# ADVANCED TOPICS IN THEORETICAL PHYSICS II

Tutorial problem set 3, 25.09.2017.

*(20 points in total)*

The in-class exercise is problem 7. Problem 8 is due at Monday, 02.10.2017.

## ■ PROBLEM 7 Transition amplitude. *(11 points)*

The transition amplitude for a particle in one dimension to go from position $x_i$ at time $t_i$ to position $x_f$ at time $t_f$ (a matrix element of the evolution operator in the position basis) can be written as a path integral,

$$\begin{aligned} F(x_f,t_f;x_i,t_i) &= \langle x_f,t_f|x_i,t_i\rangle = \langle x_f|U(t_f,t_i)|x_i\rangle \\ &= \int \mathcal{D}p(t) \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t)\, \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, \Big[p(t)\dot{x}(t) - H\Big(x(t),p(t),t\Big)\Big]\right\} , \end{aligned} \tag{7.1}$$

where this path integral is properly defined as the limit $N \to \infty$ of the discretized expression,

$$\left[\prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n\right]\left[\prod_{n=1}^{N+1}\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right]\exp\left\{\frac{i\delta t}{\hbar}\sum_{n=1}^{N+1}\left[p_n\left(\frac{x_n - x_{n-1}}{\delta t}\right) - H\Big(p_n,x_n,t_n\Big)\right]\right\} , \tag{7.2}$$

where $x_0 = x_i$ and $x_{N+1} = x_f$, and $\delta t = (t_f - t_i)/N$, $t_n = n\delta t$. This is a general result following from the canonical quantization procedure.

Given a system of a harmonic oscillator coupled to and external time-dependent source,

$$L(x,\dot{x},J) = \frac{m}{2}\dot{x}^2(t) - \frac{m\omega^2}{2}x^2(t) - J(t)x(t) , \tag{7.3}$$

calculate the transition amplitude (7.1) following the steps below.

**(a)** *(1 point)* Define the conjugate momentum and construct the Hamiltonian from (7.3).

Now you will perform all the momentum integrals in the path integral.

**(b)** *(2 points)* Complete the square for the momentum in the exponent of (7.2) and perform all the Gaussian integrals over the momenta. Make use of the following result for Gaussian integrals ($\alpha > 0, \beta \in \mathbb{R}$),

$$\int_{-\infty}^{\infty} dx\, e^{-(\alpha+i\beta)x^2} = \sqrt{\frac{\pi}{\alpha+i\beta}} , \qquad \int_{-\infty}^{\infty} dx\, e^{-i\beta x^2} = \lim_{\alpha\to 0_+}\int_{-\infty}^{\infty} dx\, e^{-(\alpha+i\beta)x^2} = \sqrt{\frac{\pi}{i\beta}} . \tag{7.4}$$

Be careful to define what the new measure is (what is meant by $\mathcal{D}x(t)$) and the overall normalization. The result for the transition amplitude is of the form

$$F_J(x_f,t_f;x_i,t_i) = \mathcal{N}\int \mathcal{D}x(t)\, \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, L(x,\dot{x},J)\right\} = \mathcal{N}\int \mathcal{D}x(t)\, \exp\left\{\frac{i}{\hbar}S[x,J]\right\} . \tag{7.5}$$

Note that this form of the path integral follows from the Hamiltonian being at most quadratic in momenta so that the momentum integrals are just Gaussian ones and can be performed exactly.

Next you will perform all the position integrals. That is possible since the Hamiltonian is also quadratic in position so that all the integrals are again Gaussian.

