Sensitivity Analysis in Bayesian networks of the noisy-OR model and its generalisations

Msc Thesis - Artificial Intelligence

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1 Introduction

Since their introduction three decades ago, probabilistic graphical models have become an essential tool in knowledge-based systems for Artificial Intelligence. In most cases, knowledge-based systems are designed to deal with real-life problems requiring considerable human knowledge and proficiency for their solutions. One popular probabilistic graphical model is called a Bayesian network. Bayesian networks are widely used in healthcare, financial advice, spam filter, image processing, robotics, etc [2, 10, 11, 12], to manage reasoning with uncertain interactions among variables in the given domain.

A Bayesian network is a compact representation of a joint probability distribution representing a set of variables and their conditional independencies via a directed acyclic graph [1, 7, 8]. The parameters in the network are specified by a set of conditional probability tables (CPT). The network serves as a tool for computing the posterior probability distribution of outcome variable(s) when observing evidence. The knowledge needed for building a Bayesian network is often obtained by domain experts and/or appropriate data. The elicitation of all the required probabilities is often the primary hindrance in building a real-world network with domain experts. To ease this elicitation task, a network engineer can use interaction models such as the noisy-OR model and its variants. An interaction model basically entails a *parameterised* CPT for a common effect variable where specific patterns of interaction among the causal influences on the effect variable hold [1, 3]. These interaction models require limited probability estimates since the remaining probabilities of the CPT are computed by the model's rules, where the model's rules are derived from the assumptions of interaction among the causal influences. However, even if the assumptions underlying the noisy-OR model hold in the application domain, the probabilities computed from the model depend on the input parameters and the accuracy of their estimates. The input parameters of the interaction model are the probability estimates taken in by the model to compute the remaining probabilities. For the noisy-OR model, the input parameters are called the *noisy-OR parameters*. However, it is not (always) possible for a network engineer to elicit accurate estimates for all input parameters. Conducting a sensitivity analysis, that is, examining the effects of changes in parameters on an output probability of interest, can give insights into the *propagation effects* due to deviating noisy-OR parameter probabilities in Bayesian networks. The propagation effects pertain to how the network's specified probabilities influence the computed output when deviations from noisy-OR parameters are assumed. However, examining the effects of variation for a wide range of possible combinations of parameters rapidly becomes infeasible [6]. The just mentioned matter indicates the need for studying generic sensitivity functions of different models to gain understanding into the consequences when input parameters of causal interaction models are possibly inaccurate. As a result, network engineers will be capable of determining how much effort they need to put into obtaining accurate probability estimates for the input parameters in order to guarantee the validity of a Bayesian network's output without executing a complete analysis.

The effects of the noisy-OR assumptions and its variants have been studied before, for example, by Woudenberg and van der Gaag [1]. They examined the following matter: how well do the computed probabilities from the model's rules approach the real probabilities if the properties underlying the noisy-OR model do not genuinely hold in the application domain? They examined whether the choice of using the noisy-OR model for the elicitation task is appropriate. We will, taking a similar approach, investigate the consequences of inaccurate estimates of the noisy-OR parameters on the output probabilities. How robust are the calculated probabilities to possible inaccuracies in the input parameters? The desired result of our investigations will help a network engineer to decide how accurate the probability estimates he/she should provide should be, and thus, the amount of time that he/she should take for the elicitation task. In contrast to Woudenberg and van der Gaag, we will assume that the choice of using the noisy-OR model for the elicitation task is indeed appropriate and therefore assume that the properties underlying the noisy-OR model hold.

This thesis is organised as follows. In Section 2, we briefly review Bayesian networks, certain causal interaction models for Bayesian networks, and how one can perform a sensitivity analysis. We conduct our research in an analytical way and will take a similar approach for different structures and parameters. This approach will be described in Section 3. In Section 4, we examine the propagation effects due to (leaky) noisy-OR parameter changes on output probabilities. Since the (leaky) noisy-OR model involves binary variables only, an interaction model called the noisy-MAX came into practice. We will study the propagation effects of this interaction model in Section 5. In Section 6, we will summarise our findings and describe the differences and similarities of our findings with [1]. We will conclude this thesis in Section 7.

2 Preliminaries

In this section, we briefly introduce some notation, review Bayesian networks, the (leaky) noisy-OR model, and how one can perform a sensitivity analysis. In the preliminaries, we mainly rely on [1, 2, 3, 6].

2.1 Notation

In this thesis, we will represent variables with capital letters and their values with lower case letters. For example, v represents a possible value of variable V. Likewise, $\mathbf{V}$ indicates a set of variables $\{V_1, ..., V_m\}$, and $\mathbf{v}$ a particular $m-$tuple $(v_1, ..., v_m)$, where v_i represents a value taken on by variable V_i in $\mathbf{V}$. Furthermore, we will use the following notation regarding the representation of values of variables:

- A variable V_i can take on values $v_i^j \in \{v_i^0, ..., v_i^{n_i}\}, n_i \geq 1$. By making n dependent of i, every V_i can have a different number of values;
- The values are ordered, meaning that $v_i^0 < ... < v_i^{n_i}$;
- In case of Boolean variables ($n_i = 1$) we simply write $\neg v_i$ instead of v_i^0, and v_i instead of v_i^1. Boolean variables are of the type true/false, present/absent or positive/negative.

2.2 Bayesian networks

A Bayesian network is a graphical representation of a joint probability distribution over a set of variables [1]. It consists of a qualitative- and quantitative part. The qualitative part is a directed acyclic graph $G = (V_G, A_G)$, containing random variables illustrated as nodes $V \in V_G$, and a set of arcs $A_G \subseteq V_G \times V_G$, which describe the (in)dependency relation among the variables [7]. In this thesis, we will write $X \rightarrow Y$ instead of (X, Y). In addition, given that X, Y and Z are nodes and $X \rightarrow Y \rightarrow Z \in G$, then X is a parent of Y and Y a parent of Z. Likewise, we say that Y is a child of X and Z a child of Y. Finally, we have that Z and Y are descendants of X, and in the same way, X and Y are ancestors of Z. The quantitative part of a Bayesian network is a set of conditional probability distributions for every variable represented in the network.

We now define the concept of a Bayesian network more formally [2]:

Definition 2.1. *A Bayesian network encodes the joint probability distribution over a set of variables $\{V_1, V_2, ..., V_m\}$, where m is finite, and decomposes it into a product of conditional probability distributions over each variable given its parents in the graph. In case of nodes with no parents, we use their prior probability distribution. The joint probability distribution over $\{V_1, V_2, ..., V_m\}$ can be obtained by taking the product of all of these prior and conditional probability distributions:*

$$Pr(v_1, v_2, ..., v_m) = \prod_{i=1}^{m} Pr(v_i | pa(V_i)) \tag{1}$$

where $pa(V_i)$ is a conjunction of value assignments of all parents of node V_i.

There exist algorithms that compute the probabilities from a Bayesian network. However, even though Bayesian networks decrease the complexity of the representation of the joint probability distribution, probabilistic inference is NP-hard [2,9].

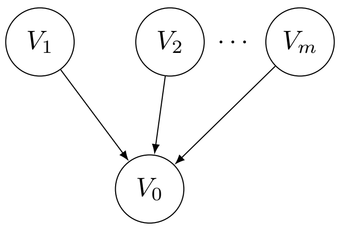

Figure 1: A causal mechanism with effect variable V_0 and m cause variables V_i.

2.3 Noisy-OR model

To simplify the quantification task, one can use a causal interaction model such as the noisy-OR model. In general, the noisy-OR model can be looked upon as a parameterised conditional probability table for a corresponding effect variable of a causal mechanism with multiple cause variables, see Figure 1. Experts' elicitation tasks become considerably more manageable with the help of a noisy-OR model because only a limited number of probabilities need to be provided. When using the noisy-OR model, it is assumed that a certain effect can be obtained with a high probability by the presence of one of the multiple causes [5]. In addition, the model assumes that if two or more causes are present, the likelihood of the concerning outcome will not decrease. The noisy-OR model possesses mere binary random variables V_i, $i \in \{0, ..., m\}$.

To express the noisy-OR model more formally, we have [1, 3]:

$$G = (V_G, A_G) \text{ with } V_G = V \text{ where } V_1 \to V_0, ..., V_m \to V_0 \in A_G \text{ and } m \geq 1.$$

Figure 1 illustrates the basic idea of such a mechanism. This directed graph G exhibits different causes V_i where $i \in \{1, ..., m\}$ and V_0 is the common effect. The noisy-OR model defines the conditional probability table (CPT) for the effect variable V_0 of a causal mechanism through [1, 3]:

- $Pr(v_0|\neg v_1, ..., \neg v_m) = 0$ (property of accountability);
- The noisy-OR parameters $q_i = Pr(v_0|\neg v_i, ..., \neg v_{i-1}, v_i, \neg v_{i+1}, ..., \neg v_m)$ associated with cause V_i, for all $i = 1, ..., m$;
- For the remaining value combinations $\mathbf{c}$ involving the presence of two or more causes we have:

  $$Pr(v_0|\mathbf{c}) = 1 - \prod_{i \in T_{\mathbf{c}}} (1 - q_i) \tag{2}$$

  where $T_{\mathbf{c}} = \{i|\mathbf{c} \wedge v_i \not\equiv \text{False}\}$.

For a causal mechanism with m modelled causes V_i, $i = \{1, .., m\}$, the noisy-OR model defines a full probability table over $m + 1$ variables, specifying a total of $2 \cdot 2^m$ probabilities. However, exactly half of the probability table is provided by that fact that $Pr(v_0|\mathbf{c}) + Pr(\neg v_0|\mathbf{c}) = 1$ and, therefore, are redundant. Of the 2^m non-redundant probabilities, the noisy-OR model needs the values of only m parameter probabilities q_i to be provided beforehand. Furthermore, the model forces the distribution $Pr(V_0|\neg v_1, ..., \neg v_m)$ to be degenerate, that is $Pr(v_0|\neg v_1, ..., \neg v_m) = 0$ and $Pr(\neg v_0|\neg v_1, ..., \neg v_m) = 1$.

In case of two causes V_1 and V_2 and their common effect V_0, the CPT leads to Table 1.

$Pr(v_0\|v_1, v_2)$	v_1	$\neg v_1$
v_2	$q_1 + q_2 - q_1 q_2$	q_2
$\neg v_2$	q_1	0

$Pr(\neg v_0\|v_1, v_2)$	v_1	$\neg v_1$
v_2	$1 - (q_1 + q_2 - q_1 q_2)$	$1 - q_2$
$\neg v_2$	$1 - q_1$	1

Table 1: CPT for a noisy-OR model with two parents.

2.4 Leaky noisy-OR model

In the noisy-OR model the property of accountability ($Pr(v_0|\neg v_1 \wedge ... \wedge \neg v_m) = 0$) is assumed. However, this is actually quite a challenging assumption. The reason for this is that it's (almost) impossible to model all the existing causes V_1,...,V_m of a common effect V_0 of a Bayesian network. A model dealing with this matter is called the *leaky* noisy-OR model. The leaky noisy-OR model assumes the following property:

$$Pr(v_0|\neg v_1 \wedge ... \wedge \neg v_m) = p$$

where the variables V_1,...,V_m again are the different modelled causes of the common effect V_0, with $p > 0$. This probability p is called the *leak* probability. The leak probability is the probability that the common effect V_0 will be present while all the modelled causes V_1,...,V_m are absent. One can also say that the leak probability is the probability that the effect v_0 occurs spontaneously.

The leaky noisy-OR model defines the CPT for the effect variable V_0 of a causal mechanism through [1, 3]:

- $Pr(v_0|\neg v_1 \wedge ... \wedge \neg v_m) = p$ (leak-probability)
- The noisy-OR parameters $q_i = Pr(v_0|\neg v_i, ..., \neg v_{i-1}, v_i, \neg v_{i+1}, ..., \neg v_m)$ associated with cause V_i, for all $i = 1, ..., m$;
- For the remaining value combinations $\mathbf{c}$ involving the presence of two or more causes we have:

  $$Pr(v_0|\mathbf{c}) = 1 - (1-p) \cdot \prod_{i \in T_{\mathbf{c}}} \frac{(1-q_i)}{(1-p)} \tag{3}$$

  where $T_{\mathbf{c}} = \{i|\mathbf{c} \wedge v_i \not\equiv \text{False}\}$.

One should take into account that during the construction of a Bayesian network, the effect of a single cause cannot be set apart from the spontaneous occurrence of the effect. Therefore, the probability assessments of the effect given only one cause are modified to account for this spontaneous leak before they are combined. Also, now the required number of probabilities to define the CPT for node V_0 is equal to $m + 1$.

2.5 Sensitivity analysis

In this section we briefly review sensitivity analysis of Bayesian networks and rely on [1, 4, 6, 8].

The reliability of the output of a Bayesian network can be examined by studying its robustness. Robustness pertains to the extent to which the network's conditional probabilities affect the output when deviations from precise estimates are assumed [4]. The probabilities of the CPT used in the Bayesian network are mostly assessed by domain experts or obtained from appropriate data. However, the chances of inaccurate obtained values for the assessed probabilities are unavoidable. Because of this reason, it is essential to study a Bayesian network's robustness. A Bayesian network can be exposed to sensitivity analyses to investigate the possible effects of these inaccurate obtained values on its output. The sensitivity analysis result is a sensitivity function $f(x)$ that demonstrates the network's output probability in the probability x being varied, which is called a one-way sensitivity analysis.

A sensitivity function is either a linear or a rectangular hyperbolic function. A linear sensitivity function has the form [1, 8]:

$$f(x) = ax + b \tag{4}$$

The linear function is derived from examining the effects of inaccuracies in the probabilities for the (possibly indirect) causes of the variable of interest where this variable has no observed descendants. We say that these linear functions capture the effects of *causal propagation.* The constants a, b are built from the network's non-varied probabilities.

A hyperbolic sensitivity function has the form:

$$f(x) = \frac{ax + b}{cx + d} \tag{5}$$

The hyperbolic function is derived from examining the effects of inaccuracies in a network's probabilities on output probabilities for variables with observed descendants. We say that these hyperbolic functions capture the effects of *diagnostic propagation*. The constants a, b, c, d are again built from the assessments for the non-varied numerical parameters. The values $f(x)$ and x both represent probabilities, indicating that $f(x)$ and x are both bounded by the unit window, meaning that $f(x) \in [0, 1]$ and $x \in [0, 1]$.

We can rewrite the hyperbolic sensitivity function and obtain:

$$f(x) = \frac{r}{x - s} + t$$

where

$$s = -\frac{d}{c},\ t = \frac{a}{c},\ \text{ and } r = \frac{bc - ad}{c^2}$$

where $x = s$ indicates the vertical asymptote of the rectangular hyperbola and $y = t$ the horizontal asymptote. The constants s, t define the general shape of the hyperbola, and the location of the quadrants of the two branches. The constant r defines the location of the vertices of the two branches. See Figure 2. Keep in mind that the vertices of a hyperbola are the exact points where the absolute value of the first derivative of the function equals 1. Depending on the quadrant of the branch under study, the vertex is located at one of the four points $(s \pm \sqrt{|r|}, t \pm \sqrt{|r|})$ [1, 8]. Note that a vertex point can lie either inside or outside the unit window and the vertical asymptote *has to* lie outside the unit window.

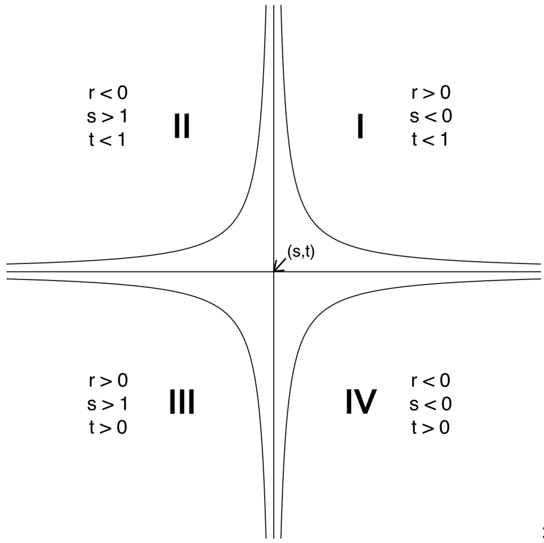

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Figure 2: Rectangular hyperbolas and their constants (the constraints on s and t are specific for sensitivity functions).

To gain more insight into a sensitivity function which is a rectangular hyperbolic function, one can study its first derivative. The first derivative of function (5) equals:

$$\frac{d}{dx} f(x) = \frac{ad - bc}{(cx + d)^2}. \tag{6}$$

3 Methods

We will conduct our research in an analytical way, taking a similar approach as Woudenberg and van der Gaag [1], but for different parameters since we consider a different research question. In addition to Woudenberg and van der Gaag, we study more models and conduct experiments with different parameter settings.

We will now present the general approach when examining the propagation effects due to noisy-OR parameter changes.

To examine the propagation effects due to noisy-OR parameter changes in Bayesian networks, we will take the following approach:

1. First, we examine the possible effects due to changes in a noisy-OR parameter on an output probability of interest pertaining to the effect variable, that is, the propagation effects in the causal direction.

   - We start by deriving a general formula for the sensitivity function under consideration. When studying the propagation effects in the causal direction, the corresponding sensitivity function is linear in x, where x is a noisy-OR parameter;
   - Subsequently, we determine the gradient of the sensitivity function and how this depends on the different parameters of the model;
   - When examining the gradient of the corresponding sensitivity function, we keep in mind the assumptions underlying the noisy-OR model, namely:

     For the causal mechanism from Figure 1 we assume:
     - The prior probability distributions for the cause variables V_i, $i \in \{1, ..., m\}$ are non-degenerate, that is $Pr(v_i) \neq 0$ and $Pr(\neg v_i) \neq 0$ for $i \in \{1, ..., m\}$;
     - $Pr(v_0|\neg v_1, ..., \neg v_m) = 0$, by the property of accountability;
     - Because the noisy-OR parameters $q_i = Pr(v_0|\neg v_i, ..., \neg v_{i-1}, v_i, \neg v_{i+1}, ..., \neg v_m)$ associated with cause V_i for all $i = 1, ..., m$, are assumed to be large since any of the factors is likely to trigger the effect [5], we mainly focus on $q_i \in [0.6, 1]$ when evaluating the propagation effects.
   - We present our observations.

