
ADVANCED TOPICS IN THEORETICAL PHYSICS II

Tutorial problem set 4, 02.10.2017.

(20 points in total)

In class problems are 9 & 10. Problem 11 is due at Monday, 09.10.2017.

■ PROBLEM 9 Anharmonic oscillator – Feynman rules *(8 points)*

Consider a one-particle system given by the action

$$S[x] = \int dt \left[\frac{m\dot{x}^2}{2} - \frac{m\omega^2}{2}x^2 - \frac{\lambda}{4!}x^4\right] , \tag{9.1}$$

where $\omega^2 > 0$ and $\lambda > 0$. In this problem you will derive derive how to calculate n-point functions as a perturbative expansion in λ.

(a) *(1 point)* Consider the system to be initially in a pure Gaussian state $|\psi\rangle$ with vanishing position and momentum expectation values, $\overline{q}(t_0) = 0 = \overline{p}(t_0)$, no squeezing $\langle\{\hat{x}(t_0), \hat{p}(t_0)\}\rangle = 0$, and variance $(\Delta x)(t_0) = \sqrt{\langle\hat{x}(t_0)^2\rangle} = \sqrt{\hbar/(2m\omega)}$ (this is actually the ground state of the free system for $\lambda = 0$.) The path integral representation of the Schwinger-Keldysh generating functional can be written as

$$\begin{aligned} Z[J_+, J_-] = &\int \mathcal{D}x_+(t)\mathcal{D}x_-(t) \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t_*} dt \Big[J_+(t)x_+(t) - J_-(t)x_-(t)\Big]\right. \\ &+ \frac{i}{\hbar}\int_{t_0}^{t_*} dt dt' \begin{bmatrix} x_+(t) & x_-(t)\end{bmatrix} \begin{bmatrix} D_0^{++}(t,t') & D_0^{+-}(t,t') \\ D_0^{-+}(t,t') & D_0^{--}(t,t')\end{bmatrix} \begin{bmatrix} x_+(t') \\ x_-(t')\end{bmatrix} \\ &\left.+ \frac{i}{\hbar}\int_{t_0}^{t_*} dt \left(-\frac{\lambda}{4!}\right)\Big[x_+^4(t) - x_-^4(t)\Big]\right\} . \end{aligned} \tag{9.2}$$

Remind yourself what are $D_0^{ab}(t,t')$ $(a, b = \pm)$ from the lectures. Show that the full generating functional can formally be written as

$$Z[J_+, J_-] = \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t_*} dt \left(-\frac{\lambda}{4!}\right)\left[\left(\frac{\hbar}{i}\frac{\delta}{\delta J_+(t)}\right)^4 - \left(-\frac{\hbar}{i}\frac{\delta}{\delta J_-(t)}\right)^4\right]\right\} Z_0[J_+, J_-] , \tag{9.3}$$

where $Z_0[J_+, J_-]$ is the generating functional (9.2) for $\lambda = 0$,

$$Z_0[J_+, J_-] = \exp\left\{-\frac{1}{2\hbar^2}\int_{t_0}^{t_*} dt dt' \begin{bmatrix} J_+(t) & -J_-(t)\end{bmatrix} \begin{bmatrix} i\Delta_0^{++}(t,t') & i\Delta_0^{+-}(t,t') \\ i\Delta_0^{-+}(t,t') & i\Delta_0^{--}(t,t')\end{bmatrix} \begin{bmatrix} J_+(t') \\ -J_-(t')\end{bmatrix}\right\} . \tag{9.4}$$