**(c)** *(1 point)* Make a variable substitution in (7.5) which is a just a shift of the variable,

$$x(t) = \overline{x}(t) + \delta x(t) \ , \tag{7.6}$$

(or in the discretized form $x_n = \overline{x}_n + \delta x_n$, $\overline{x}_n = \overline{x}(t_n)$) where $\overline{x}(t)$ is defined to satisfy the classical equation of motion,

$$\left. \frac{\delta S[x, J]}{\delta x(t)} \right|_{x=\overline{x}} = 0 \ , \tag{7.7}$$

and the boundary conditions $\overline{x}(t_i) = x_i$, $\overline{x}(t_f) = x_f$. Show that the path integral can now be written as

$$F(x_i, t_i; x_f, t_f) = \exp\left[\frac{i}{\hbar} S[\overline{x}, J]\right] \times \mathcal{F}(t_f, t_i) \ , \tag{7.8}$$

where the *fluctuating factor* is

$$\mathcal{F}(t_i, t_f) = \mathcal{N} \int\limits_{\delta x(t_i)=0}^{\delta x(t_f)=0} \mathcal{D}\delta x(t) \ \exp\left[\frac{i}{\hbar} S[\delta x, 0]\right] \ . \tag{7.9}$$

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**(d)** *(1 point)* The classical path of the particle $\overline{x}(t)$ satisfies the classical equation of motion. It is possible to find a (formal) solution of that equation for arbitrary external source $J(t)$. Separate the solution $x = x_h + x_p$ into the homogeneous part $\overline{x}_h(t)$ that satisfy a homogeneous equation,

$$\left[\frac{d^2}{dt^2} + \omega^2\right] \overline{x}_h(t) = 0 \tag{7.10}$$

with the boundary conditions, $\overline{x}_h(t_i) = x_i$ and $\overline{x}_h(t_f) = x_f$ and the particular part $\overline{x}_p(t)$ satisfying an inhomogeneous equation,

$$\left[\frac{d^2}{dt^2} + \omega^2\right] \overline{x}_p(t) = -\frac{J(t)}{m} \tag{7.11}$$

with the boundary conditions $\overline{x}_p(t_i) = 0 = \overline{x}_p(t_f)$. Solve for the homogeneous part. Show that the solution for the inhomogeneous part can be written as,

$$\overline{x}_p(t) = -\int_{t_i}^{t_f} dt' \, G(t, t') \frac{J(t')}{m} \ , \tag{7.12}$$

where $G(t, t')$ is the appropriate Green function satisfying the same boundary conditions as the particular part, *i.e.* $G(t_i, t') = 0 = G(t_f, t')$,

$$G(t, t') = -\Theta(t' - t) \frac{\sin\omega(t_f - t') \sin\omega(t - t_i)}{\omega \sin\omega(t_f - t_i)} - \Theta(t - t') \frac{\sin\omega(t_f - t) \sin\omega(t' - t_i)}{\omega \sin\omega(t_f - t_i)} \ . \tag{7.13}$$

**(e)** *(2 points)* Show that $S[\overline{x}, J]$ that appears in the phase in (7.8) is

$$\begin{aligned} S[\overline{x}, J] = {}& \frac{1}{2m} \int\limits_{t_i}^{t_f} dt dt' \, J(t) G(t, t') J(t') - x_f \int\limits_{t_i}^{t_f} dt \, \frac{\sin\omega(t - t_i)}{\sin\omega(t_f - t_i)} \, J(t) - x_i \int\limits_{t_i}^{t_f} dt \, \frac{\sin\omega(t_f - t)}{\sin\omega(t_f - t_i)} J(t) \\ & + \frac{m\omega}{2}(x_f^2 + x_i^2) \cot\omega(t_f - t_i) - \frac{m\omega \, x_i x_f}{\sin\omega(t_f - t_i)} \ . \end{aligned} \tag{7.14}$$

**(f)** *(4 points)* Show that the fluctuating factor is

$$\mathcal{F}(t_i, t_f) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega(t_f - t_i)}} \ . \tag{7.15}$$