2. Subsequently, we examine the possible effects due to changes in a noisy-OR parameter on an output probability of interest pertaining to a parent of the observed effect variable, that is, the propagation effects in the diagnostic direction.

   - We again derive a general expression for the sensitivity function, which in this case is a rectangular hyperbolic function;
   - From generic research of sensitivity functions from Bayesian networks, we know that the effect of deviations in the x-value on the output probability of interest mainly depends on the location of the vertex of the corresponding hyperbola branch. Therefore, analogous to Woudenberg and van der Gaag [1], we examine the influence of the involved parameters' values on the vertex's location;
   - After that, we plot several example sensitivity functions to support the mentioned findings and gain more insight into the propagation effects in the entire interval $[0, 1]$;
   - Since we assume that noisy-OR parameters generally take on high values in practice, we then focus on the interval for $x \in [0.6, 1]$, where x is a noisy-OR parameter;
   - In addition to Woudenberg and van der Gaag [1], we investigate the derivative of the corresponding sensitivity function;
   - With the help of WOLFRAM MATHEMATICA we obtain the value of the maximum gradient of the sensitivity function under study in the interval $x \in [0.6, 1]$;
   - We present our observations.

We will carry out this procedure for various Bayesian network's graphical structures for different interaction models and parameter settings. The assumptions will differ for different interaction models and experiments. For example, the assumptions underlying the leaky noisy-OR and noisy-MAX model are different than for the noisy-OR model. For each case, we will clearly indicate the underlying assumptions beforehand.

4 Propagation effects due to (leaky) noisy-OR parameter changes

The (leaky) noisy-OR model is a helpful interaction model contributing to ease the elicitation task: they require a limited number of parameter probabilities, indicating clear engineering advantages. However, the probability estimates of the input parameters of the (leaky) noisy-OR model can be inaccurate. For that reason, we will examine the effects of changes in noisy-OR parameters on specific probabilities. These effects pertain to how the network's specified probabilities influence the computed output when deviations from noisy-OR parameters are assumed. We will call these effects the propagation effects. By examining the propagation effects of noisy-OR parameter changes on probabilities computed from a Bayesian network, we want to identify conditions under which the use of inaccurate input parameters can result in different output probabilities, and therefore, possibly harm the validity of a Bayesian network's output. These conditions can help a network engineer to determine how much effort he/she should put into acquiring precise estimates for the noisy-OR parameters.

4.1 Propagation effects due to noisy-OR parameter changes: independent causes

We start by considering the conditional probability tables for the three variables of the basic mechanism from Figure 3. This basic mechanism can also be comprehended as a mechanism consisting of two parents (the cause variables C_1 and C_2) and their child (the effect variable E).

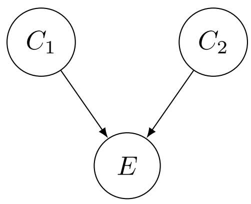

Figure 3: A basic causal mechanism with the effect variable E and cause variables C_1, C_2.

For the basic mechanism from Figure 3 we now make the following assumptions:

- The prior probability distributions for cause variables C_1 and C_2 are non-degenerate, that is $Pr(c_i) \neq 0$ and $Pr(\neg c_i) \neq 0$ for $i = 1, 2$;
- $Pr(e|\neg c_1, \neg c_2) = 0$, by the property of accountability;
- Because noisy-OR parameters are assumed to be large since any of the factors is likely to trigger the effect [5], we mainly focus on $Pr(e|c_1, \neg c_2), Pr(e|\neg c_1, c_2) \in [0.6, 1]$ in our research. We specifically use this constraint when evaluating the propagation effects.

4.1.1 Propagation effects in the causal direction

First, we examine the possible effects on the probability $Pr(e)$ due to changes in a noisy-OR parameter.

Theorem 4.1. *Consider the causal mechanism in Figure 3 and assume it models a noisy-OR. Let $x = Pr(e|\neg c_1, c_2)$ be the noisy-OR parameter associated with cause C_2. Then the sensitivity function $Pr(e)(x)$ has the following form:*

$$Pr(e)(x) = xPr(c_2)\Big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\Big) + Pr(e|c_1, \neg c_2)Pr(c_1) \tag{7}$$

Proof:
We have that probability $Pr(e)$ is equal to:

$$\begin{aligned} Pr(e) = {} & Pr(e|c_1, c_2)Pr(c_1)Pr(c_2) + Pr(c_2|\neg c_1, c_2)Pr(\neg c_1)Pr(c_2) + Pr(e|c_1, \neg c_2)Pr(c_1)Pr(\neg c_2) \\ & + Pr(e|\neg c_1, \neg c_2)Pr(\neg c_1)Pr(\neg c_2) \end{aligned} \tag{8}$$

Note that $Pr(e|\neg c_1, \neg c_2)Pr(\neg c_1)Pr(\neg c_2) = 0$ since $Pr(e|\neg c_1, \neg c_2) = 0$.

Using the noisy-OR model described in Section 2.3, we first compute $Pr(e|c_1, c_2)$:

$$\begin{aligned} Pr(e|c_1, c_2) &= 1 - \big((1 - Pr(e|\neg c_1, c_2))(1 - Pr(e|c_1, \neg c_2))\big) \\ &= 1 - \big((1 - x)(1 - Pr(e|c_1, \neg c_2))\big) \\ &= 1 - \big(1 - Pr(e|c_1, \neg c_2) - x + xPr(e|c_1, \neg c_2)\big) \\ &= Pr(e|c_1, \neg c_2) + x - xPr(e|c_1, \neg c_2) \\ &= x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2). \end{aligned} \tag{9}$$

From Equation (8) we now have:

$$\begin{aligned} Pr(e)(x) &= \Big(x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)\Big)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2) + Pr(e|c_1, \neg c_2)Pr(c_1)Pr(\neg c_2) \\ &= Pr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) + xPr(c_1)Pr(c_2) - xPr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2) \\ &+ Pr(e|c_1, \neg c_2)Pr(c_1)Pr(\neg c_2) \\ &= x\Big(Pr(c_1)Pr(c_2) - Pr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) + Pr(\neg c_1)Pr(c_2)\Big) + Pr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) \\ &+ Pr(e|c_1, \neg c_2)Pr(c_1)Pr(\neg c_2) \\ &= x\Big(Pr(c_1)Pr(c_2) - Pr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) + Pr(\neg c_1)Pr(c_2)\Big) \\ &+ Pr(e|c_1, \neg c_2)\Big(Pr(c_1)Pr(c_2) + Pr(c_1)Pr(\neg c_2)\Big) \\ &= xPr(c_2)\Big(Pr(c_1) - Pr(e|c_1, \neg c_2)Pr(c_1) + Pr(\neg c_1)\Big) + Pr(e|c_1, \neg c_2)\Big(Pr(c_1)Pr(c_2) + Pr(c_1)Pr(\neg c_2)\Big) \\ &= xPr(c_2)\Big(Pr(c_1) - Pr(e|c_1, \neg c_2)Pr(c_1) + Pr(\neg c_1)\Big) + Pr(e|c_1, \neg c_2)\Big(Pr(c_1)\big(Pr(c_2) + Pr(\neg c_2)\big)\Big) \\ &= xPr(c_2)\Big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\Big) + Pr(e|c_1, \neg c_2)Pr(c_1) \end{aligned}$$

The last 2 lines can been deduced because $Pr(c_i) + Pr(\neg c_i) = 1$ for $i = 1, 2$. □

To say something about the propagation effects we will examine the first derivative, that is in this case, the gradient of a single-variable function. The gradient of Equation (7) describes the effect that a deviation of x can have on the prior output probability of interest, namely $Pr(e)$. Thus, a small gradient expresses the information that a possibly large deviation of the probability x will still have only a minor effect on the output probability of interest.

To better convey the size of the gradient in natural language, we consider the gradient ∇ of the sensitivity function under study to be:

- small, when $|\nabla| \leq 0.25$;
- moderate, when $|\nabla| \in (0.25, 0.75)$;
- large, when $|\nabla| \geq 0.75$.

We will, in the same way, use these gradations for describing the propagation effects.

Corollary 4.1.1. *The gradient of the sensitivity function from Equation (7) is:*

$$\frac{d}{dx}Pr(e)(x) = Pr(c_2)\Big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\Big)$$

Observation: One can see that the gradient of Equation (7) is large when at least the probability $Pr(c_2)$ is large and $Pr(c_1)$ and/or $Pr(e|c_1, \neg c_2)$ is/are small. Note that the gradient of Equation (7) is in the interval $(0, 1)$ because $Pr(c_2), Pr(e|c_1, \neg c_2)$ and $Pr(c_1)$ are all probabilities and $Pr(c_1), Pr(c_2)$ fall within the interval $(0, 1)$ by assumption. Since $Pr(e|c_1, \neg c_2)$ is a noisy-OR parameter, and we assume that noisy-OR parameters possess a probability in the interval $[0.6, 1]$, we conclude that the gradient is large when cause C_2 is likely to be present and C_1 absent. In addition we have that the smaller the noisy-OR parameter $Pr(e|c_1, \neg c_2)$, the larger the gradient. We conclude that if cause C_1 is likely to be absent and cause C_2 present, the gradient of Equation (7) will be large.
Analogous observations hold for the sensitivity function obtained for the probability of interest $Pr(e)$ when $x = Pr(e|c_1, \neg c_2)$; c_1 and c_2 merely exchange roles.

Remark. *We have also used the words "large" and "small" for describing the prior probabilities* $Pr(c_1)$ *and* $Pr(c_2)$*, and noisy-OR parameters. We will use "large" and "small" in a more abstract way to indicate the prior probabilities and thus, not provide concrete values for large or small prior probabilities. Because if we, for example, have* $Pr(c_2) = 0.75$ *and* $Pr(c_1) = 0.25$ *in Equation (7), the gradient will attain a maximum value of:* $\frac{d}{dx}Pr(e)(x) = Pr(c_2)\Big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\Big) = 0.75(1 - 1 \cdot 0.25) = 0.5625$*, which by assumption is moderate. Therefore, we leave it to the reader to provide specific values for these prior probabilities. However, to indicate some clarification: the prior probability should be at least* ≥ 0.75 *and* ≤ 0.25 *in order to be stated as "large" or "small, respectively. In like manner, we will not provide concrete values for describing the noisy-OR probabilities. The noisy-OR parameters are by assumption in the interval* $[0.6, 1]$*, indicating that our focus is on rather large probabilities.*

Example 4.1. *As an example, consider the following parameter setting:* $Pr(c_1) = 0.1$*,* $Pr(c_2) = 0.9$ *and* $Pr(e|c_1, \neg c_2) = 0.85$*. The gradient of Equation (7) equals* $Pr(c_2)\big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\big) = 0.9(1 - 0.85 \cdot 0.1) = 0.8235$*. Thus, for these parameter values we consider the gradient, and thus the propagation effects, to be large. If we now set the parameter settings to* $Pr(c_1) = 0.5$*,* $Pr(c_2) = 0.5$*, and* $Pr(e|c_1, \neg c_2) = 0.6$*, the gradient equals:* $Pr(c_2)\big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\big) = 0.5(1 - 0.6 \cdot 0.5) = 0.35$*. As a consequence, we consider the propagation effects to be moderate.*

4.1.2 Propagation effects in the causal direction conditioned on one cause

We further examine the effects of causal propagation through the causal mechanism of Figure 3 by assuming the actual presence or absence of one of the causes C_1 or C_2, that is, by considering $Pr(e|c_i)$ or $Pr(e|\neg c_i)$ for $i = 1, 2$, for the probability of interest.
We first examine $Pr(e|c_2)$ as a function of the probability $x = Pr(e|\neg c_1, c_2)$, the result is a linear function in x.

Theorem 4.2. *Consider the causal mechanism in Figure 3 and assume it models a noisy-OR. Let* $x = Pr(e|\neg c_1, c_2)$ *be the noisy-OR parameter associated with cause* C_2*. Then the sensitivity function* $Pr(e|c_2)(x)$ *has the following form:*

$$Pr(e|c_2)(x) = x\Big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\Big) + Pr(e|c_1, \neg c_2)Pr(c_1) \tag{10}$$

Proof:
We have:

$$\begin{aligned}
Pr(e|c_2)(x) &= Pr(e|c_1, c_2)Pr(c_1) + xPr(\neg c_1) \text{ (by conditioning and independence of } C_1 \text{ and } C_2) \\
&= \Big(x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)\Big)Pr(c_1) + xPr(\neg c_1) \text{ (by Equation (9))} \\
&= x\Big(Pr(c_1) - Pr(e|c_1, \neg c_2)Pr(c_1) + Pr(\neg c_1)\Big) + Pr(e|c_1, \neg c_2)Pr(c_1) \\
&= x\Big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\Big) + Pr(e|c_1, \neg c_2)Pr(c_1).
\end{aligned}$$

We made use of the fact that $Pr(c_1) + Pr(\neg c_1) = 1$. □

Observation: The gradient of Equation (10) is in the interval (0,1). The gradient is large when the probabilities $Pr(c_1)$ and/or $Pr(e|c_1, \neg c_2)$ are/is small. However, since the parameter $Pr(e|c_1, \neg c_2)$ is a noisy-OR parameter, which are assumed to be large, we conclude that the propagation effects are large when the prior probability $Pr(c_1)$ is small, and in addition we have that the smaller the noisy-OR parameter $Pr(e|c_1, \neg c_2)$, the larger the propagation effects. The difference compared to the result derived in Section 4.1.1 is the absence of the prior probability $Pr(c_2)$ in the gradient. This means that the gradient of Equation (10) is large whenever the prior probability $Pr(c_1) \leq 0.25$, regardless of the probability of the noisy-OR parameter $Pr(e|c_1, \neg c_2)$.
Analogous observations hold for the function $Pr(e|c_1)(x)$ where $x = Pr(e|c_1, \neg c_2)$.

We now examine $Pr(e|c_2)$ as a function of the probability $Pr(e|c_1, \neg c_2)$, that is the noisy-OR parameter associated with cause C_1. The result is again a linear function in x.

Theorem 4.3. *Consider the causal mechanism in Figure 3 and assume it models a noisy-OR. Let* $x = Pr(e|c_1, \neg c_2)$ *be the noisy-OR parameter associated with cause* C_1*. Then the sensitivity function* $Pr(e|c_2)(x)$ *has the following form:*

$$Pr(e|c_2)(x) = xPr(c_1)\Big(1 - Pr(e|\neg c_1, c_2)\Big) + Pr(e|\neg c_1, c_2) \tag{11}$$

Proof:
We have:

$$Pr(e|c_2)(x) = Pr(e|c_1, c_2)Pr(c_1) + Pr(e|\neg c_1, c_2)Pr(\neg c_1)$$

Note that we substitute $Pr(e|c_1, c_2)$ with a different equation than Equation (9). Since $x = Pr(e|c_1, \neg c_2)$ we therefore obtain using the noisy-OR model described in Section 2.3:

$$\begin{aligned} Pr(e|c_1, c_2) &= 1 - \big((1 - Pr(e|\neg c_1, c_2))(1 - Pr(e|c_1, \neg c_2))\big) \\ &= 1 - \big((1 - x)(1 - Pr(e|c_1, \neg c_2))\big) \\ &= 1 - \big(1 - Pr(e|c_1, \neg c_2) - x + xPr(e|c_1, \neg c_2)\big) \\ &= Pr(e|c_1, \neg c_2) + x - xPr(e|c_1, \neg c_2) \\ &= x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2). \end{aligned} \tag{12}$$

We now have:

$$\begin{aligned} Pr(e|c_2)(x) &= \Big(x\big(1 - Pr(e|\neg c_1, c_2)\big) + Pr(e|\neg c_1, c_2)\Big)Pr(c_1) + Pr(e|\neg c_1, c_2)Pr(\neg c_1) \\ &= xPr(c_1) - xPr(e|\neg c_1, c_2)Pr(c_1) + Pr(e|\neg c_1, c_2)Pr(c_1) + Pr(e|\neg c_1, c_2)Pr(\neg c_1) \\ &= xPr(c_1)\Big(1 - Pr(e|\neg c_1, c_2)\Big) + Pr(e|\neg c_1, c_2). \quad \square \end{aligned}$$

Observation: The gradient of Equation (11) is in the interval $(0, 1)$. The gradient is large when the probability $Pr(c_1)$ is large and $Pr(e|\neg c_1, c_2)$ is small. The difference compared to Equation (10) is that now a large value for $Pr(c_1)$ results in a large gradient. This can be easily explained by the fact that $x = Pr(e|c_1, \neg c_2)$ which is the noisy-OR parameter associated with cause C_1. Since $Pr(c_1) \in (0, 1)$ and $Pr(e|c_1, \neg c_2)$ is a noisy-OR parameter, which by our assumption lie in the interval $[0.6, 1]$, we obtain that the gradient of Equation (10) lies in the interval $[0, 0.4)$ and thus we conclude that the propagation effects can become small or moderate at most.
Analogous observations hold for $Pr(e|c_1)$ as a function of the probability $Pr(e|\neg c_1, c_2)$.