The elements of the free Keldysh propagator are,

$$i\Delta_0^{++}(t;t') = \langle\psi|T\big[\hat{x}_0(t)\hat{x}_0(t')\big]|\psi\rangle = i\Delta_{\rm F}(t;t') = \frac{\hbar}{2m\omega}\,{\rm e}^{-i\omega|t-t'|} , \tag{9.5}$$

$$i\Delta_0^{--}(t;t') = \langle\psi|\overline{T}\big[\hat{x}_0(t)\hat{x}_0(t')\big]|\psi\rangle = i\Delta_{\rm D}(t;t') = \frac{\hbar}{2m\omega}\,{\rm e}^{i\omega|t-t'|} , \tag{9.6}$$

$$i\Delta_0^{+-}(t;t') = \langle\psi|\big[\hat{x}_0(t')\hat{x}_0(t)\big]|\psi\rangle = i\Delta_0^{-}(t;t') = \frac{\hbar}{2m\omega}\,{\rm e}^{i\omega(t-t')} , \tag{9.7}$$

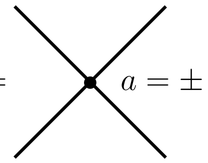
$$i\Delta_0^{-+}(t;t') = \langle\psi|\big[\hat{x}_0(t)\hat{x}_0(t')\big]|\psi\rangle = i\Delta_0^{+}(t;t') = \frac{\hbar}{2m\omega}\,{\rm e}^{-i\omega(t-t')} , \tag{9.8}$$

where the index 0 on the position operators stands for free in the sense that these are the operators for $\lambda = 0$, *i.e.* $\hat{x}(t) = \hat{x}_0(t) + \delta_\lambda \hat{x}(t)$, where $\delta_\lambda \hat{x}(t) \to 0$ in the limit when $\lambda \to 0$.

(b) *(2 points)* In cases where λ can be considered a small parameter, the form (9.3) of the generating functional is very convenient for generating a perturbative expansion for n-point functions. By expanding it to a particular order in λ we get a generating functional which generates n-point functions to a given order. This expansion can be represented in a compact notation by Feynman diagrams in pretty much the same way as is standardly done for the in-out generating functional. The difference here is that vertices and endpoints of propagators have $\pm$ polarities associated to them. These are not physical polarities, but are a consequence of the "doubling of variables" we have introduced in order to construct the generating functional in the first place.

Using the following simple Feynman rules,

$i\Delta_0^{ab}(t_1, t_2) =$ 

; $-ai\lambda/\hbar =$ 

write down all the diagrams contributing to $\langle\psi|\hat{x}(t_1)\hat{x}(t_2)|\psi\rangle$ up to and including the order of λ^2. Be careful about polarity of the external and internal legs.

(c) *(4 points)* Calculate $\langle\psi|\hat{x}(t_1)\hat{x}(t_2)|\psi\rangle$ to order λ. You should get that the one-loop contribution to the Wightman function contains a term that linearly grows in time, $\propto \lambda(t_1 - t_2)$. Is this what you would expect on physical grounds? Give an argument to why this perturbative result must fail for large $(t_1 - t_2)$.

(d) *(1 point)* Calculate how the squeezing of the state evolves in time, *i.e.* calculate $\langle\psi|\{\hat{x}(t), \hat{p}(t)\}|\psi\rangle$. Do this by using the fact that $\hat{p}(t) = m\frac{d}{dt}\hat{x}(t)$, so that

$$\langle\psi|\{\hat{x}(t_1), \hat{p}(t_2)\}|\psi\rangle = m\frac{d}{dt_2}\langle\psi|\{\hat{x}(t_1), \hat{x}(t_2)\}|\psi\rangle \ . \tag{9.9}$$

Note that in the limit $\lambda \to 0$ the specified initial state is the ground state of the system, and evolves trivially in time. Therefore no squeezing is generated. The squeezing is generated by the non-quadratic (interaction) part of the potential.