Make use of the following result for $N$-dimensional Gaussian integrals

$$\int_{-\infty}^{\infty} dx_1 dx_2 \dots dx_N \, \exp\left[-\mathbf{x}^\top \mathbf{A}\, \mathbf{x}\right] = \frac{\pi^{N/2}}{\sqrt{\det \mathbf{A}}} \ , \tag{7.16}$$

where $\mathbf{x}^\top = (x_1 \ x_2 \ \dots \ x_N)$ and $\mathbf{A}$ is an $N \times N$ symmetric matrix, $\mathbf{A}^\top = \mathbf{A}$. The problem of evaluating the path integral for the fluctuating factor then reduces to evaluating the determinant of the operator in the action. This determinant can be evaluated in multiple ways. Here we will do it by calculating the determinant in the discretized form and taking the continuum limit in the end. Matrix $\mathbf{A}_N$ is of the form

$$\mathbf{A}_N = \begin{pmatrix} a & b & 0 & \dots \\ b & a & b & \dots \\ 0 & b & a & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \ . \tag{7.17}$$

Find what $a$ and $b$ are. Show that recurrence relation satisfied by the determinant of this matrix is

$$D_{N+2} = a D_{N+1} - b^2 D_N \ , \qquad D_N = \det \mathbf{A}_N \tag{7.18}$$

by expanding the determinant in minoras. Find the initial conditions for this recurrence relation ($D_2$ and $D_3$ and from them infer $D_1$ and $D_0$ by extension). Next, find $\alpha$ and $\beta$ such that the recurrence relation can be cast into

$$D_{N+2} - \alpha D_{N+1} = \beta (D_{N+1} - \alpha D_N) \ , \tag{7.19}$$

which implies

$$D_{N+2} - \alpha D_{N+1} = \beta^{N+1} (D_1 - \alpha D_0) \ . \tag{7.20}$$

Derive a similar result for the combination $D_{N+2} - \beta D_{n+1}$ and show that

$$D_N = \frac{\beta^N - \alpha^N}{\beta - \alpha} \ . \tag{7.21}$$

Show that, in the limit when $\delta t \to 0$ (or equivalently when $N \to \infty$), this leads to

$$\det \mathbf{A}_N = \left[\frac{m}{2i\hbar \delta t}\right]^N \frac{\sin\omega(t_f - t_i)}{\omega \delta t} \ , \tag{7.22}$$

from where (7.15) follows.

## ■ PROBLEM 8 Transition amplitudes, transition probabilities and expectation values. *(9 points)*

The most general pure Gaussian state (in position representation) can be written as

$$\psi(x) = \mathcal{N} \exp\left[-(1 + 2i\eta)\frac{(x - x_0)^2}{4\hbar\sigma} + i\frac{p_0 x}{\hbar}\right] \ , \tag{8.1}$$

where the four parameters $x_0, p_0, \sigma, \eta$ specify the physical properties of the Gaussian state.

**(a)** *(1point)* Find the normalization $\mathcal{N}$. Relate parameters $x_0, p_0, \sigma, \eta$ to physical quantities by calculating all the 1-point and coincidence 2-point functions.

**(b)** *(4 points)* Use the result of Problem 7 to find the transition amplitude for a particle that starts out in the ground state of a harmonic oscillator at $t_i$ to be found again in the ground state at some $t_f$, *i.e.* calculate,

$$Z[J] = \langle 0, t_f | 0, t_i \rangle = \langle 0 | U(t_f, t_i) | 0 \rangle \ . \tag{8.2}$$

The harmonic oscillator ground state wave function is given by

$$\psi_0(x) = \langle x | 0 \rangle = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp\left[ -\frac{m\omega x^2}{2\hbar} \right] \ . \tag{8.3}$$

This exercise is on the longer side, but involves just evaluating Gaussian integrals and the use of trigonometric identities. The final result is of the form

$$Z[J] = e^{i\varphi} \exp\left\{ -\frac{1}{2\hbar^2} \int_{t_i}^{t_f} dt dt' \, J(t) i\Delta_F(t,t') J(t') \right\} \ . \tag{8.4}$$

Determine what $\Delta_F(t,t')$ is and which equation it satisfies. Can you say what is the meaning of the phase $\varphi$?