Finally, we examine $Pr(e|\neg c_2)(x)$ for $x = Pr(e|c_1, \neg c_2)$. Note that if we choose the noisy-OR parameter associated with cause C_2, that is $x = Pr(e|\neg c_1, c_2)$, the sensitivity function $Pr(e|\neg c_2)(x)$ would be a constant function since $Pr(e|\neg c_2)(x) = Pr(e|c_1, \neg c_2)Pr(c_1) + Pr(e|\neg c_1, \neg c_2)Pr(c_1)$, and thus, is independent of the term $x = Pr(e|\neg c_1, c_2)$.

Theorem 4.4. *Consider the causal mechanism in Figure 3 and assume it models a noisy-OR. Let* $x = Pr(e|c_1, \neg c_2)$ *be the noisy-OR parameter associated with cause* C_1*. Then the sensitivity function* $Pr(e|\neg c_2)(x)$ *has the following form:*

$$Pr(e|\neg c_2)(x) = xPr(c_1) \tag{13}$$

Proof
We have:

$$\begin{aligned} Pr(e|\neg c_2)(x) &= xPr(c_1) + Pr(e|\neg c_1, \neg c_2)Pr(\neg c_1) \\ &= xPr(c_1) + 0 \\ &= xPr(c_1). \quad \square \end{aligned}$$

Observation: One can observe that only one of the noisy-OR parameters is present in Equation (13), whereby $x = Pr(e|c_1, \neg c_2)$. We note that the gradient of Equation (13) lies in the interval (0,1) and is equal to the prior probability $Pr(c_1)$, meaning that the propagation effects solely depend on the value of the prior probability $Pr(c_1)$.

Analogous observations to Theorem 4.4 hold for function $Pr(e|\neg c_1)(x)$ when $x = Pr(e|\neg c_1, c_2)$, and a constant function when $x = Pr(e|c_1, \neg c_2)$.

Thus far, we have considered the consequences of deviating noisy-OR probabilities upon causal propagation through the basic causal mechanism in Figure 3. It seems that large propagation effects on the probability of interest can be expected if the cause associated with the noisy-OR parameter under study has a large prior probability of being present and the other cause has a large prior probability of being absent, see Theorem 4.1. For clarification: if $x = Pr(e|\neg c_1, c_2)$, then large propagation effects occur when cause C_1 is likely to be absent and cause C_2 present. We have that the larger these prior probabilities, the larger the gradient of the sensitivity function under study, and thus the larger the propagation effects on the probability of interest. In addition, we found that the smaller the noisy-OR parameter involved that is not set to x, the larger the propagation effects. Furthermore, we observed that by actually establishing the presence or absence of cause C_i, the gradient no longer reveals a dependency of $Pr(c_i)$ or $Pr(\neg c_i)$, respectively. See Theorem 4.2, 4.3, and 4.4.

4.1.3 Propagation effects in the diagnostic direction

Till so far, we have investigated the effects of a deviating noisy-OR parameter on an output probability of interest upon causal propagation. Now we will examine the propagation effects in the diagnostic direction, that is, upon propagating evidence for the effect variable to an unobserved cause variable. We again consider the conditional probability tables for the three variables of the basic mechanism in Figure 3.

4.1.3.1 Sensitivity function $\mathbf{Pr(c_2|e)(x)}$ with $\mathbf{x = Pr(e|\neg c_1, c_2)}$

First we examine $Pr(c_2|e)$ as a function of the noisy-OR parameter $Pr(e|\neg c_1, c_2)$. The result is sensitivity function $Pr(c_2|e)(x)$ which is hyperbolic in the probability x.

Theorem 4.5. *Consider the causal mechanism in Figure 3 and assume it models a noisy-OR. Let $x = Pr(e|\neg c_1, c_2)$ be the noisy-OR parameter associated with cause C_2. Then the sensitivity function $Pr(c_2|e)(x)$ has the following form:*

$$Pr(c_2|e)(x) = \frac{x + \frac{\beta}{1-\beta}}{x + \frac{\beta}{Pr(c_2)(1-\beta)}} \tag{14}$$

where $\beta = Pr(e|c_1, \neg c_2)Pr(c_1)$.

Proof: We have for $x = Pr(e|\neg c_1, c_2)$:

$$\begin{aligned} Pr(c_2|e)(x) &= \frac{Pr(c_2, e)(x)}{Pr(e)(x)} \\ &= \frac{Pr(e, c_1, c_2)(x) + Pr(e, \neg c_1, c_2)(x)}{Pr(e)(x)} \\ &= \frac{Pr(e|c_1, c_2)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2)}{Pr(e)(x)} \end{aligned}$$

Where $Pr(e)(x) = Pr(c_2, e)(x) + Pr(\neg c_2, e)(x)$ and $Pr(e|c_1, c_2)$ is dependent of x.

We use that $Pr(e|c_1, c_2) = x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)$ (Eq. 9), and substitute:

$$\begin{aligned}
Pr(c_2|e)(x) &= \frac{\Big(x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)\Big)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2)}{Pr(e)(x)} \\
&= \frac{Pr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) + xPr(c_1)Pr(c_2) - xPr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2)}{Pr(c_2, e)(x) + Pr(\neg c_2, e)(x)} \\
&= \frac{Pr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) + xPr(c_1)Pr(c_2) - xPr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2)}{Pr(c_2, e)(x) + Pr(e|c_1, \neg c_2)Pr(c_1)Pr(\neg c_2) + Pr(e|\neg c_1, \neg c_2)Pr(\neg c_1)Pr(\neg c_2)} \\
&= \frac{Pr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) + xPr(c_1)Pr(c_2) - xPr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2)}{\big(x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)\big)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2) + Pr(e|c_1, \neg c_2)Pr(c_1)Pr(\neg c_2) + 0} \\
&= \frac{x\big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\big) + Pr(e|c_1, \neg c_2)Pr(c_1)}{x\big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\big) + \frac{Pr(e|c_1, \neg c_2)Pr(c_1)}{Pr(c_2)}} \cdot \frac{Pr(c_2)}{Pr(c_2)} \\
&= \frac{x + \frac{Pr(e|c_1, \neg c_2)Pr(c_1)}{1 - Pr(e|c_1, \neg c_2)Pr(c_1)}}{x + \frac{Pr(e|c_1, \neg c_2)Pr(c_1)}{Pr(c_2)\big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\big)}} \\
&= \frac{x + \frac{\beta}{1-\beta}}{x + \frac{\beta}{Pr(c_2)(1-\beta)}}
\end{aligned}$$

where $\beta = Pr(e|c_1, \neg c_2)Pr(c_1)$. $\square$

Observation: Since Equation (14) is a hyperbolic function, we use the properties of hyperbolic functions described in Section 2.5, and discover that the vertical asymptote of Equation (14) lies at $x = s = -\frac{\beta}{Pr(c_2)(1-\beta)}$. Because $\frac{\beta}{Pr(c_2)(1-\beta)} > 0$, the asymptote is located to the left of the unit window and the horizontal asymptote lies at $t = 1$. As a result, we find that Equation (14) is a fragment of a fourth-quadrant hyperbola branch.

From generic research of sensitivity functions from Bayesian networks, we find that the effect of deviations in the x-value on the output probability of interest mainly depends on the location of the vertex of the corresponding hyperbola branch [1]. Generally, we have that the closer the vertex of the fourth-quadrant hyperbola branch lies to the upper-left corner of the unit window, the larger the propagation effects. Equation (14) has its vertex at:

$$(s + \sqrt{|r|}, 1 - \sqrt{|r|}) = \Big(- \frac{\beta}{Pr(c_2)(1-\beta)} + \sqrt{\Big(\frac{\beta}{Pr(c_2)(1-\beta)} - \frac{\beta}{1-\beta}\Big)}, 1 - \sqrt{\Big(\frac{\beta}{Pr(c_2)(1-\beta)} - \frac{\beta}{1-\beta}\Big)}\Big)$$

The vertex is located within the unit window for some values of $\frac{\beta}{1-\beta}, \frac{\beta}{Pr(c_2)(1-\beta)}$ with $\frac{\beta}{Pr(c_2)(1-\beta)} < \sqrt{\frac{\beta}{Pr(c_2)(1-\beta)} - \frac{\beta}{1-\beta}} < 1$. To obtain $\frac{\beta}{Pr(c_2)(1-\beta)} < \sqrt{\frac{\beta}{Pr(c_2)(1-\beta)} - \frac{\beta}{1-\beta}}$ given that $\frac{\beta}{Pr(c_2)(1-\beta)} \geq \frac{\beta}{1-\beta}$, we discover that merely rather small values of $\frac{\beta}{1-\beta}$ produce a vertex with an x-coordinate in the unit range. In addition, the vertex only approaches the upper-left corner of the unit window, if in addition the difference $\frac{\beta}{Pr(c_2)(1-\beta)} - \frac{\beta}{1-\beta}$ is rather small. We observe that Equation (14) approaches 1 if the value $\frac{\beta}{Pr(c_2)(1-\beta)} - \frac{\beta}{1-\beta}$ is small. Hence, in order to acquire a vertex approaching the upper-left corner of the unit window, we find that $Pr(c_1)$ and $Pr(e|c_1, \neg c_2)$ need to be small and $Pr(c_2)$ large.

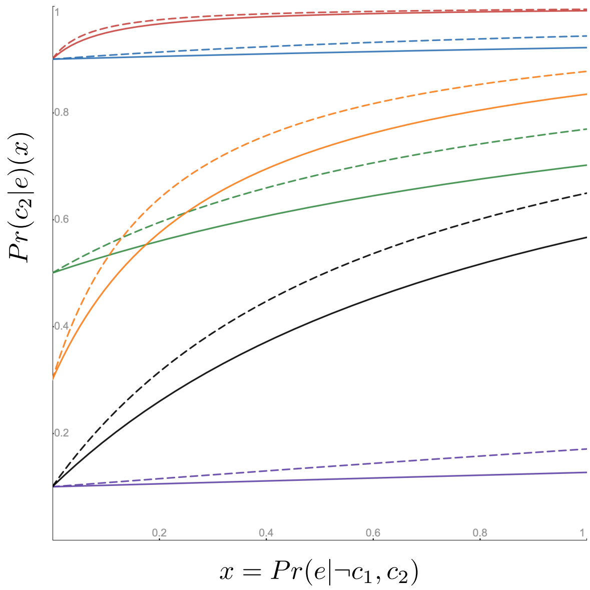

Figure 4: Several example sensitivity functions adhering to Theorem 4.5. (See Table 2 for parameter settings)

Parameter	Red	Red dashed	Green	Green dashed	Purple	Purple dashed	Orange	Orange dashed	Black	Black dashed	Blue	Blue dashed
$Pr(e\|c_1, \neg c_2)$	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6
$Pr(c_1)$	0.1	0.1	0.5	0.5	0.9	0.9	0.1	0.1	0.1	0.1	0.9	0.9
$Pr(c_2)$	0.9	0.9	0.5	0.5	0.1	0.1	0.3	0.3	0.1	0.1	0.9	0.9

Table 2: Parameter settings for sensitivity functions from Figure 4

To support the above mentioned findings and gain more insight into the effects we consider concrete parameter settings, see Figure 4. We observe, when focusing on the entire interval $x = Pr(e|\neg c_1, c_2) \in [0, 1]$, that when the prior probability $Pr(c_1)$ is small and the prior probability $Pr(c_2)$ is small/moderate, the largest propagation effects occur, see orange (dashed) and black (dashed). In addition, we observe a minor influence of the noisy-OR parameter $Pr(e|c_1, \neg c_2)$; we note that slightly larger propagation effects in the entire interval $[0, 1]$ occur when $Pr(e|c_1, \neg c_2)$ is "small", this effect is conveyed by the solid versus dashed function for each colour. Since by our assumption the noisy-OR parameter $Pr(e|c_1, \neg c_2)$ lies in the interval $[0.6, 1]$, we conclude that a smaller noisy-OR parameter $Pr(e|c_1, \neg c_2)$ provides *slightly* larger propagation effects in the entire interval $x = Pr(e|\neg c_1, c_2) \in [0, 1]$. Furthermore, we have that the vertical offset on the y-axis is greatly influenced by $Pr(c_2)$; see red (dashed), green (dashed) and purple (dashed). If the probability $Pr(c_2)$ approaches 1, and hence the value $\frac{\beta}{Pr(c_2)(1-\beta)} - \frac{\beta}{1-\beta}$ becomes very small, the x-coordinate of the vertex will indeed approach 0.

However, probabilities of noisy-OR parameters are assumed to be large [5], which indicates that we should especially focus on the propagation effects for $x = Pr(e|\neg c_1, c_2) \geq 0.6$. As one can observe in Figure 4, large propagation effects only occur, with a particular parameter settings, when $x = Pr(e|\neg c_1, c_2)$ is less than 0.4 (see for example orange and black function). These results show that the propagation effects, for $x = Pr(e|\neg c_1, c_2)$ larger than 0.6, are moderate or small. To gain better insight into Equation's (14) behaviour in the interval $x = Pr(e|\neg c_1, c_2) \geq 0.6$, we compute its first derivative. As we have mentioned, the sensitivity functions corresponding to Equation (14) are a fragment of a fourth-quadrant hyperbola branch, and as a consequence, we know that the first derivative $\frac{d}{dx} Pr(c_2|e)(x) > 0$ for all $x \in [0, 1]$.

Corollary 4.5.1. *The first derivative of the sensitivity function from Equation (14) is:*

$$\frac{d}{dx}Pr(c_2|e)(x) = \frac{Pr(e|c_1,\neg c_2)Pr(c_1)\big(1-Pr(c_2)\big)}{Pr(c_2)\big(1-Pr(e|c_1,\neg c_2)Pr(c_1)\big)\Big(x+\frac{Pr(e|c_1,\neg c_2)Pr(c_1)}{Pr(c_2)\big(1-Pr(e|c_1,\neg c_2)Pr(c_1)\big)}\Big)^2}$$

$$= \frac{\beta\big(1-Pr(c_2)\big)}{Pr(c_2)(1-\beta)\Big(x+\frac{\beta}{Pr(c_2)\big(1-\beta\big)}\Big)^2} \quad (15)$$

where $\beta = Pr(e|c_1,\neg c_2)Pr(c_1)$*.*

Now, for specific parameter settings for some of the functions demonstrated in Figure 4, namely the green (dashed), orange (dashed) and black (dashed) function, we plot the derivatives of Equation 15. See Figure 5.

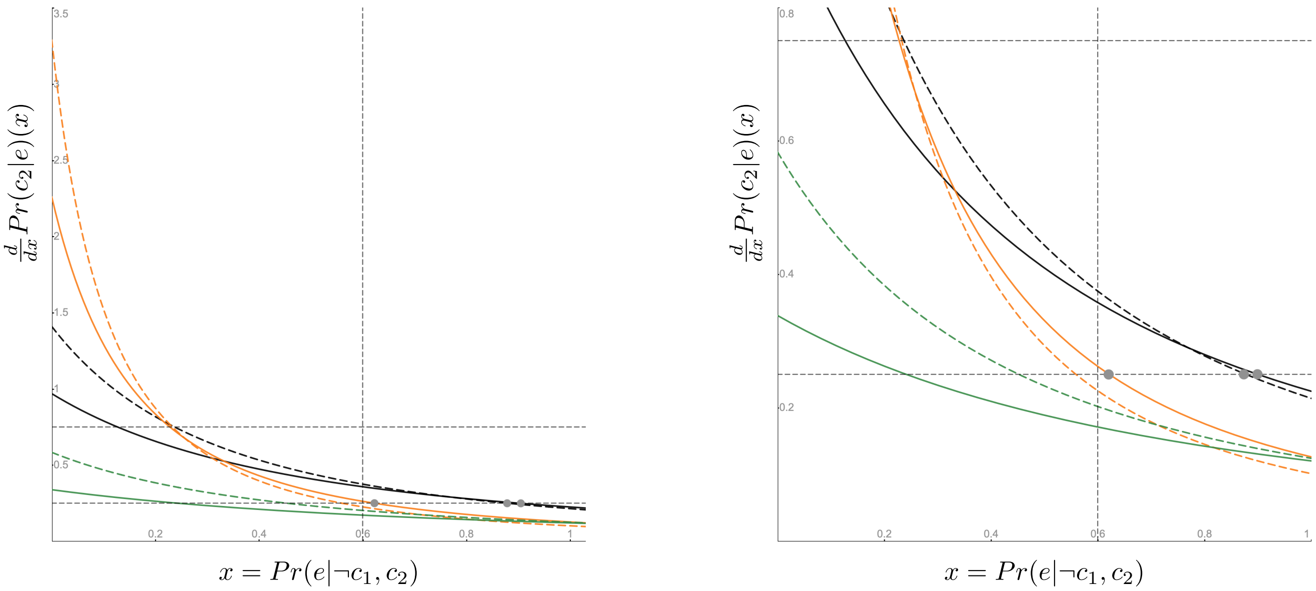

Figure 5: Several examples of Equation (15) restricted to the window $x \in [0,1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [0,3.5]$ (left) and the window $x \in [0,1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [0,0.8]$ (right). (See Table 2 for parameter settings)

In Figure 5, the horizontal lines at $y = 0.25$ and $y = 0.75$ indicate the boundaries between what we consider to be a small, moderate or large gradient and therefore whether the propagation effects are small, moderate, or large. The black dashed function has moderate propagation effects in the interval $x \in [0.6, 0.8776)$ and small propagation effects in the interval $x \in [0.8776, 1]$. The black function has moderate propagation effects in the interval $x \in [0.6, 0.8998)$ and small propagation effects in the interval $x \in [0.8998, 1]$. The orange function has moderate propagation effects in the interval $x \in [0.6, 0.6215)$ and small propagation effects in the interval $x \in [0.6215, 1]$. The orange dashed function has small propagation effects in the entire interval $x \in [0.6, 1]$.

