equation (1) that the scale factor reads

$$a^3(t) = \frac{8\pi G_N}{c^2\Lambda}\rho_0 \sinh^2\left(\frac{\sqrt{3\Lambda}\, t}{2}\right), \tag{2}$$

where $a(0) = 0$, $a(t_0) = 1$, $\rho(t) = \rho_0/a(t)^3$ (nonrelativistic matter). By inverting this expression, show that the age of the Universe can be written as,

$$t_0 = \frac{1}{3H_0}\frac{1}{\sqrt{\Omega_\Lambda}} \ln\left[\frac{1+\sqrt{\Omega_\Lambda}}{1-\sqrt{\Omega_\Lambda}}\right], \tag{3}$$

where $\Omega_\Lambda = \Lambda/(3H_0^2)$. Do you get the expected result in the limit when $\Omega_\Lambda \to 0$? What is t_0 for $\Omega_\Lambda = 0.73 \pm 0.04$ (WMAP value)? How does your result compare with the age of the Universe quoted by the WMAP team, $t_0 = 13.7 \pm 0.2$ Gy?

15. *Proper length. Angular diameter and luminosity distance.* (*9 points + 3 bonus points)*

Assume that the metric tensor is given by the FLRW line element, which in spherical coordinates reads,

$$ds^2 = c^2dt^2 - a^2\Big(\frac{dr^2}{1-\kappa r^2} + r^2(d\theta)^2 + r^2\sin^2(\theta)(d\phi)^2\Big) \tag{4}$$

where κ is the curvature of spatial sections. The luminosity distance d_L is defined in terms of the measured flux $\mathcal{F}$ and emitted luminosity, $\mathcal{L}$, as follows, $\mathcal{L} = \mathcal{F}/(4\pi d_L^2)$. From this it follows that d_L is related to the coordinate distance to the source r as follows,

$$d_L = a_0 \frac{a_0}{a_*} r = a_0 r(1 + z_*)\,, \tag{5}$$

where $a_0 = a(t_0) = 1$ is the scale factor today, $a_* = a(t_*)$ is the scale factor at the time t_* the light is emitted, and $z = z(t_*)$ denotes the redshift at the source. For a null geodesic in a FLRW universe, the *proper length* ℓ to the source located at a coordinate distance $r = r_*$ is,

$$\ell(t_0, t_*) = c(\eta_0 - \eta_*) = \int_{t_*}^{t_0} \frac{cdt}{a(t)} = \int_0^r \frac{dr'}{\sqrt{1 - \kappa r'^2}}\,, \tag{6}$$

where η_0 and η_* are the conformal times today and at the source at the time of emission, respectively. In curved space-times it is useful to define the *angular diameter distance, $\theta = \ell/R$*, where $R = |\kappa|^{-1/2}$.

(a) *(3 points)*

Show that the luminosity distance can be expressed in terms of the proper distance, $\ell = c(\eta_0 - \eta_*) = R\theta$ as follows,

$$d_L(z) = (1+z)R\sinh\Big(\frac{c}{H_0 R}\int_0^z \frac{dz'}{E(z')}\Big) \qquad \text{(open universe)} \tag{7}$$

$$d_L(z) = (1+z)\frac{c}{H_0}\int_0^z \frac{dz'}{E(z')} \qquad \text{(flat universe)} \tag{8}$$

$$d_L(z) = (1+z)R\sin\Big(\frac{c}{H_0 R}\int_0^z \frac{dz'}{E(z')}\Big) \qquad \text{(closed universe)} \tag{9}$$

where R is the radius of curvature, $R^2 = 1/|\kappa|$, $z = z(t_*)$ is the redshift at the emission time, and $E = E(z')$ is a function defined in terms of the matter content of the Universe by the following rewriting of the Friedmann equation,

$$H(z) \equiv \frac{\dot{a}}{a} = H_0 E(z)\,, \qquad E(z)^2 = H_0^{-2}\Big(\frac{8\pi G_N}{3c^2}(\rho_m + \rho_Q) + \frac{\Lambda}{3} - \frac{c^2\kappa}{a^2}\Big)\,. \tag{10}$$

(b) *(3 points)*

Evaluate the integral in Eqs. (7–9) for the following cases, (1) $\Omega_\Lambda = \Lambda/(3H_0^2) = 1, \Omega_m = (8\pi G_N \rho_m)/(3c^2 H_0^2) = 0$, $\Omega_\kappa = -c^2\kappa/H_0^2 = 0$, $\Omega_Q = (8\pi G_N \rho_Q)/(3c^2 H_0^2) = 0$; (2) $\Omega_Q = 0$, $\Omega_\Lambda = 0$, $\Omega_\kappa = 1 - \Omega_m$, $0 \le \Omega_m \le 1$; (3) $\Omega_Q = 0$, $\Omega_m = 0$, $\Omega_\kappa = 1 - \Omega_\Lambda$, $0 \le \Omega_\Lambda \le 1$.

(c) *(3 points + 3 bonus points)* (Bonus points are for the last term in the expansion in Eq. (14).)

Expand the integral for the luminosity distance,

$$\frac{d_L(z)}{1+z} = R\sinh\left(\frac{1}{R}\int_{t_*}^{t_0}\frac{cdt'}{a(t')}\right) \qquad \text{(open universe)} \tag{11}$$

$$\frac{d_L(z)}{1+z} = \int_{t_*}^{t_0}\frac{cdt'}{a(t')} \qquad \text{(flat universe)} \tag{12}$$

$$\frac{d_L(z)}{1+z} = R\sin\left(\frac{1}{R}\int_{t_*}^{t_0}\frac{cdt'}{a(t')}\right) \qquad \text{(closed universe)}, \tag{13}$$

in a Taylor series around $z=0$, and show that the expansion can be written as

$$d_L(z) = \frac{cz}{H_0}\left\{1+\frac{1}{2}(1-q_0)z - \frac{1}{6}\left[1-q_0-3q_0^2+j_0+\frac{c^2\kappa}{H_0^2}\right]z^2\right\} \tag{14}$$

where κ is the curvature of the Universe ($\kappa = 1/R^2$ for a closed universe, $\kappa = -1/R^2$ for an open universe and $\kappa = 0$ for a spatially flat universe), $H_0 = \dot{a}(t_0)/a_0$, $a_0 = 1$, $q_0 = -\ddot{a}(t_0)H_0^{-2}$ and $j_0 = \dddot{a}(t_0)H_0^{-3}$ is known as the jerk or jolt parameter.

Hint. First expand the scale factor in the integrand in terms of the powers of $(t-t_0)$, and then integrate. In the resulting expression, reexpress the powers of $(t-t_0)$ in terms of z.

NB: bonus points are extra points (to the 30% for homeworks) that you can earn.