■ PROBLEM 10 Rotating the Schwinger-Keldysh functional *(4 points)*

The setup of the Schwinger-Keldysh generating functional with doubling of the variables is not fundamental. In the free generating functional

$$Z[J_+, J_-] = \int \mathcal{D}x_+(t)\mathcal{D}x_-(t) \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t_*} dt \Big[J_+(t)x_+(t) - J_-(t)x_-(t)\Big] \right.$$
$$\left. + \frac{i}{\hbar}\int_{t_0}^{t_*} dt dt' \begin{bmatrix} x_+(t) & x_-(t) \end{bmatrix} \begin{bmatrix} D_0^{++}(t,t') & D_0^{+-}(t,t') \\ D_0^{-+}(t,t') & D_0^{--}(t,t') \end{bmatrix} \begin{bmatrix} x_+(t') \\ x_-(t') \end{bmatrix} \right\} \tag{10.1}$$

rotate the variables x_+ and x_- into the *averaged variable* and *fluctuations*,

$$\overline{x} = \frac{x_+ + x_-}{2} \ , \qquad \Delta x = x_+ - x_- \ , \tag{10.2}$$

respectively, and show that it can be written as

$$Z[\bar{J}, \Delta J] = \int \mathcal{D}\overline{x}(t)\mathcal{D}\Delta x(t) \exp\left\{ \frac{i}{\hbar}\int_{t_0}^{t_*} dt \Big[\bar{J}(t)\overline{x}(t) + \Delta J(t)\Delta x(t) \Big] \right.$$

$$\left. + \frac{i}{\hbar}\int_{t_0}^{t_*} dt dt' \begin{bmatrix} \Delta x(t) & \overline{x}(t) \end{bmatrix} \begin{bmatrix} D_0^{11}(t,t') & D_0^{21}(t,t') \\ D_0^{12}(t,t') & D_0^{22}(t,t') \end{bmatrix} \begin{bmatrix} \Delta x(t') \\ \overline{x}(t') \end{bmatrix} \right\} . \tag{10.3}$$

What are the operators $D_0^{ij}(t,t')$? What are the new currents $\bar{J}$ and ΔJ?

We already know that the solution for the path integral is (9.4). Show that it can be written in terms of the new currents as

$$Z_0[\bar{J}, \Delta J] = \exp\left\{ -\frac{1}{2\hbar^2}\int_{t_0}^{t_*} dt dt' \begin{bmatrix} \Delta J(t) & \bar{J}(t) \end{bmatrix} \begin{bmatrix} 0 & i\Delta^a(t,t') \\ i\Delta^r(t,t') & 2F(t,t') \end{bmatrix} \begin{bmatrix} \Delta J(t) \\ \bar{J}(t) \end{bmatrix} \right\} , \tag{10.4}$$

where $F(t;t') = (1/2)\langle\{\hat{x}(t), \hat{x}(t')\}\rangle$ is the statistical (Hadamard) function and $\Delta^r(t;t')$ and $\Delta^a(t;t')$ are the retarded and advanced propagator, respectively. Is any information about the 2-point functions of the system lost by performing this rotation? Prove that indeed the upper left element of the rotated Keldysh propagator is zero. Does this result hold in general? Can you give a physical interpretation of that result?

■ PROBLEM 11 Inverted anharmonic oscillator *(9 points + 4* bonus points)*

The action for a particle of mass m in an inverted harmonic oscillator potential with a quartic potential is given by

$$S[x] = \int dt \left[\frac{m\dot{x}^2}{2} + \frac{m\mu^2}{2}x^2 - \frac{\lambda}{4!}x^4 \right] , \tag{11.1}$$

where $\mu^2 > 0$ and $\lambda > 0$.

(a) *(1 point)* Draw the classical potential. Show that the positions of the stable minima of the potential are

$$\chi_\pm = \pm\sqrt{\frac{6\, m\mu^2}{\lambda}} . \tag{11.2}$$

What is the frequency of small oscillations around the minima?