We observe two remarkable differences when comparing the propagation effects in the interval $x \in [0.6, 1]$ to the entire interval $x \in [0, 1]$. First, we saw when focusing on the entire interval $x \in [0, 1]$ that the largest propagation effects occur when the prior probability $Pr(c_1)$ is small, $Pr(c_2)$ is small/moderate, and the noisy-OR parameter $Pr(e|c_1,\neg c_2)$ is small. However, with the help of Figure 5 and only focusing on the interval $x \in [0.6, 1]$, we don't observe that a smaller noisy-OR parameter $Pr(e|c_1,\neg c_2)$ leads to larger propagation effects. We even observe for some functions the opposite effect in the interval $x \in [0.6, 1]$, namely that a larger value for the noisy-OR parameter $Pr(e|c_1,\neg c_2)$ provides larger propagation effects. The second observation is that when only focusing on the interval $x \in [0.6, 1]$, we have that the smaller the prior probability $Pr(c_2)$, the larger the propagation effects.

With the help of WOLFRAM MATHEMATICA, we find a maximum of $\max \frac{d}{dx}Pr(c_2|e)(x) = 0.416666$ in the interval $x = Pr(e|\neg c_1, c_2) \in [0.6, 1]$ of Equation (15) with the following parameter setting (see Appendix A.1):

$Pr(e|c_1, \neg c_2) = 0.775347, Pr(c_1) = 3.28813 \cdot 10^{-7}$ and $Pr(c_2) = 4.24907 \cdot 10^{-7}$.

This maximum lies at $x = 0.6$.

Since the prior probabilities $Pr(c_1)$ and $Pr(c_2)$ are now extremely small, we put a lower bound on these priors, namely $Pr(c_1), Pr(c_2) > 0.01$. We find a maximum of $\max \frac{d}{dx} Pr(c_2|e)(x) = 0.412496$ in the interval $x = Pr(e|\neg c_1, c_2) \in [0.6, 1]$ of Equation (15) with the following parameter setting (see Appendix A.1):

$Pr(e|c_1, \neg c_2) = 0.600043, Pr(c_1) = 0.0100007$ and $Pr(c_2) = 0.0100003$.

We observe that this maximum value of the derivative is a little lower and the value of the noisy-OR parameter $Pr(e|c_1, \neg c_2)$ becomes 0.600043. This maximum again lies at $x = 0.6$.

Finally, we put a lower bound of $Pr(c_1), Pr(c_2) > 0.05$ on these priors. We now find a maximum of $\max \frac{d}{dx} Pr(c_2|e)(x) = 0.395741$ in the interval $x = Pr(e|\neg c_1, c_2) \in [0.6, 1]$ of Equation (15) with the following parameter setting (see Appendix A.1):

$Pr(e|c_1, \neg c_2) = 0.600012, Pr(c_1) = 0.050001$ and $Pr(c_2) = 0.0500003$.

We again observe that the maximum value of the derivative is lower and the value of the noisy-OR parameter $Pr(e|c_1, \neg c_2)$ becomes 0.600012. This maximum lies at $x = 0.6$.

We conclude that Equation (14) shows that the strongest effects on the output probability $Pr(c_2|e)$ in the interval $x = Pr(e|\neg c_1, c_2) \in [0.6, 1]$ can be expected, based on the following:

- The prior probabilities $Pr(c_1)$ and $Pr(c_2)$ are small; that is, causes C_1 and C_2 are likely to be absent. In addition, we observe that when we put a lower bound on the priors of C_1 and C_2, a smaller value of the noisy-OR parameter $Pr(e|c_1, \neg c_2)$ will provide larger propagation effects.

We found that when examining $Pr(c_2|e)$ as a function of the probability $Pr(e|\neg c_1, c_2)$, the propagation effects can become moderate at most

4.1.3.2 Sensitivity function $\mathbf{Pr(c_2|e)(x)}$ with $\mathbf{x = Pr(e|c_1, \neg c_2)}$

We now examine $Pr(c_2|e)$ as a function of the probability $Pr(e|c_1, \neg c_2)$, that is the noisy-OR parameter associated with cause C_1. The result is function $Pr(c_2|e)(x)$ which is hyperbolic in the probability x.

Theorem 4.6. *Consider the causal mechanism in Figure 3 and assume it models a noisy-OR. Let $x = Pr(e|c_1, \neg c_2)$ be the noisy-OR parameter associated with cause C_1. Then the sensitivity function $Pr(c_2|e)(x)$ has the following form:*

$$Pr(c_2|e)(x) = \frac{x + \gamma}{\beta x + \gamma} \tag{16}$$

where $\gamma = \frac{Pr(e|\neg c_1, c_2)}{Pr(c_1)\left(1 - Pr(e|\neg c_1, c_2)\right)}$ *and* $\beta = \frac{1 - Pr(c_2)Pr(e|\neg c_1, c_2)}{Pr(c_2)\left(1 - Pr(e|\neg c_1, c_2)\right)}$

Proof:
We have:

$$Pr(c_2|e)(x) = \frac{Pr(e|c_1, c_2)Pr(c_1)Pr(c_2) + Pr(e|\neg c_1, c_2)Pr(\neg c_1)Pr(c_2)}{Pr(e)(x)}$$

where $Pr(e|c_1, c_2)$ is dependent of x. $Pr(e)(x)$ is equal to:

$$\begin{aligned} Pr(e)(x) &= Pr(c_2, e)(x) + Pr(\neg c_2, e)(x) \\ &= Pr(e|c_1, c_2)Pr(c_1)Pr(c_2) + Pr(e|\neg c_1, c_2)Pr(\neg c_1)Pr(c_2) + xPr(c_1)Pr(\neg c_2) + 0 \end{aligned}$$

and using Eq. (9) we get:

$$= \Big(x\big(1 - Pr(e|\neg c_1, c_2)\big) + Pr(e|\neg c_1, c_2)\Big)Pr(c_1)Pr(c_2) + Pr(e|\neg c_1, c_2)Pr(\neg c_1)Pr(c_2) + xPr(c_1)Pr(\neg c_2)$$

Then, by dividing the denominator and numerator by $Pr(c_1)Pr(c_2)$ we get:

$$
\begin{aligned}
Pr(c_2|e)(x) &= \frac{x - xPr(e|\neg c_1, c_2) + Pr(e|\neg c_1, c_2) + Pr(e|\neg c_1, c_2)\frac{Pr(\neg c_1)}{Pr(c_1)}}{x - xPr(e|\neg c_1, c_2) + Pr(e|\neg c_1, c_2) + Pr(e|\neg c_1, c_2)\frac{Pr(\neg c_1)}{Pr(c_1)} + x\frac{Pr(\neg c_2)}{Pr(c_2)}} \\
&= \frac{x\Big(1 - Pr(e|\neg c_1, c_2)\Big) + Pr(e|\neg c_1, c_2) + Pr(e|\neg c_1, c_2)\frac{Pr(\neg c_1)}{Pr(c_1)}}{x\Big(1 - Pr(e|\neg c_1, c_2) + \frac{Pr(\neg c_2)}{Pr(c_2)}\Big) + Pr(e|\neg c_1, c_2) + Pr(e|\neg c_1, c_2)\frac{Pr(\neg c_1)}{Pr(c_1)}}
\end{aligned}
$$

divide both the denominator and numerator by $\big(1 - Pr(e|\neg c_1, c_2)\big)$:

$$
\begin{aligned}
&= \frac{x + \frac{Pr(e|\neg c_1,c_2)\left(1+\frac{Pr(\neg c_1)}{Pr(c_1)}\right)}{1-Pr(e|\neg c_1,c_2)}}{x + \frac{Pr(e|\neg c_1,c_2)\left(1+\frac{Pr(\neg c_1)}{Pr(c_1)}\right)}{1-Pr(e|\neg c_1,c_2)} + \frac{xPr(\neg c_2)}{Pr(c_2)\big(1-Pr(e|\neg c_1,c_2)\big)}} \\
&= \frac{x + \frac{Pr(e|\neg c_1,c_2)}{Pr(c_1)\big(1-Pr(e|\neg c_1,c_2)\big)}}{x + \frac{Pr(e|\neg c_1,c_2)}{Pr(c_1)\big(1-Pr(e|\neg c_1,c_2)\big)} + \frac{xPr(\neg c_2)}{Pr(c_2)\big(1-Pr(e|\neg c_1,c_2)\big)}} \\
&= \frac{x+\gamma}{x + \gamma + \frac{xPr(\neg c_2)}{Pr(c_2)\big(1-Pr(e|\neg c_1,c_2)\big)}} \\
&= \frac{x+\gamma}{\beta x + \gamma}
\end{aligned}
$$

where $\gamma = \frac{Pr(e|\neg c_1,c_2)}{Pr(c_1)\big(1-Pr(e|\neg c_1,c_2)\big)}$ and
$\beta = \frac{Pr(\neg c_2)}{Pr(c_2)\big(1-Pr(e|\neg c_1,c_2)\big)} + 1 = \frac{Pr(c_2)+Pr(\neg c_2)-Pr(c_2)Pr(e|\neg c_1,c_2)}{Pr(c_2)\big(1-Pr(e|\neg c_1,c_2)\big)} = \frac{1-Pr(c_2)Pr(e|\neg c_1,c_2)}{Pr(c_2)\big(1-Pr(e|\neg c_1,c_2)\big)}$. □

Observation: We find that Equation (16) is a hyperbolic function, and therefore we use the properties of hyperbolic functions described in Section 2.5. Note that Equation (16) is not defined for $Pr(e|\neg c_1, c_2) = 1$, since then the denominator equals 0. The vertical asymptote of Equation (16) lies at $x = s = -\frac{\gamma}{\beta}$. Because $\frac{\gamma}{\beta} > 0$, the vertical asymptote is located to the left of the unit window and the horizontal asymptote lies at $t = \frac{1}{\beta}$, where $0 < \frac{1}{\beta} < 1$. As a result, we find that Equation (16) is a fragment of a first-quadrant hyperbola branch. We now find that the closer the vertex of the first-quadrant hyperbola branch lies to the point $(0, \frac{1}{\beta})$, the larger the propagation effects. Equation (16) has its vertex at:

$$
(s + \sqrt{|r|}, t + \sqrt{|r|}) = \left(-\frac{\gamma}{\beta} + \sqrt{\left|\frac{\gamma(\beta-1)}{\beta^2}\right|}, \frac{1}{\beta} + \sqrt{\left|\frac{\gamma(\beta-1)}{\beta^2}\right|} \right)
$$

We have that the vertex is located within the unit window for values of β, γ with $\frac{\gamma}{\beta} < \sqrt{\left|\frac{\gamma(\beta-1)}{\beta^2}\right|} < 1$. To have a vertex closely located to the point $(0, \frac{1}{\beta})$ we find that $\frac{\gamma}{\beta}$ and $\sqrt{\left|\frac{\gamma(\beta-1)}{\beta^2}\right|}$ should approach 0. Thus, for $\frac{\gamma}{\beta}$ to approach 0, we need γ to be small and β large. Consequently, large propagation effects occur when the vertex of the sensitivity function under study is closely located to the point $(0, 0)$. For $\gamma \to 0$, $Pr(c_1)$ has to be large and $Pr(e|\neg c_1, c_2)$ small. On the other hand, for β to become large, we need that $Pr(c_2)$ is small. The same holds for the term $\sqrt{\left|\frac{\gamma(\beta-1)}{\beta^2}\right|}$. For $\sqrt{\left|\frac{\gamma(\beta-1)}{\beta^2}\right|}$ to approach 0, we need again that γ should be small and β large.

To support the above mentioned conditions and gain more insight in Equation's (16) behaviour, we look at concrete parameter settings of several hyperbola branches restricted to the unit window. See Figure 6. We observe, when focusing on the entire interval $x = Pr(e|c_1, \neg c_2) \in [0, 1]$, that Equation (16) shows that the strongest effect on the output probability $Pr(c_2|e)$ in the entire interval $x \in [0, 1]$ can be expected based on the following:

- $Pr(c_1)$ is large and $Pr(c_2)$ is small, that is, cause C_1 is likely to be present and cause C_2 absent;
- The smaller the noisy-OR parameter $Pr(e|\neg c_1, c_2)$ the larger the propagation effects on the output probability. However, this influence is relatively small.

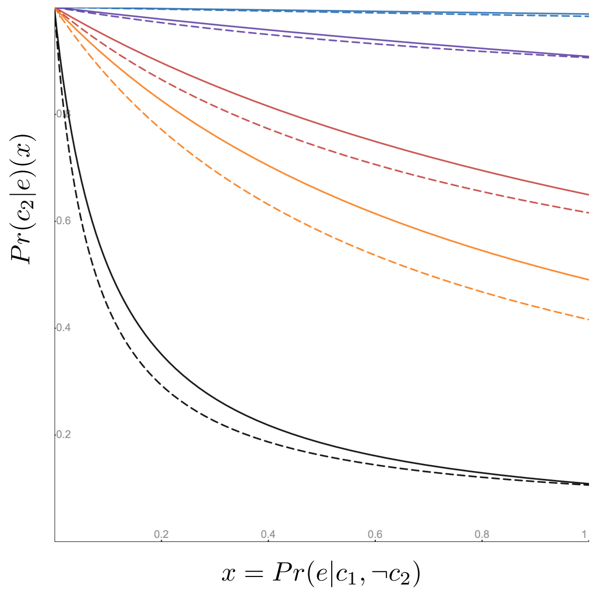

Figure 6: Several example sensitivity functions adhering to Theorem 4.6. (See Table 3 for parameter settings)

Parameter	Blue	Blue dashed	Red	Red dashed	Black	Black dashed	Orange	Orange dashed	Purple	Purple dashed
$Pr(e\|\neg c_1, c_2)$	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6
$Pr(c_1)$	0.1	0.1	0.5	0.5	0.9	0.9	0.1	0.1	0.9	0.9
$Pr(c_2)$	0.9	0.9	0.5	0.5	0.1	0.1	0.1	0.1	0.9	0.9

Table 3: Parameter settings for sensitivity functions from Figure 6

We will now focus on probabilities for $x = Pr(e|c_1, \neg c_2) \geq 0.6$. As we observe in Figure 6, large propagation effects only occur, with a particular parameter settings, when x is less than 0.5. These results show that the propagation effects, for $x = Pr(e|c_1, \neg c_2)$ larger than 0.6, are moderate or small. To gain better insight into Equation's (16) behaviour in the interval $x = Pr(e|c_1, \neg c_2) \geq 0.6$, we compute its first derivative. The sensitivity functions corresponding to Equation (16) are a fragment of a first-quadrant hyperbola branch, and as a consequence, we know that the first derivative $\frac{d}{dx}Pr(c_2|e)(x) < 0$ for all $x \in [0, 1]$.

Corollary 4.6.1. *The first derivative of the sensitivity function from Equation (16) is:*

$$\frac{d}{dx}Pr(c_2|e)(x) = \frac{\frac{Pr(e|\neg c_1, c_2)}{Pr(c_1)(1-Pr(e|\neg c_1,c_2))}\left(1 - \frac{1-Pr(c_2)Pr(e|\neg c_1,c_2)}{Pr(c_2)(1-Pr(e|\neg c_1,c_2))}\right)}{\left(\frac{1-Pr(c_2)Pr(e|\neg c_1,c_2)}{Pr(c_2)(1-Pr(e|\neg c_1,c_2))} \cdot x + \frac{Pr(e|\neg c_1,c_2)}{Pr(c_1)(1-Pr(e|\neg c_1,c_2))}\right)^2}$$

$$= \frac{\gamma(1-\beta)}{(\beta x + \gamma)^2} \tag{17}$$

where $\gamma = \frac{Pr(e|\neg c_1,c_2)}{Pr(c_1)\left(1-Pr(e|\neg c_1,c_2)\right)}$ *and* $\beta = \frac{1-Pr(c_2)Pr(e|\neg c_1,c_2)}{Pr(c_2)\left(1-Pr(e|\neg c_1,c_2)\right)}$.

Now, for specific parameter settings for some of the functions demonstrated in Figure 6, namely the black (dashed), orange (dashed) and red (dashed) function, we plot the derivatives of Equation 17. See Figure 7.

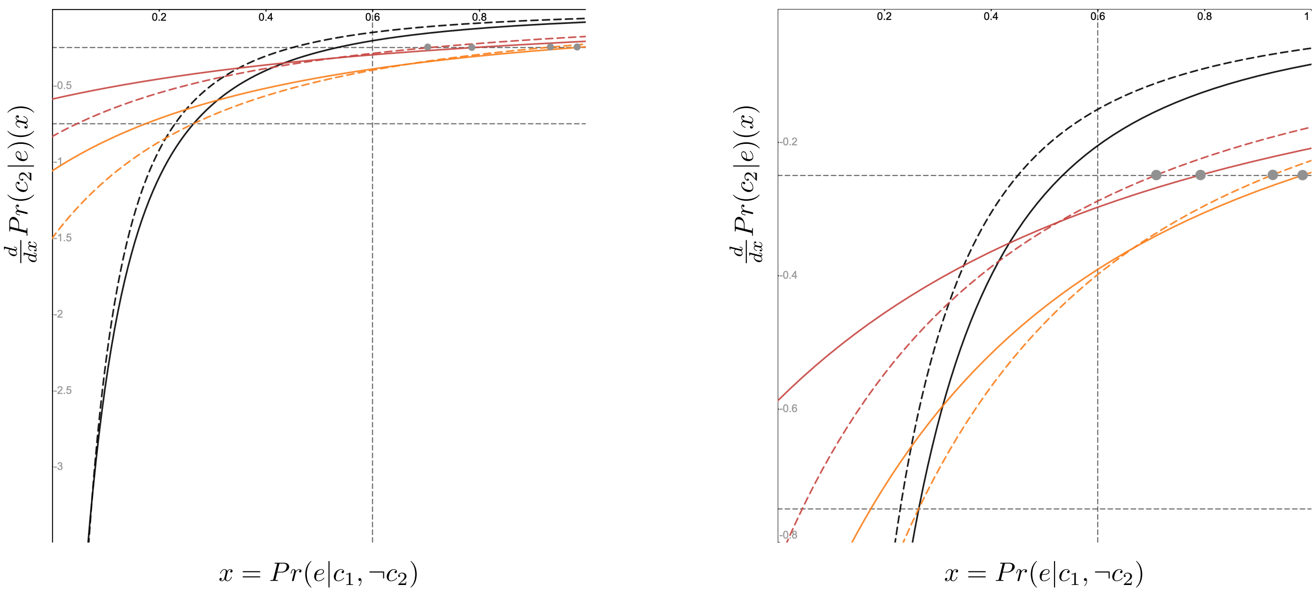

Figure 7: Several examples of Equation (17) restricted to the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [-3.5, 0]$ (left) and the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [-0.8, 0]$ (right). (See Table 3 for parameter settings)

In Figure 7, the horizontal lines at $y = -0.25$ and $y = -0.75$ indicate the boundaries between what we consider to be a small, moderate or large gradient. The black (dashed) function has small propagation effects in the entire interval $x \in [0.6, 1]$. The red function has moderate propagation effects in the interval $x \in [0.6, 0.7893)$ and small propagation effects in the interval $x \in [0.7893, 1]$. The red dashed function has moderate propagation effects in the interval $x \in [0.6, 0.7078)$ and small propagation effects in the interval $x \in [0.7078, 1]$. The orange function has moderate propagation effects in the interval $x \in [0.6, 0.9828)$ and small propagation effects in the interval $x \in [0.9828, 1]$. Finally, the orange dashed function has moderate propagation effects in the interval $x \in [0.6, 0.9252)$ and small propagation effects in the interval $x \in [0.9252, 1]$.