One can use the functional above to generate proper vacuum expectation values. The following trick is encountered in almost every treatment of path integrals. It can be shown that $Z[J]$ generates the following quantities

$$\begin{aligned} \left[ \frac{1}{i} \frac{\delta}{\delta J(t_1)} \right] \cdots \left[ \frac{1}{i} \frac{\delta}{\delta J(t_n)} \right] Z[J] \bigg|_{J=0} &= \langle 0, t_f | T \Big[ \hat{x}(t_1) \dots \hat{x}(t_n) \Big] | 0, t_i \rangle \\ &= \langle 0, t_i | U(t_f, t_i) T \Big[ \hat{x}(t_1) \dots \hat{x}(t_n) \Big] | 0, t_i \rangle \ . \end{aligned} \tag{8.5}$$

This is not exactly the expectation value of the time-ordered product of position operators, there is an extra evolution operator there. But, since the ground state $|0, t_i\rangle$ is an eigenstate of the (time-independent) Hamiltonian, the action of the evolution operator on it is trivial,

$$\langle 0, t_i | U(t_f, t_i) = e^{-\frac{i}{\hbar} E_0 (t_f - t_i)} \langle 0, t_i | = \langle 0, t_f | 0, t_i \rangle \langle 0, t_i | = Z[0] \langle 0, t_i | \ , \tag{8.6}$$

where $E_0$ is the energy of the ground state. Therefore, this extra evolution operator just factors out as a phase, and the *normalized* generating functional does indeed generate proper expectation values

$$\left[ \frac{1}{i} \frac{\delta}{\delta J(t_1)} \right] \cdots \left[ \frac{1}{i} \frac{\delta}{\delta J(t_n)} \right] \frac{Z[J]}{Z[0]} \bigg|_{J=0} = \langle 0, t_i | T \Big[ \hat{x}(t_1) \dots \hat{x}(t_n) \Big] | 0, t_i \rangle \ . \tag{8.7}$$

Note that this works only for eigenstates of the Hamiltonian. Had we chosen some other non-stationary states or different initial and final states then $Z[J]/Z[0]$ would not generate expectation values.

**(c)** *(4 points)* There is a way to construct a generating functional for the $n$-point functions that works for an arbitrary state which goes under the name of the Schwinger-Keldysh (or in-in, or closed-time-path) generating functional, and it was defined in the lecture. Using the transition amplitude from Problem 7 it can be written as

$$Z[J_+, J_-] = \int_{-\infty}^{\infty} dx_1 dx_2 dx_* \, \psi^*(x_2) \psi(x_1) \, F^*_{J_-}(x_*, t_*; x_2, t_i) F_{J_+}(x_*, t_*, x_1, t_i) \ , \tag{8.8}$$

where $t_i$ is the initial time. Calculate these three integrals to obtain the generating functional for an arbitrary Gaussian state (8.1). *Hint*: Calculate the integral over the return point ($x = x_*$) first.

The Schwinger-Keldysh generating functional generates the following expectation values

$$\begin{aligned} \left[ -\frac{1}{i} \frac{\delta}{\delta J_-(t_1)} \right] \cdots \left[ -\frac{1}{i} \frac{\delta}{\delta J_-(t_n)} \right] \left[ \frac{1}{i} \frac{\delta}{\delta J_+(t_{n+1})} \right] \cdots \left[ \frac{1}{i} \frac{\delta}{\delta J_+(t_{n+m})} \right] Z[J_+, J_-] \bigg|_{J_+ = J_- = 0} \\ = \langle \psi | \overline{T} \Big[ \hat{x}(t_1) \dots \hat{x}(t_n) \Big] T \Big[ \hat{x}(t_{n+1}) \dots \hat{x}(t_{n+m}) \Big] | \psi \rangle \ , \end{aligned} \tag{8.9}$$

and this is true for an arbitrary state $|\psi\rangle$ (it need not be Gaussian, but if it is non-Gaussian we do not know how to calculate it exactly).