(b) *(3 points)* Consider first the system to be initially in the state defined in Problem 10 (a), $\overline{x}(t_0) = 0 = \overline{p}(t_0)$, $\langle\{\hat{x}(t_0), \hat{p}(t_0)\}\rangle = 0$, $(\Delta x)(t_0) = \sqrt{\hbar/(2m\omega)}$. How would the spread of the wave packet Δx evolve in the case $\lambda = 0$ (remember previous problem sets). How do you expect the spread of the wave packet Δx to evolve for $\lambda > 0$? Calculate $(\Delta x)^2$ perturbatively to linear order in λ. The 'free' generating functional here is the same as in (9.4), just with the following propagator functions

$$i\Delta_0^{++}(t,t') = \frac{\hbar}{2m\mu}\Big[\cosh\mu(t+t') - i\sinh\mu|t-t'| \Big] , \tag{11.3}$$

$$i\Delta_0^{--}(t,t') = \frac{\hbar}{2m\mu}\Big[\cosh\mu(t+t') + i\sinh\mu|t-t'| \Big] , \tag{11.4}$$

$$i\Delta_0^{+-}(t,t') = \frac{\hbar}{2m\mu}\Big[\cosh\mu(t+t') + i\sinh\mu(t-t') \Big] , \tag{11.5}$$

$$i\Delta_0^{-+}(t,t') = \frac{\hbar}{2m\mu}\Big[\cosh\mu(t+t') - i\sinh\mu(t-t') \Big] . \tag{11.6}$$

Note the time dependence of the coincident limit of these functions. Estimate the time of breakdown of perturbation theory. This is the time of symmetry breaking.

Hint: The computation here is the same (up until plugging in the propagators) as the one in Problem 9 (c). That means the symmetry factors of the Feynman diagrams are the same.

(c) *(2 points)* Now consider an initial Gaussian state with the same properties as in the previous problem, except that it is centered around the right minimum of the potential, $\overline{x}(t_0) = \chi_+$.

In this case one can construct the free generating functional in the straightforward manner, just that the propagators will be different to account for different initial conditions. A better way of doing this is to redefine the variable so that formally we have the same initial conditions as in the previous problem. Then we can also use the propagators of the form (9.5)-(9.8), no need to calculate the new ones.

Introduce a new variable y by performing the constant shift (of the zero of the coordinate system),

$$x(t) = y(t) + \chi_+ \ . \tag{11.7}$$

It is obvious that $\overline{y}(t_0) = 0$, and the other initial conditions remain unchanged. Find the action (11.1) written in terms of y variable. The quadratic part of this action now corresponds to a stable harmonic oscillator. What is the frequency Ω of that oscillator? By coupling y to an external source one can construct the Schwinger-Keldysh generating functional for n-point functions of y's. The free generating functional here is the same as (9.4), just ω is substituted by Ω everywhere, including the functions (9.5)-(9.8).

Show that the full generating functional can be written as

$$Z[J_+, J_-] = \exp\left\{ \frac{i}{\hbar} \int_{t_0}^{t_*} dt \left(-\frac{\lambda}{4!}\right) \left[\left(\frac{\hbar}{i} \frac{\delta}{\delta J_+(t)}\right)^4 - \left(-\frac{\hbar}{i} \frac{\delta}{\delta J_-(t)}\right)^4 \right] \right\}$$
$$\times \exp\left\{ \frac{i}{\hbar} \int_{t_0}^{t_*} dt \left(-\sqrt{\frac{m\mu^2\lambda}{6}}\right) \left[\left(\frac{\hbar}{i} \frac{\delta}{\delta J_+(t)}\right)^3 - \left(-\frac{\hbar}{i} \frac{\delta}{\delta J_-(t)}\right)^3 \right] \right\} Z_0[J_+, J_-] \ . \tag{11.8}$$

(d) *(3 points)* Calculate $\langle \hat{x}(t) \rangle = \chi_+ + \langle \hat{y}(t) \rangle$ to linear order in λ. Comment on the quantum nature of this result.

(e) *(4^* bonus points)* Apply the linearized one-loop 1PI effective action discussed in the lecture to the problem in (c). Derive the linearized equation of motion from it, solve it numerically and plot the result.