We now compare the propagation effects in the interval $x \in [0.6, 1]$ to the entire interval $x \in [0, 1]$. We saw that for the entire interval $x \in [0, 1]$, the largest propagation effects occur when the prior probability $Pr(c_1)$ is large and $Pr(c_2)$ is small, and the noisy-OR parameter $Pr(e|\neg c_1, c_2)$ is small. However, with the help of Figure 7, and only focusing on the interval $x \in [0.6, 1]$, we don't observe that a smaller noisy-OR parameter $Pr(e|c_1, \neg c_2)$ necessarily leads to larger propagation effects. We even see for some functions the opposite effect in the interval $x \in [0.6, 1]$, namely that a larger value for the noisy-OR parameter $Pr(e|c_1, \neg c_2)$ provides larger propagation effects. We also observe that the strongest propagation effects in the interval $x \in [0.6, 1]$ occur when both $Pr(c_1)$ and $Pr(c_2)$ are small, see orange (dashed) function.

With the help of WOLFRAM MATHEMATICA, we find an absolute maximum of $\max |\frac{d}{dx}Pr(c_2|e)(x)| = |0.416666|$ in the interval $x = Pr(e|c_1, \neg c_2) \in [0.6, 1]$ of Equation (17) with several parameter settings (see Appendix A.2). For example, one of the parameter settings is:

$$Pr(e|\neg c_1, c_2) = 0.9999, Pr(c_1) = 0.385073 \text{ and } Pr(c_2) = 0.187681.$$

This maximum lies at $x = 0.6$.

We conclude that Equation (16) shows that the strongest effects on the output probability $Pr(c_2|e)$ in the interval $x = Pr(e|c_1, \neg c_2) \in [0.6, 1]$ can be expected, based on the following:

- The prior probability $Pr(c_2)$ is small and $Pr(c_1)$ is small/moderate. In addition we observe that the largest propagation effects happen when the noisy-OR parameter $Pr(e|\neg c_1, c_2)$ is large.

We find that the absolute maximum value of the corresponding derivatives of Equation (16) in the interval $[0.6, 1]$

is equivalent to the maximum value obtained for the corresponding derivatives of Equation (14) in Section 4.1.3.1. Altogether, we again find when studying $Pr(c_2|e)(x)$ for $x = Pr(e|c_1, \neg c_2) \in [0.6, 1]$ that the propagation effects can become again moderate at most.

4.1.3.3 Sensitivity function $\mathbf{Pr(c_2|\neg e)(x)}$ with $\mathbf{x = Pr(e|\neg c_1, c_2)}$

We now examine $Pr(c_2|\neg e)$ as a function of the probability $Pr(e|\neg c_1, c_2)$, that is the noisy-OR parameter for cause C_2. The result is sensitivity function $Pr(c_2|\neg e)(x)$ which is hyperbolic in the probability x.

Theorem 4.7. *Consider the causal mechanism in Figure 3 and assume it models a noisy-OR. Let $x = Pr(e|\neg c_1, c_2)$ be the noisy-OR parameter associated with cause C_2. Then the sensitivity function $Pr(c_2|\neg e)(x)$ has the following form:*

$$Pr(c_2|\neg e)(x) = \frac{x-1}{x-\frac{\alpha}{\gamma}} \tag{18}$$

where $\gamma = 1 - Pr(e|c_1, \neg c_2)Pr(c_1)$ and $\alpha = 1 + Pr(c_1)\Big(\frac{Pr(\neg c_2)-Pr(e|c_1,\neg c_2)}{Pr(c_2)}\Big)$.

Proof:
First, we note the following:

- $x = Pr(e|\neg c_1, c_2)$ and $Pr(\neg e|c_1, c_2) = 1 - \Big(x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)\Big)$ since $Pr(\neg e|c_1, c_2) = 1 - Pr(e|c_1, c_2)$;
- $Pr(\neg e|\neg c_1, c_2) = 1 - Pr(e|\neg c_1, c_2) = 1 - x$;
- $Pr(\neg e|c_1, \neg c_2) = 1 - Pr(e|c_1, \neg c_2)$.

We have:

$$\begin{aligned}
Pr(c_2|\neg e)(x) &= \frac{Pr(\neg e|c_1, c_2)Pr(c_1)Pr(c_2) + Pr(\neg e|\neg c_1, c_2)Pr(\neg c_1)Pr(c_2)}{Pr(\neg e)(x)} \\
&= \frac{\Bigg(1 + x\big(Pr(e|c_1, \neg c_2) - 1\big) - Pr(e|c_1, \neg c_2)\Bigg)Pr(c_1)Pr(c_2) + (1-x)Pr(\neg c_1)Pr(c_2)}{Pr(\neg e)(x)}
\end{aligned}$$

and $Pr(\neg e)(x)$ is equal to:

$$\begin{aligned}
Pr(\neg e)(x) &= Pr(c_2, \neg e)(x) + Pr(\neg c_2, \neg e)(x) \\
&= Pr(\neg e|c_1, c_2)Pr(c_1)Pr(c_2) + Pr(\neg e|\neg c_1, c_2)Pr(\neg c_1)Pr(c_2) + Pr(\neg e|c_1, \neg c_2)Pr(c_1)Pr(\neg c_2) \\
&= \Bigg(1 + x\big(Pr(e|c_1, \neg c_2) - 1\big) - Pr(e|c_1, \neg c_2)\Bigg)Pr(c_1)Pr(c_2) + (1-x)Pr(\neg c_1)Pr(c_2) \\
&+ \big(1 - Pr(e|c_1, \neg c_2)\big)Pr(c_1)Pr(\neg c_2) \\
&= Pr(c_1)Pr(c_2) + xPr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) - xPr(c_1)Pr(c_2) - Pr(e|c_1, \neg c_2)Pr(c_1)Pr(c_2) + Pr(\neg c_1)Pr(c_2) \\
&- xPr(\neg c_1)Pr(c_2) + Pr(c_1)Pr(\neg c_2) - Pr(e|c_1, \neg c_2)Pr(c_1)Pr(\neg c_2)
\end{aligned}$$

Note that the numerator of the sensitivity function $Pr(c_2|\neg e)(x)$ equals the denominator minus the last 2 terms, that is $Pr(c_1)Pr(\neg c_2) - Pr(e|c_1,\neg c_2)Pr(c_1)Pr(\neg c_2)$. We have:

$$Pr(c_2|\neg e)(x) = \frac{xPr(e|c_1,\neg c_2)Pr(c_1)Pr(c_2) - xPr(c_2) + Pr(c_2) - Pr(e|c_1,\neg c_2)Pr(c_1)Pr(c_2)}{xPr(e|c_1,\neg c_2)Pr(c_1)Pr(c_2) - xPr(c_2) + Pr(c_2) + Pr(c_1)Pr(\neg c_2) - Pr(e|c_1,\neg c_2)Pr(c_1)}$$

(divide both the numerator and denominator by $Pr(c_2)$)

$$= \frac{xPr(e|c_1,\neg c_2)Pr(c_1) - x + 1 - Pr(e|c_1,\neg c_2)Pr(c_1)}{xPr(e|c_1,\neg c_2)Pr(c_1) - x + 1 + Pr(c_1)\frac{Pr(\neg c_2)}{Pr(c_2)} - \frac{Pr(e|c_1,\neg c_2)Pr(c_1)}{Pr(c_2)}}$$

$$= \frac{x\Big(Pr(e|c_1,\neg c_2)Pr(c_1) - 1\Big) + 1 - Pr(e|c_1,\neg c_2)Pr(c_1)}{x\Big(Pr(e|c_1,\neg c_2)Pr(c_1) - 1\Big) + 1 + Pr(c_1)\Big(\frac{Pr(\neg c_2) - Pr(e|c_1,\neg c_2)}{Pr(c_2)}\Big)}$$

$$= \frac{x + \frac{1 - Pr(e|c_1,\neg c_2)Pr(c_1)}{Pr(e|c_1,\neg c_2)Pr(c_1) - 1}}{x + \frac{1 + Pr(c_1)\Big(\frac{Pr(\neg c_2) - Pr(e|c_1,\neg c_2)}{Pr(c_2)}\Big)}{Pr(e|c_1,\neg c_2)Pr(c_1) - 1}}$$

$$= \frac{x + \frac{\gamma}{-\gamma}}{x + \frac{1 + Pr(c_1)\Big(\frac{Pr(\neg c_2) - Pr(e|c_1,\neg c_2)}{Pr(c_2)}\Big)}{-\gamma}}$$

$$= \frac{x - 1}{x + \frac{1 + Pr(c_1)\Big(\frac{Pr(\neg c_2) - Pr(e|c_1,\neg c_2)}{Pr(c_2)}\Big)}{-\gamma}}$$

$$= \frac{x - 1}{x + \frac{\alpha}{-\gamma}} = \frac{x - 1}{x - \frac{\alpha}{\gamma}}$$

where $\gamma = 1 - Pr(e|c_1,\neg c_2)Pr(c_1)$ and $\alpha = 1 + Pr(c_1)\Big(\frac{Pr(\neg c_2) - Pr(e|c_1,\neg c_2)}{Pr(c_2)}\Big)$. □

Observation: Equation (18) is hyperbolic in the probability $x = Pr(e|\neg c_1, c_2)$. Building on the properties of hyperbolic functions described in Section 2.5, we find that the vertical asymptote equals $s = \frac{\alpha}{\gamma}$ where $\alpha, \gamma > 0$. To show that $\frac{\alpha}{\gamma} \geq 1$, we proof that $\alpha \geq \gamma$:

$1 + Pr(c_1)\Big(\frac{Pr(\neg c_2) - Pr(e|c_1,\neg c_2)}{Pr(c_2)}\Big) \geq 1 - Pr(e|c_1,\neg c_2)Pr(c_1)$, subtract 1 from both sides & divide by $Pr(c_1)$, we get:
$\frac{1 - Pr(c_2) - Pr(e|c_1,\neg c_2)}{Pr(c_2)} \geq -Pr(e|c_1,\neg c_2)$, multiply both sides by $Pr(c_2)$ and we find:
$1 - Pr(c_2) - Pr(e|c_1,\neg c_2) \geq -Pr(c_2)Pr(e|c_1,\neg c_2)$

Since $Pr(c_2)$ and $Pr(e|c_1,\neg c_2)$ are probabilities and by assumption $Pr(c_2) \in (0,1)$, we indeed find that $\alpha \geq \gamma$. Since the vertical asymptote *has to* lie outside the unit window, we also have to add the restriction $Pr(e|c_1,\neg c_2) \neq 1$.

Because of above, we find that the vertical asymptote is located to the right of the unit window. Since the horizontal asymptote t of Equation (18) equals $t = 1$, we conclude that Equation (18) is a fragment of a third-quadrant hyperbola branch. We derive that the closer the vertex of the third-quadrant hyperbola branch lies to the point $(1,1)$, the larger the propagation effects of Equation (18). We find that Equation (18) has its vertex at:

$$(s - \sqrt{|r|}, 1 - \sqrt{|r|}) = \Big(\frac{\alpha}{\gamma} - \sqrt{|\frac{\alpha}{\gamma} - 1|}, 1 - \sqrt{|\frac{\alpha}{\gamma} - 1|}\Big)$$

Since $\frac{\alpha}{\gamma} > \sqrt{|\frac{\alpha}{\gamma} - 1|}$ for $\frac{\alpha}{\gamma} > 1$, we have that $\frac{\alpha}{\gamma}$ should approach 1 in order to have a vertex closely located to the point $(1,1)$. As a consequence, $Pr(c_2)$ should have a large value, and $Pr(c_1)$ a small value. The influence of $Pr(e|\neg c_1, c_2)$ is relatively small.

In order to confirm the above observations and gain more insight into the propagation effects, we consider concrete parameter settings for the parameters $Pr(e|c_1, \neg c_2), Pr(c_1)$ and $Pr(c_2)$. We make some plots of several hyperbola branches restricted to the unit window. See Figure 8.

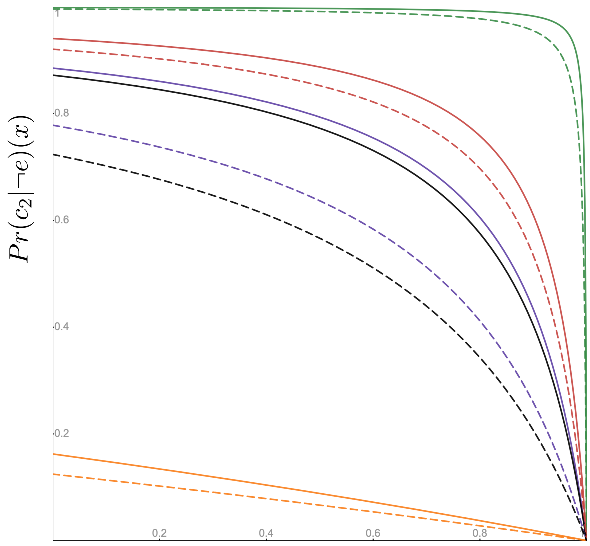

Figure 8: Several example sensitivity functions adhering to Theorem 4.7. (See Table 4 for parameter settings)

Parameter	Green	Green dashed	Purple	Purple dashed	Orange	Orange dashed	Red	Red dashed	Black	Black dashed
$Pr(e\|c_1, \neg c_2)$	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6
$Pr(c_1)$	0.1	0.1	0.5	0.5	0.9	0.9	0.9	0.9	0.1	0.1
$Pr(c_2)$	0.9	0.9	0.5	0.5	0.1	0.1	0.9	0.9	0.1	0.1

Table 4: Parameter settings for sensitivity functions from Figure 8

With help of Figure 8 we conclude that Equation (18) demonstrates that the strongest effects on the output probability $Pr(c_2|\neg e)$ in the entire interval $x = Pr(e|\neg c_1, c_2) \in [0, 1]$ can be expected based on the following:

- $Pr(c_1)$ is small and $Pr(c_2)$ large, that is, cause C_1 is likely to be absent and C_2 present;
- The larger the noisy-OR parameter $Pr(e|c_1, \neg c_2)$, the larger the propagation effects on the output probability. However, this influence is relatively small.

Since we focus on probabilities for $x = Pr(e|\neg c_1, c_2) \geq 0.6$, we observe that the propagation effects are not only large for the above mentioned parameter settings, but also when for example $Pr(c_1) = Pr(c_2) = 0.5$. We observe that for many parameter settings small deviations in the noisy-OR parameter $x = Pr(e|\neg c_1, c_2)$ for $x \in [0.6, 1]$ will have a large effect on the output probability $Pr(c_2|\neg e)$.

To gain better insight into Equation's (18) behaviour we compute its first derivative. The sensitivity functions corresponding to Equation (18) are a fragment of a third-quadrant hyperbola branch, and as a consequence, we know that $\frac{d}{dx} Pr(c_2|\neg e)(x) < 0$ for all $x = Pr(e|\neg c_1, c_2) \in [0, 1]$.

Corollary 4.7.1. *The first derivative of the sensitivity function from Equation (18) is:*

$$
\begin{aligned}
\frac{d}{dx}Pr(c_2|\neg e)(x) &= \frac{1 - \frac{1+Pr(c_1)\frac{Pr(\neg c_2)-Pr(e|c_1,\neg c_2)}{Pr(c_2)}}{1-Pr(e|c_1,\neg c_2)Pr(c_1)}}{\left(x + \frac{1+Pr(c_1)\frac{Pr(\neg c_2)-Pr(e|c_1,\neg c_2)}{Pr(c_2)}}{Pr(e|c_1,\neg c_2)Pr(c_1)-1}\right)^2} \\
&= \frac{1-\frac{\alpha}{\gamma}}{\left(x-\frac{\alpha}{\gamma}\right)^2}
\end{aligned}
\tag{19}
$$

where $\gamma = 1 - Pr(e|c_1,\neg c_2)Pr(c_1)$ *and* $\alpha = 1 + Pr(c_1)\left(\frac{Pr(\neg c_2)-Pr(e|c_1,\neg c_2)}{Pr(c_2)}\right)$.

We plot the derivatives of the specific parameter settings shown in Table 4 of Equation (19). See Figure 9.

