Problem set 9 for Cosmology (ns-tp430m)

Problems are due at Thu Apr 24. In total 14 points

17. The fate of our Universe. (*7 points*)

Our Universe is filled mostly with dark and baryonic matter, $\Omega_{\rm m} \simeq 0.31$, and dark energy, which can be well described by cosmological constant, $\Omega_\Lambda \simeq 0.68$. On the order hand, the spatial curvature of the Universe is very small, $|\Omega_\kappa| < 0.01$.

Consider a universe with a constant spatial curvature κ filled with non-relativistic matter, a cosmological constant that can be caracterised by $\{\Omega_{\rm m}, \Omega_\Lambda, \Omega_\kappa\}$, such that $\Omega_{\rm m} + \Omega_\Lambda + \Omega_\kappa = 1$, $\Omega_{\rm m} \geq 0$ and $\Omega_\Lambda \geq 0$, but no further constraints imposed on $\{\Omega_{\rm m}, \Omega_\Lambda, \Omega_\kappa\}$.

(a) (*2 points)*

Show that when the corresponding Friedmann equation is solved, one gets for the scalar factor as a function of conformal time the following integral relation,

$$H_0\eta = \int_{a_{\rm min}}^{a} \frac{d\tilde{a}}{\sqrt{\Omega_{\rm m}\tilde{a} + \Omega_\Lambda\tilde{a}^4 + \Omega_\kappa\tilde{a}^2}}\,, \tag{1}$$

while, when expressed as a function of physical time, one gets,

$$H_0 t = \int_{a_{\rm min}}^{a} \frac{d\tilde{a}}{\sqrt{(\Omega_{\rm m}/\tilde{a}) + \Omega_\Lambda\tilde{a}^2 + \Omega_\kappa}}\,, \tag{2}$$

where $a_{\rm min}$ is the scale factor at the creation of the Universe (which may – but need not to – be zero).

(b) (*2 points)*

Find the condition between $\Omega_{\rm m}$ and Ω_Λ (recall that $\Omega_\kappa = 1 - \Omega_{\rm m} - \Omega_\Lambda$) for which (A) the Universe will recollapse into a big crunch (BC) and (B) will expand forever.

(c) (*3 points)*

By making use of Mathematica, Maple or Matlab, and the knowledge from part (b), plot $a(\eta)$ and $a(t)$ by integrating (1) and (2) for (A) the case when the universe recollapses into a BC and (B) for the case when it expands forever.

18. Killing vectors of a negatively curved FLRW space-time. (*7 points*)

Consider a spatially homogeneous, expanding universe with a negative spatial curvature $\kappa < 0$.

(a) *(2 points)*

Show that the (comoving) spatial part of the metric $d\ell^2 = d\chi^2 + R_c \sinh^2(\chi/R_c)d\Omega_2^2$ ($d\Omega_2^2 = d\theta^2 + \sin^2(\theta)d\phi^2$) can be embedded into a four dimensional flat space with a signature $(+,+,+,-)$,

$$d\ell^2 = dX_1^2 + dX_2^2 + dX_3^2 - dX_4^2\,, \tag{3}$$

with the constraint,

$$X_4^2 - (X_1^2 + X_2^2 + X_3^2) = R_c^2 = \frac{1}{-\kappa}\,. \tag{4}$$

Hint: Show this by constructing explicit coordinate transformations, $X_I(\chi,\theta,\phi)$ ($I = 1,2,3,4$).

(b) *(2 points)*

Show that the symmetry group of this embedding is $SO(3,1)$, which is also the symmetry group of the original metric. (One can show that there are *no* symmetries associated with the $g_{0\mu}$ part of the metric - you need not show that.)

(c) *(3 points)*

Based on this, construct all Killing vectors in coordinates X_I ($I = 1,2,3,4$). Make use of the coordinate transformations from part (a) to reexpress these Killing vectors in spherical coordinates $\{\chi,\theta,\phi\}$. Based on your analysis, answer the following questions: How many Killing vectors (i.e. symmetries) has a general FLRW space-time? Is the counting of the symmetries for the $\kappa = 0$ and $\kappa > 0$ cases identical as for the $\kappa < 0$ case?