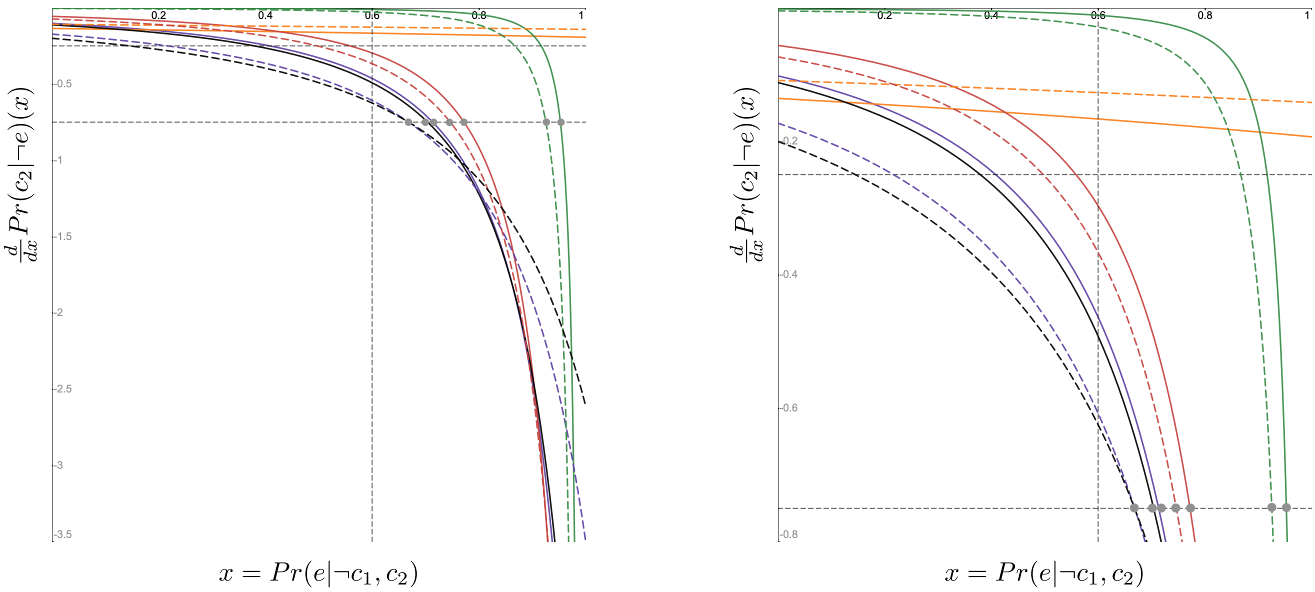

Figure 9: Several examples of Equation (19) restricted to the window $x \in [0,1]$ and $\frac{d}{dx}Pr(c_2|\neg e)(x) \in [-3.5, 0]$ (left) and the window $x \in [0,1]$ and $\frac{d}{dx}Pr(c_2|\neg e)(x) \in [-0.8, 0]$ (right). (See Table 4 for parameter settings)

In Figure 9, it is demonstrated that for the orange (dashed) function the first derivative $\frac{d}{dx}Pr(c_2|\neg e)(x) \in [-0.1, -0.25]$ for $x \in [0,1]$. The purple (dashed), black (dashed), red (dashed) and green (dashed) function all obtain values $\frac{d}{dx}Pr(c_2|\neg e)(x) > |0.75|$, in the interval where $x > 0.6$. We conclude that the propagation effects of $Pr(c_2|\neg e)(x)$ where $x = Pr(e|\neg c_1, c_2)$ in the interval $x \in [0.6, 1]$ can become large for a wide range of parameter settings.

When we investigate the possible maximum value of max $\frac{d}{dx}Pr(c_2|\neg e)(x)$ in the interval $x = Pr(e|\neg c_1, c_2) \in [0.6, 1]$, we find that the derivative of $\frac{d}{dx}Pr(c_2|\neg e)(x)$ can go to infinity for some parameter settings. Some functions of Equation (18) approach the vertical asymptote, and thus $\frac{d}{dx}Pr(c_2|\neg e)(x)$ can go to infinity. The propagation effects can become extremely large.

The findings demonstrated in Section 4.1.3.1 are in line with the results discovered in Section 4.1.3.2. Namely, when examining $Pr(c_2|e)$ as a function of the probability $Pr(e|c_i, \neg c_j)$ where $i, j = 1, 2$, we found that the propagation effects in the diagnostic direction due to inaccurate estimates of a noisy-OR parameter can become moderate at most when keeping in mind the underlying properties of the noisy-OR model. We discovered that the derivative can have a maximum value of $|0.416666|$. On the other hand, the results in Section 4.1.3.3 show that the propagation effects in the diagnostic direction due to inaccurate estimates of a noisy-OR parameter can become large for a wide range of parameter settings when examining $Pr(c_2|\neg e)$ as a function of the probability $Pr(e|\neg c_1, c_2)$. We conclude that the propagation effects in the diagnostic direction are highly dependent on which output probability we look at and which noisy-OR parameter we vary.

4.2 Propagation effects due to noisy-OR parameter changes: dependent causes

So far, we have studied the effects of a deviating noisy-OR parameter upon propagation through the basic causal mechanism exhibited in Figure 3. In this section, we consider a causal mechanism involving a direct connection, and therefore, a possibly dependency between their pair of cause variables, meaning there is no a priory independence of the two cause variables. See Figure 10. An extra arc $C_1 \rightarrow C_2$ is added and we make the following assumptions:

- The prior probability distribution over C_1 is non-degenerate;
- The conditional distribution over C_2 given C_1 is non-degenerate;
- $Pr(e|\neg c_1, \neg c_2) = 0$;
- Because noisy-OR parameters are assumed to be large [5], we mainly focus on $Pr(e|c_1, \neg c_2), Pr(e|\neg c_1, c_2) \in [0.6, 1]$ in our research. We specifically use this constraint when evaluating the propagation effects.

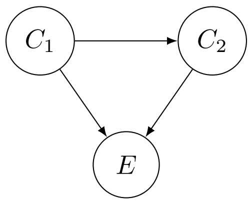

Figure 10: A causal mechanism with effect variable E and possibly dependent cause variables.

Furthermore, we will carry on with the gradation of the gradient described in Section 4.1.1 of a sensitivity function under study. We consider the gradient ∇ to be small when $|\nabla| \leq 0.25$, moderate when $|\nabla| \in (0.25, 0.75)$, and large when $|\nabla| \geq 0.75$.

4.2.1 Propagation effects in the causal direction

We start by investigating the possible propagation effects on the probability $Pr(e)$ due to changes in a noisy-OR parameter.

Theorem 4.8. *Consider the causal mechanism in Figure 10 and assume it models a noisy-OR. Let $x = Pr(e|\neg c_1, c_2)$ be the noisy-OR parameter associated with cause C_2. Then the sensitivity function $Pr(e)(x)$ has the following form:*

$$Pr(e)(x) = x\Big(Pr(c_2) - Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1)\Big) + Pr(e|c_1, \neg c_2)Pr(c_1) \quad (20)$$

Proof:

We have $Pr(e)(x) =$

$= Pr(e|c_1, c_2)Pr(c_2|c_1)Pr(c_1) + xPr(c_2|\neg c_1)Pr(\neg c_1) + Pr(e|c_1, \neg c_2)Pr(\neg c_2|c_1)Pr(c_1) + Pr(e|\neg c_1, \neg c_2)Pr(\neg c_2|\neg c_1)Pr(\neg c_1)$

(where $Pr(e|c_1, c_2) = \Big(x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)\Big)$ by Eq. 9,

and the last term $Pr(e|\neg c_1, \neg c_2)Pr(\neg c_2|\neg c_1)Pr(\neg c_1) = 0$ since $Pr(e|\neg c_1, \neg c_2) = 0$)

$= \Big(x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)\Big)Pr(c_2|c_1)Pr(c_1) + xPr(c_2|\neg c_1)Pr(\neg c_1) + Pr(e|c_1, \neg c_2)Pr(\neg c_2|c_1)Pr(c_1) + 0$

$= Pr(c_2|c_1)Pr(c_1)Pr(e|c_1, \neg c_2) + xPr(c_2|c_1)Pr(c_1) - xPr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1)$

$+ xPr(c_2|\neg c_1)Pr(\neg c_1) + Pr(e|c_1, \neg c_2)Pr(\neg c_2|c_1)Pr(c_1)$

$= x\Big(Pr(c_2|c_1)Pr(c_1)\Big(1 - Pr(e|c_1, \neg c_2)\Big) + Pr(c_2|\neg c_1)Pr(\neg c_1)\Big) + Pr(e|c_1, \neg c_2)Pr(c_1)\Big(Pr(c_2|c_1) + Pr(\neg c_2|c_1)\Big)$

we note that $Pr(c_2|c_1)Pr(c_1) + Pr(c_2|\neg c_1)Pr(\neg c_1) = Pr(c_2)$ and $Pr(c_2|c_1) + Pr(\neg c_2|c_1) = 1$, and therefore we obtain:

$= x\Big(Pr(c_2) - Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1)\Big) + Pr(e|c_1, \neg c_2)Pr(c_1).$ $\square$

Observation: The above function shows that strong effects on the output probability $Pr(e)$ can be expected when:

- $Pr(c_2)$ has a large value and at least one of the values $Pr(e|c_1, \neg c_2), Pr(c_2|c_1)$, or $Pr(c_1)$ is small.

Since $Pr(e|c_1, \neg c_2)$ is a noisy-OR parameter, which are assumed to be large, we conclude that the gradient is large when the prior probability $Pr(c_2)$ is large and $Pr(c_1)$ and/or $Pr(c_2|c_1)$ is/are small. In addition we have that the smaller the noisy-OR parameter $Pr(e|c_1, \neg c_2)$, the larger the propagation effects. The gradient is in the interval (0,1). Note that the gradient here equals $(Pr(c_2) - Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1))$ and in Equation (7) (the basic causal mechanism exhibited in Figure 3) it equals $\big(Pr(c_2) - Pr(e|c_1, \neg c_2)Pr(c_2)Pr(c_1)\big)$ which means that the prior probability $Pr(c_2)$ in Equation (7) is replaced by the conditional probability $Pr(c_2|c_1)$. We remark that $Pr(c_1), Pr(c_2)$ and $Pr(c_2|c_1)$ can not be regarded as entirely independent from each other since $Pr(c_2) = Pr(c_2|c_1)Pr(c_1) + Pr(c_2|\neg c_1)Pr(\neg c_1)$.

We conclude that the effect of changes in the noisy-OR parameter on the probability $Pr(e)$ of the model shown in Figure 10, are in line with the basic causal mechanism shown in Figure 3. However, the gradient in Equation (20) differs from the gradient in Equation (7). The size in which the gradient of Equation (20) differs, depends on the value of $Pr(c_2|c_1)$ compared to $Pr(c_2)$. If $Pr(c_2|c_1) < Pr(c_2)$, the propagation effects of Equation (20) will be larger than of Equation (7). However, if $Pr(c_2|c_1) > Pr(c_2)$ the gradient of Equation (20) will be smaller than of Equation (7), and thus the propagation effects will be smaller. Moreover, we find that the gradient of Equation (20) equals:

$Pr(c_2) - Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1)$, we can rewrite this as:

$Pr(c_2|c_1)Pr(c_1) + Pr(c_2|\neg c_1)Pr(\neg c_1) - Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) =$

$= Pr(c_2|\neg c_1)Pr(\neg c_1) + Pr(c_2|c_1)Pr(c_1)\big(1 - Pr(e|c_1, \neg c_2)\big)$

and thus, we find that the gradient of Equation (20) is $\geq Pr(c_2|\neg c_1)Pr(\neg c_1)$.

We discovered that the results of the presence of possibly dependent cause variables are largely in line with the results of Section 4.1.1, where there are no dependent causes. For the propagation effects in the causal direction, we observed that the gradient from Equation (20) possesses the term $Pr(c_2|c_1)$ instead of $Pr(c_2)$ in Equation (7). The amount in which the gradient from Equation (20) is larger/smaller than of Equation (7), depends on the value of $Pr(c_2|c_1)$ compared to $Pr(c_2)$.

4.2.2 Propagation effects in the diagnostic direction

In this section, we investigate the consequences of varying a noisy-OR parameter upon propagation in the diagnostic direction, that is, upon propagating evidence for the effect variable to an unobserved cause variable. We still consider the conditional probability tables for the three variables of the causal mechanism from Figure 10.

4.2.2.1 Sensitivity function $\mathbf{Pr(c_2|e)(x)}$ with $\mathbf{x = Pr(e|\neg c_1, c_2)}$

We examine $Pr(c_2|e)$ as a function of the probability $Pr(e|\neg c_1, c_2)$. The result is function $Pr(c_2|e)(x)$ which is hyperbolic in the probability x.

Theorem 4.9. *Consider the causal mechanism in Figure 10 and assume it models a noisy-OR. Let $x = Pr(e|\neg c_1, c_2)$ be the noisy-OR parameter associated with cause C_2. Then the sensitivity function $Pr(c_2|e)(x)$ has the following form:*

$$Pr(c_2|e)(x) = \frac{x + \frac{\alpha}{\gamma-\alpha}}{x + \frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)}} \tag{21}$$

where $\alpha = Pr(e|c_1, \neg c_2)$ and $\gamma = \frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)}$.

Proof: We have:

$$Pr(c_2|e) = \frac{Pr(c_2, e)}{Pr(c_2, e) + Pr(\neg c_2, e)}, \text{ and therefore}$$

$$Pr(c_2|e)(x) = \frac{Pr(e|c_1, c_2)Pr(c_2|c_1)Pr(c_1) + xPr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2, e)(x) + Pr(\neg c_2, e)(x)}.$$

We substitute the value $Pr(e|c_1, c_2)$ with $x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)$ (Eq. (9)), and obtain:

$$\begin{aligned}
Pr(c_2|e)(x) &= \frac{\Big(x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)\Big)Pr(c_2|c_1)Pr(c_1) + xPr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(e)(x)} \\
&= \frac{Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) + xPr(c_2|c_1)Pr(c_1) - xPr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) + xPr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2, e)(x) + Pr(\neg c_2, e)(x)}
\end{aligned}$$

Where $Pr(c_2, e)(x) = \Big(x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)\Big)Pr(c_2|c_1)Pr(c_1) + xPr(c_2|\neg c_1)Pr(\neg c_1)$, and $Pr(\neg c_2, e)(x) = Pr(e|c_1, \neg c_2)Pr(\neg c_2|c_1)Pr(c_1) + Pr(e|\neg c_1, \neg c_2)Pr(\neg c_2|\neg c_1)Pr(\neg c_1) = Pr(e|c_1, \neg c_2)Pr(\neg c_2|c_1)Pr(c_1)$. Divide both numerator and denominator by $Pr(c_1)$, then $Pr(c_2|e)(x) =$

$$\begin{aligned}
&= \frac{Pr(e|c_1, \neg c_2)Pr(c_2|c_1) + xPr(c_2|c_1) - xPr(e|c_1, \neg c_2)Pr(c_2|c_1) + \frac{xPr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_1)}}{Pr(e|c_1, \neg c_2)Pr(c_2|c_1) + xPr(c_2|c_1) - xPr(e|c_1, \neg c_2)Pr(c_2|c_1) + \frac{xPr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_1)} + Pr(e|c_1, \neg c_2)Pr(\neg c_2|c_1)} \\
&= \frac{Pr(e|c_1, \neg c_2) + x - xPr(e|c_1, \neg c_2) + \frac{xPr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}{Pr(e|c_1, \neg c_2) + x - xPr(e|c_1, \neg c_2) + \frac{xPr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)} + \frac{Pr(e|c_1, \neg c_2)Pr(\neg c_2|c_1)}{Pr(c_2|c_1)}} \\
&= \frac{x\Big(1 - Pr(e|c_1, \neg c_2) + \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}\Big) + Pr(e|c_1, \neg c_2)}{x\Big(1 - Pr(e|c_1, \neg c_2) + \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}\Big) + Pr(e|c_1, \neg c_2) + \frac{Pr(e|c_1, \neg c_2)Pr(\neg c_2|c_1)}{Pr(c_2|c_1)}} \\
&= \frac{x + \frac{Pr(e|c_1, \neg c_2)}{1 - Pr(e|c_1, \neg c_2) + \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}}{x + \frac{Pr(e|c_1, \neg c_2) + \frac{Pr(e|c_1, \neg c_2)Pr(\neg c_2|c_1)}{Pr(c_2|c_1)}}{1 - Pr(e|c_1, \neg c_2) + \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}} \\
&= \frac{x + \frac{Pr(e|c_1, \neg c_2)}{\frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)} - Pr(e|c_1, \neg c_2)}}{x + \frac{Pr(e|c_1, \neg c_2)\Big(1 + \frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)}\Big)}{\frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)} - Pr(e|c_1, \neg c_2)}} \quad \text{(because } Pr(c_2|c_1)Pr(c_1) + Pr(c_2|\neg c_1)Pr(\neg c_1) = Pr(c_2)\text{)} \\
&= \frac{x + \frac{Pr(e|c_1, \neg c_2)}{\frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)} - Pr(e|c_1, \neg c_2)}}{x + \frac{Pr(e|c_1, \neg c_2)}{Pr(c_2|c_1)\big(\frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)} - Pr(e|c_1, \neg c_2)\big)}} \quad \text{(since } Pr(c_2|c_1) + Pr(\neg c_2|c_1) = 1\text{)} \\
&= \frac{x + \frac{\alpha}{\gamma-\alpha}}{x + \frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)}}
\end{aligned}$$

where $\alpha = Pr(e|c_1, \neg c_2)$ and $\gamma = \frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)}$. □

Observation: Since Equation (21) is a hyperbolic function, we use the properties of hyperbolic functions described in Section 2.5, and discover that the vertical asymptote of Equation (21) lies at $x = s = -\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)}$ and the horizontal asymptote equals $t = 1$. Because $Pr(c_2) > Pr(c_2|c_1)Pr(c_1)$, we find that $\frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)} > 1$, and thus $\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)} > 0$. Consequently, we find that Equation (21) is a fragment of a fourth-quadrant hyperbola branch, and as a consequence, we know that the first derivative $\frac{d}{dx}Pr(c_2|e)(x) > 0$ for all $x \in [0, 1]$.

Equation (21) has its vertex at:

$$(s + \sqrt{|r|}, 1 - \sqrt{|r|}) = \Big(-\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)} + \sqrt{\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)} - \frac{\alpha}{\gamma-\alpha}}, 1 - \sqrt{\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)} - \frac{\alpha}{\gamma-\alpha}}\Big)$$

The vertex is located within the unit window for some values of $\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)}, \frac{\alpha}{\gamma-\alpha}$ with $\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)} < \sqrt{\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)} - \frac{\alpha}{\gamma-\alpha}} < 1$. To obtain $\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)} < \sqrt{\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)} - \frac{\alpha}{\gamma-\alpha}}$ given that $\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)} \geq \frac{\alpha}{\gamma-\alpha}$, we discover that merely rather small values of $\frac{\alpha}{\gamma-\alpha}$ produce a vertex with an x-coordinate in the unit range. In order to have a small value for $\frac{\alpha}{\gamma-\alpha} = \frac{Pr(e|c_1,\neg c_2)}{\frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)} - Pr(e|c_1,\neg c_2)}$, the prior probability $Pr(c_2)$ has to be large and $Pr(c_1)$ has to be small. In addition, the vertex only approaches the upper-left corner of the unit window if the difference $\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)} - \frac{\alpha}{\gamma-\alpha}$ is rather small, since then Equation (21) approaches 1. We find that the difference $\frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)} - \frac{\alpha}{\gamma-\alpha}$ is small when the conditional probability $Pr(c_2|c_1)$ is large.

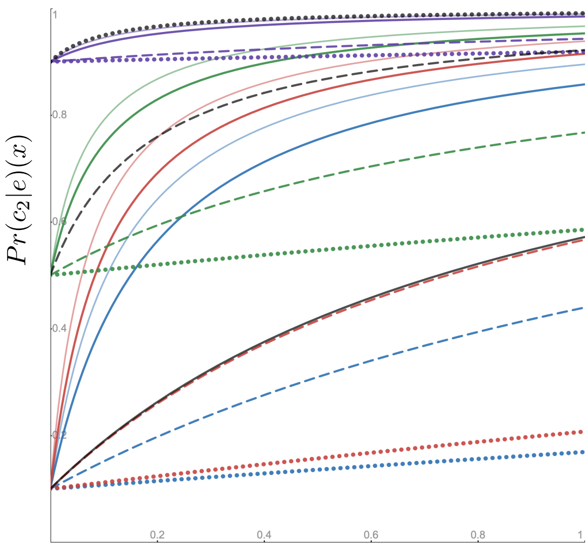

Figure 11: Several example sensitivity functions adhering to Theorem 4.9. (See Table 5 for parameter settings)

Parameter	Blue	Blue light	Blue dashed	Blue dotted	Red	Red light	Red dashed	Red dotted	Green	Green light
$Pr(e\|c_1, \neg c_2)$	0.85	0.6	0.85	0.85	0.85	0.6	0.85	0.85	0.85	0.6
$Pr(c_2\|c_1)$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.5	0.5
$Pr(c_2\|\neg c_1)$	0.5	0.5	0.5	0.5	0.9	0.9	0.9	0.9	0.9	0.9
$Pr(c_1)$	0.1	0.1	0.5	0.9	0.1	0.1	0.5	0.9	0.1	0.1
$Pr(c_2)$	0.46	0.46	0.3	0.14	0.82	0.82	0.5	0.18	0.86	0.86

[1] We choose 0.098 because otherwise this function would overlap with the red dashed function.

Parameter	Green dashed	Green dotted	Purple	Purple light	Purple dashed	Purple dotted	Black	Black dashed	Black dotted
$Pr(e\|c_1, \neg c_2)$	0.85	0.85	0.85	0.6	0.85	0.85	0.85	0.85	0.85
$Pr(c_2\|c_1)$	0.5	0.5	0.5	0.5	0.5	0.5	0.1	0.5	0.9
$Pr(c_2\|\neg c_1)$	0.9	0.9	0.1	0.1	0.1	0.1	0.1	0.5	0.9
$Pr(c_1)$	0.5	0.9	0.1	0.1	0.5	0.9	0.098[1]	0.1	0.1
$Pr(c_2)$	0.7	0.54	0.14	0.14	0.3	0.46	0.1	0.5	0.9

Table 5: Parameter settings for sensitivity functions from Figure 11

In order to support the above mentioned findings and gain more insight in what the influences are of the values of the parameters on the probability $Pr(c_2|e)$, we make some plots of the sensitivity function with different input parameters $Pr(e|c_1, \neg c_2), Pr(c_2|\neg c_1), Pr(c_2|c_1)$, $Pr(c_1)$ and $Pr(c_2)$ as provided in Table 5. The value of $Pr(c_2)$ is determined by $Pr(c_2|c_1)$, $Pr(c_2|\neg c_1)$ and $Pr(c_1)$, since $Pr(c_2) = Pr(c_2|c_1)Pr(c_1) + Pr(c_2|\neg c_1)Pr(\neg c_1)$; for sake of completeness we listed all parameters in Table 5. Furthermore, we especially want to examine the propagation effects when $Pr(c_2|c_1) \neq Pr(c_2|\neg c_1)$, since we want to investigate the effect of a dependency between the cause variables. For the cases where $Pr(c_2|c_1) = Pr(c_2|\neg c_1)$ we refer to Chapter 4.1. However, for comparison we plot three functions where $Pr(c_2|c_1) = Pr(c_2|\neg c_1)$, see black, black dashed, and black dotted.
The results of the effects of different concrete parameter settings are exhibited in Figure 11. We observe that function $Pr(c_2|e)(x)$ demonstrates that strong effects on the output probability $Pr(c_2|e)$ in the entire interval $x \in [0, 1]$ can be expected based on the following:

- $Pr(c_2|c_1)$ and $Pr(c_1)$ have small values and $Pr(c_2|\neg c_1)$ has a large value;
- The smaller the noisy-OR parameter $Pr(e|c_1, \neg c_2)$, the larger the propagation effects on the output probability. This effect is conveyed by the solid versus light function for each colour.

Again, one should recall that the focus is on probabilities for $x = Pr(e|\neg c_1, c_2) \geq 0.6$. In Figure 11, it is shown that large propagation effects only occur for small values for $x = Pr(e|\neg c_1, c_2)$ (see for example blue, green and red functions). These results show that the propagation effects, for $x = Pr(e|\neg c_1, c_2)$ larger than 0.6, are moderate or small. However, to gain better insight into Equation's (21) behaviour in the interval $x = Pr(e|\neg c_1, c_2) \geq 0.6$, we compute its first derivative. As we have mentioned, the sensitivity functions corresponding to Equation (21) are a fragment of a fourth-quadrant hyperbola branch, and as a consequence, we know that the first derivative $\frac{d}{dx}Pr(c_2|e)(x) > 0$ for all $x \in [0, 1]$.

Corollary 4.9.1. *The first derivative of the sensitivity function from Equation (21) is:*

$$\begin{aligned}\frac{d}{dx}Pr(c_2|e)(x) &= \frac{Pr(e|c_1, \neg c_2)(1 - Pr(c_2|c_1))}{Pr(c_2|c_1)\Big(\frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)} - Pr(e|c_1, \neg c_2)\Big) \cdot \Big(x + \frac{Pr(e|c_1,\neg c_2)}{Pr(c_2|c_1)\big(\frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)} - Pr(e|c_1,\neg c_2)\big)}\Big)^2} \\ &= \frac{\alpha(1 - Pr(c_2|c_1))}{Pr(c_2|c_1)(\gamma - \alpha)\Big(x + \frac{\alpha}{Pr(c_2|c_1)(\gamma-\alpha)}\Big)^2}\end{aligned} \quad (22)$$

where $\gamma = \frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)}$ *and* $\alpha = Pr(e|c_1, \neg c_2)$.

Now, for the red (light/dashed), blue (light/dashed) and green (light/dashed) functions demonstrated in Figure 11, we plot the derivatives of Equation (22).

In Figure 12, it is shown that for the parameter settings of the red (light) and green (light/dashed) function the first derivative $\frac{d}{dx}Pr(c_2|e)(x) < 0.25$ for $x \in [0.6, 1]$. We observe moderate and small propagation effects for the blue (dashed) and red dashed function. Finally, when only concentrating on the interval $x \in [0.6, 1]$, we don't observe that a smaller noisy-OR parameter $Pr(e|c_1, \neg c_2)$ leads to larger propagation effects anymore. The propagation effects can even become larger in the interval $x \in [0.6, 1]$ due to a larger noisy-OR parameter $x = Pr(e|c_1, \neg c_2)$.

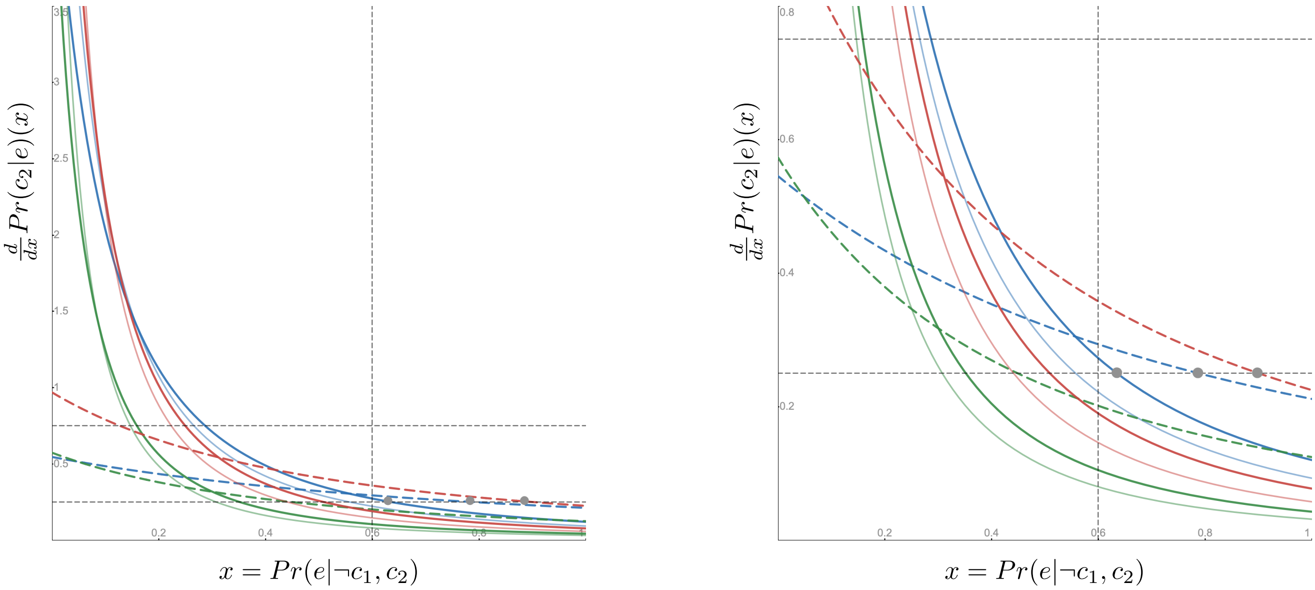

Figure 12: Several examples of Equation (22) restricted to the window $x \in [0,1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [0,3.5]$ (left) and the window $x \in [0,1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [0,0.8]$ (right). (See Table 5 for parameter settings)

With the help of WOLFRAM MATHEMATICA, we find a maximum of max $\frac{d}{dx}Pr(c_2|e)(x) = 0.416666$ in the interval $x = Pr(e|\neg c_1, c_2) \in [0.6,1]$ of Equation (22) with several parameter settings (see Appendix B.1). For example, one of the parameter settings is:

$Pr(e|c_1,\neg c_2) = 0.84354$, $Pr(c_2|c_1) = 2.08187 \cdot 10^{-7}$, $Pr(c_2|\neg c_1) = 0.9285467399$, $Pr(c_1) = 0.397759$ and $Pr(c_2) = 0.559209$.

This maximum lies at $x = 0.6$. We conclude that our findings concerning the propagation effects of Equation (21) in the interval $x = Pr(e|\neg c_1, c_2) \in [0.6,1]$ are moderate or small, which are in line with the results derived in Section 4.1.3.1. We discovered, when studying the effect of possibly dependent cause variables, that Equation (22) where $x = Pr(e|\neg c_1, c_2) \in [0.6,1]$ can attain maximum value of 0.416666, indicating that the propagation effects are in line with Section 4.1.3.1.

4.2.2.2 Sensitivity function $\mathbf{Pr(c_2|e)(x)}$ with $\mathbf{x = Pr(e|c_1, \neg c_2)}$

We now examine $Pr(c_2|e)$ as a function of the probability $Pr(e|c_1,\neg c_2)$, that is the noisy-OR parameter associated with cause C_1. The result is function $Pr(c_2|e)(x)$ which is hyperbolic in the probability x.

Theorem 4.10. *Consider the causal mechanism in Figure 10 and assume it models a noisy-OR. Let $x = Pr(e|c_1,\neg c_2)$ be the noisy-OR parameter associated with cause C_1. Then the sensitivity function $Pr(c_2|e)(x)$ has the following form:*

$$Pr(c_2|e)(x) = \frac{x+\beta}{x\alpha+\beta} \tag{23}$$

where $\alpha = \frac{1-Pr(e|\neg c_1,c_2)Pr(c_2|c_1)}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}$ *and* $\beta = \frac{Pr(e|\neg c_1,c_2)\frac{Pr(c_2)}{Pr(c_1)}}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}$.

Proof: We have:

$Pr(c_2|e)(x)$ with $x = Pr(e|c_1, \neg c_2)$, since

$Pr(c_2|e) = \dfrac{Pr(c_2,e)}{Pr(c_2,e)+Pr(\neg c_2,e)}$, we find

$$Pr(c_2|e)(x) = \frac{Pr(e|c_1,c_2)Pr(c_2|c_1)Pr(c_1) + Pr(e|\neg c_1,c_2)Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2,e)(x)+Pr(\neg c_2,e)(x)}$$

where $Pr(e|c_1,c_2)$ is dependent of x.
We substitute the value $Pr(e|c_1,c_2)$ with $x\big(1-Pr(e|\neg c_1,c_2)\big)+Pr(e|\neg c_1,c_2)$ (Eq. (12)), and obtain:
$Pr(c_2|e)(x) =$

$$= \frac{\Big(x\big(1-Pr(e|\neg c_1,c_2)\big)+Pr(e|\neg c_1,c_2)\Big)Pr(c_2|c_1)Pr(c_1) + Pr(e|\neg c_1,c_2)Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(e)(x)}$$

$$= \frac{Pr(e|\neg c_1,c_2)Pr(c_2|c_1)Pr(c_1) + xPr(c_2|c_1)Pr(c_1) - xPr(e|\neg c_1,c_2)Pr(c_2|c_1)Pr(c_1) + Pr(e|\neg c_1,c_2)Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2,e)(x)+Pr(\neg c_2,e)(x)}.$$

Where $Pr(c_2,e)(x) = \Big(x\big(1-Pr(e|\neg c_1,c_2)\big)+Pr(e|\neg c_1,c_2)\Big)Pr(c_2|c_1)Pr(c_1) + Pr(e|\neg c_1,c_2)Pr(c_2|\neg c_1)Pr(\neg c_1)$,
and $Pr(\neg c_2,e)(x) = Pr(e|c_1,\neg c_2)Pr(\neg c_2|c_1)Pr(c_1) + Pr(e|\neg c_1,\neg c_2)Pr(\neg c_2|\neg c_1)Pr(\neg c_1) = xPr(\neg c_2|c_1)Pr(c_1)$.
Divide both numerator and denominator by $Pr(c_1)$, then $Pr(c_2|e)(x) =$

$$= \frac{xPr(c_2|c_1) - xPr(e|\neg c_1,c_2)Pr(c_2|c_1) + Pr(e|\neg c_1,c_2)Pr(c_2|c_1) + Pr(e|\neg c_1,c_2)Pr(c_2|\neg c_1)\frac{Pr(\neg c_1)}{Pr(c_1)}}{xPr(c_2|c_1) - xPr(e|\neg c_1,c_2)Pr(c_2|c_1) + Pr(e|\neg c_1,c_2)Pr(c_2|c_1) + Pr(e|\neg c_1,c_2)Pr(c_2|\neg c_1)\frac{Pr(\neg c_1)}{Pr(c_1)} + xPr(\neg c_2|c_1)}$$

$$= \frac{x\Big(Pr(c_2|c_1) - Pr(e|\neg c_1,c_2)Pr(c_2|c_1)\Big) + Pr(e|\neg c_1,c_2)Pr(c_2|c_1) + Pr(e|\neg c_1,c_2)Pr(c_2|\neg c_1)\frac{Pr(\neg c_1)}{Pr(c_1)}}{x\Big(Pr(c_2|c_1) - Pr(e|\neg c_1,c_2)Pr(c_2|c_1)\Big) + Pr(e|\neg c_1,c_2)Pr(c_2|c_1) + Pr(e|\neg c_1,c_2)Pr(c_2|\neg c_1)\frac{Pr(\neg c_1)}{Pr(c_1)} + xPr(\neg c_2|c_1)}$$

$$= \frac{x + \frac{Pr(e|\neg c_1,c_2)Pr(c_2|c_1)+Pr(e|\neg c_1,c_2)Pr(c_2|\neg c_1)\frac{Pr(\neg c_1)}{Pr(c_1)}}{Pr(c_2|c_1)-Pr(e|\neg c_1,c_2)Pr(c_2|c_1)}}{x + \frac{Pr(e|\neg c_1,c_2)Pr(c_2|c_1)+Pr(e|\neg c_1,c_2)Pr(c_2|\neg c_1)\frac{Pr(\neg c_1)}{Pr(c_1)}}{Pr(c_2|c_1)-Pr(e|\neg c_1,c_2)Pr(c_2|c_1)} + x\frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)-Pr(e|\neg c_1,c_2)Pr(c_2|c_1)}}$$

$$= \frac{x + \frac{Pr(e|\neg c_1,c_2)\left(Pr(c_2|c_1)+Pr(c_2|\neg c_1)\frac{Pr(\neg c_1)}{Pr(c_1)}\right)}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}}{x + \frac{Pr(e|\neg c_1,c_2)\left(Pr(c_2|c_1)+Pr(c_2|\neg c_1)\frac{Pr(\neg c_1)}{Pr(c_1)}\right)}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))} + x\frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}}$$

$$= \frac{x + \frac{Pr(e|\neg c_1,c_2)\frac{Pr(c_2)}{Pr(c_1)}}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}}{x + \frac{Pr(e|\neg c_1,c_2)\frac{Pr(c_2)}{Pr(c_1)}}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))} + x\frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}}$$

$$= \frac{x+\beta}{x\alpha+\beta}$$

where $\alpha = 1 + \frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))} = \frac{1-Pr(e|\neg c_1,c_2)Pr(c_2|c_1)}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}$ and $\beta = \frac{Pr(e|\neg c_1,c_2)\frac{Pr(c_2)}{Pr(c_1)}}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}$. □

Observation: We have that Equation (23) is a hyperbolic function, and therefore we use the properties of hyperbolic functions described in Section 2.5. We find that the vertical asymptote of Equation (23) lies at $x = s = -\frac{\beta}{\alpha}$. Note that Equation (23) is not defined for $Pr(e|\neg c_1,c_2) = 1$. Because $\frac{\beta}{\alpha} > 0$, the vertical asymptote is located to the left of the unit window and the horizontal asymptote lies at $t = \frac{1}{\alpha}$, where $\alpha > 1$, and thus $t < 1$. As a result, we find that Equation (23) is a fragment of a first-quadrant hyperbola branch. Furthermore, we have that the closer the vertex of the first-quadrant hyperbola branch lies to the point $(0, \frac{1}{\alpha})$, the larger the propagation effects of Equation (23).

Equation (23) has its vertex at:

$$(s+\sqrt{|r|}, t+\sqrt{|r|}) = \left(-\frac{\beta}{\alpha} + \sqrt{\left|\frac{\beta(1-\alpha)}{\alpha^2}\right|}, \frac{1}{\alpha} + \sqrt{\left|\frac{\beta(1-\alpha)}{\alpha^2}\right|}\right)$$

Now, we have that the vertex is located within the unit window for values of α, β with $\frac{\beta}{\alpha} < \sqrt{\left|\frac{\beta(1-\alpha)}{\alpha^2}\right|} < 1$. To have a vertex close located to the point $(0, \frac{1}{\alpha})$ we find that $\frac{\beta}{\alpha}$ and $\sqrt{\left|\frac{\beta(1-\alpha)}{\alpha^2}\right|}$ should approach 0. This means that β should be small and α should be large, then we have that the term $\frac{\beta}{\alpha}$ approaches 0. Consequently, large propagation effects occur when the vertex of the sensitivity function under study is closely located to the point $(0, 0)$. For $\beta \to 0$ we need that $Pr(c_1)$ is large and $Pr(c_2), Pr(c_2|c_1)$ and $Pr(e|\neg c_1, c_2)$ are small. In addition, for α to become large, we need that $Pr(c_2|c_1)$ and $Pr(e|\neg c_1, c_2)$ are small. The same holds for the term $\sqrt{\left|\frac{\beta(1-\alpha)}{\alpha^2}\right|}$. For $\sqrt{\left|\frac{\beta(1-\alpha)}{\alpha^2}\right|}$ to approach 0, we need again that β should be small and α large.

In order to gain better insight in what the influences are of the values of the parameters on the probability $Pr(c_2|e)$, we look at concrete parameter settings of several hyperbola branches restricted to the unit window. We make some plots of Equation (23) with different input parameters $Pr(e|c_1, \neg c_2)$, $Pr(c_2|c_1)$, $Pr(c_2|\neg c_1)$ and $Pr(c_1)$, whereby the value of $Pr(c_2)$ is determined by $Pr(c_2|c_1)$, $Pr(c_2|\neg c_1)$ and $Pr(c_1)$. For sake of completeness all mentioned probabilities are listed in Table 6.

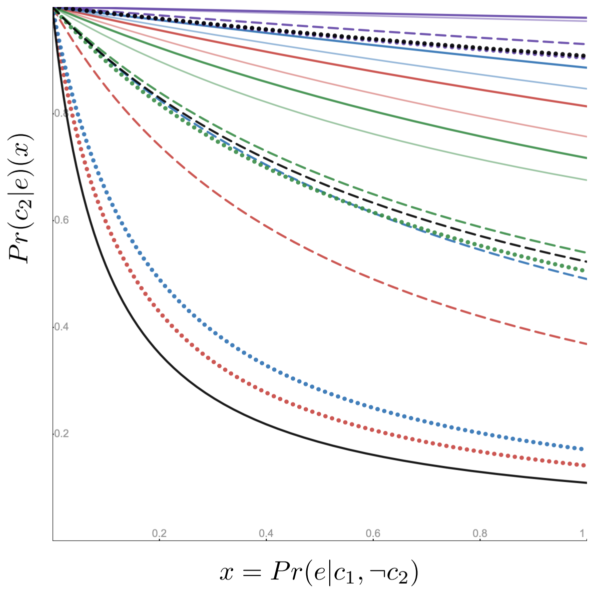

Figure 13: Several example sensitivity functions adhering to Theorem 4.10. (See Table 6 for parameter settings)

Parameter	Red	Red light	Red dashed	Red dotted	Blue	Blue light	Blue dashed	Blue dotted	Green	Green light
$Pr(e\|c_1, \neg c_2)$	0.85	0.6	0.85	0.85	0.85	0.6	0.85	0.85	0.85	0.6
$Pr(c_2\|c_1)$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.5	0.5
$Pr(c_2\|\neg c_1)$	0.5	0.5	0.5	0.5	0.9	0.9	0.9	0.9	0.1	0.1
$Pr(c_1)$	0.1	0.1	0.5	0.9	0.1	0.1	0.5	0.9	0.1	0.1
$Pr(c_2)$	0.46	0.46	0.3	0.14	0.82	0.82	0.5	0.18	0.14	0.14

Parameter	Green dashed	Green dotted	Purple	Purple light	Purple dashed	Purple dotted	Black	Black dashed	Black dotted
$Pr(e\|c_1, \neg c_2)$	0.85	0.85	0.85	0.6	0.85	0.85	0.85	0.85	0.85
$Pr(c_2\|c_1)$	0.5	0.5	0.9	0.9	0.9	0.9	0.1	0.5	0.9
$Pr(c_2\|\neg c_1)$	0.9	0.9	0.1	0.1	0.1	0.1	0.1	0.5	0.9
$Pr(c_1)$	0.5	0.9	0.1	0.1	0.5	0.9	0.9	0.9	0.9
$Pr(c_2)$	0.7	0.46	0.54	0.54	0.5	0.86	0.1	0.5	0.9

Table 6: Parameter settings for sensitivity functions from Figure 13

The results are exhibited in Figure 13. We observe that Equation (23) demonstrates that the largest propagation effects in the entire interval $x \in [0, 1]$ can be expected based on the following:

- $Pr(c_2|\neg c_1)$ and $Pr(c_2|c_1)$ have small values and $Pr(c_1)$ has a large value;
- The smaller the noisy-OR parameter $Pr(e|\neg c_1, c_2)$, the larger the propagation effects on the output probability $Pr(c_2|e)$. This effect is conveyed by the solid versus light function for each colour.

When we focus on the interval where $x = Pr(e|c_1, \neg c_2) \geq 0.6$ in Figure 13, we note that the propagation effects can become small and moderate at most. In order to gain more insight into the propagation effects for $x \geq 0.6$ of Equation (23), we compute its first derivative. Since the sensitivity functions corresponding to Equation (23) are a fragment of a first-quadrant hyperbola branch, we know that the first derivative $\frac{d}{dx}Pr(c_2|e)(x) < 0$ for all $x \in [0, 1]$.

Corollary 4.10.1. *The first derivative of the sensitivity function from Equation (23) is:*

$$\begin{aligned}\frac{d}{dx}Pr(c_2|e)(x) &= \frac{\frac{Pr(e|\neg c_1,c_2)\frac{Pr(c_2)}{Pr(c_1)}}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}\left(1-\frac{1-Pr(e|\neg c_1,c_2)Pr(c_2|c_1)}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}\right)}{\left(x\frac{1-Pr(e|\neg c_1,c_2)Pr(c_2|c_1)}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}+\frac{Pr(e|\neg c_1,c_2)\frac{Pr(c_2)}{Pr(c_1)}}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}\right)^2} \\ &= \frac{\beta(1-\alpha)}{(x\alpha+\beta)^2}\end{aligned} \tag{24}$$

where $\alpha = \frac{1-Pr(e|\neg c_1,c_2)Pr(c_2|c_1)}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}$ *and* $\beta = \frac{Pr(e|\neg c_1,c_2)\frac{Pr(c_2)}{Pr(c_1)}}{Pr(c_2|c_1)(1-Pr(e|\neg c_1,c_2))}$.

Now, for specific parameter settings for some of the functions demonstrated in Figure 13, namely the red (dashed/dotted), blue (dashed/dotted) and green (dashed/dotted) function we plot the derivatives of Equation (24). See Figure 14.

With the help of WOLFRAM MATHEMATICA, we find an absolute maximum of $\max |\frac{d}{dx}Pr(c_2|e)(x)| = |0.416666|$ in the interval $x = Pr(e|c_1, \neg c_2) \in [0.6, 1]$ of Equation (24) with several parameter settings (see Appendix B.2). For example, one of the parameter settings is:

$$Pr(e|\neg c_1, c_2) = 0.9999, Pr(c_2|c_1) = 0.184973,\ Pr(c_2|\neg c_1) = 0.3883,\ Pr(c_1) = 0.560874 \text{ and } Pr(c_2) = 0.274277.$$

This maximum lies at $x = 0.6$. Again, the noisy-OR parameter $Pr(e|\neg c_1, c_2)$ is extremely large; a somehow similar parameter setting corresponds to the same gradient's maximum value of Equation (16) in Section 4.1.3.2. We conclude that the effect of changes in the noisy-OR parameter $Pr(e|c_1, \neg c_2)$ on the probability $Pr(c_2|e)$ of the causal mechanism involving possibly dependent cause variables, are in line with the basic causal mechanism shown in Figure 3, see Section 4.1.3.2. We note that the propagation effects in the interval $x = Pr(e|c_1, \neg c_2) \in [0.6, 1]$ can become moderate at most.

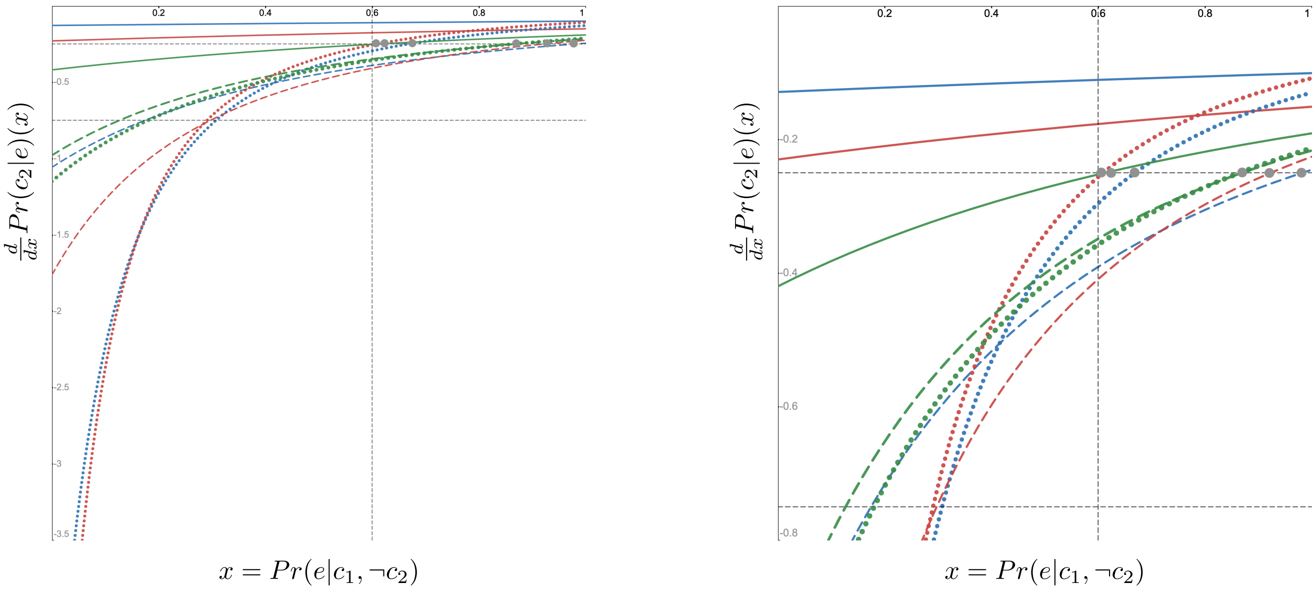

Figure 14: Several examples of Equation (24) restricted to the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [-3.5, 0]$ (left) and the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [-0.8, 0]$ (right). (See Table 6 for parameter settings)

4.2.2.3 Sensitivity function $\mathbf{Pr(c_2|\neg e)(x)}$ with $\mathbf{x = Pr(e|\neg c_1, c_2)}$

We now examine $Pr(c_2|\neg e)$ as a function of the probability $Pr(e|\neg c_1, c_2)$, that is the noisy-OR parameter associated with cause C_2. The result is function $Pr(c_2|\neg e)(x)$ which is hyperbolic in the probability x.

Theorem 4.11. *Consider the causal mechanism in Figure 10 and assume it models a noisy-OR. Let $x = Pr(e|\neg c_1, c_2)$ be the noisy-OR parameter associated with cause C_2. Then the sensitivity function $Pr(c_2|\neg e)(x)$ has the following form:*

$$Pr(c_2|\neg e)(x) = \frac{x + \gamma}{x + \gamma + \alpha} \tag{25}$$

where $\gamma = \frac{Pr(c_2)}{Pr(e|c_1,\neg c_2)Pr(c_2|c_1)Pr(c_1) - Pr(c_2)}$ *and* $\alpha = \frac{Pr(\neg c_2|c_1)Pr(c_1)\left(1 - Pr(e|c_1,\neg c_2)\right)}{Pr(e|c_1,\neg c_2)Pr(c_2|c_1)Pr(c_1) - Pr(c_2)}$.

Proof:
Since $Pr(e|\mathbf{c}) + Pr(\neg e|\mathbf{c}) = 1$, we again note the following (see also Section 4.1.3.3):

- $Pr(\neg e|c_1, c_2) = 1 - \Big(x\big(1 - Pr(e|c_1, \neg c_2)\big) + Pr(e|c_1, \neg c_2)\Big)$;
- $Pr(\neg e|\neg c_1, c_2) = 1 - Pr(e|\neg c_1, c_2) = 1 - x$;
- $Pr(\neg e|c_1, \neg c_2) = 1 - Pr(e|c_1, \neg c_2)$.

We have:

$$\begin{aligned} Pr(c_2|\neg e)(x) &= \frac{Pr(\neg e|c_1, c_2)Pr(c_2|c_1)Pr(c_1) + Pr(\neg e|\neg c_1, c_2)Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(\neg e)(x)} \\ &= \frac{\Big(1 + x\big(Pr(e|c_1, \neg c_2) - 1\big) - Pr(e|c_1, \neg c_2)\Big)Pr(c_2|c_1)Pr(c_1) + (1 - x)Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(\neg e)(x)} \end{aligned}$$

and $Pr(\neg e)(x)$ is equal to:

$$\begin{aligned}
Pr(\neg e)(x) &= Pr(c_2, \neg e)(x) + Pr(\neg c_2, \neg e)(x) \\
&= Pr(\neg e|c_1, c_2)Pr(c_2|c_1)Pr(c_1) + Pr(\neg e|\neg c_1, c_2)Pr(c_2|\neg c_1)Pr(\neg c_1) + Pr(\neg e|c_1, \neg c_2)Pr(\neg c_2|c_1)Pr(c_1) \\
&= \Big(1 + x\big(Pr(e|c_1, \neg c_2) - 1\big) - Pr(e|c_1, \neg c_2)\Big)Pr(c_2|c_1)Pr(c_1) + (1 - x)Pr(c_2|\neg c_1)Pr(\neg c_1) \\
&+ \big(1 - Pr(e|c_1, \neg c_2)\big)Pr(\neg c_2|c_1)Pr(c_1) \\
&= Pr(c_2|c_1)Pr(c_1) + xPr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) - xPr(c_2|c_1)Pr(c_1) - Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) \\
&+ Pr(c_2|\neg c_1)Pr(\neg c_1) - xPr(c_2|\neg c_1)Pr(\neg c_1) + Pr(\neg c_2|c_1)Pr(c_1) - Pr(e|c_1, \neg c_2)Pr(\neg c_2|c_1)Pr(c_1)
\end{aligned}$$

Note that the numerator of the sensitivity function $Pr(c_2|\neg e)(x)$ equals the denominator minus the last 2 terms, that is $Pr(\neg c_2|c_1)Pr(c_1) - Pr(e|c_1, \neg c_2)Pr(\neg c_2|c_1)Pr(c_1)$. Dividing both the numerator and denominator by $Pr(c_2|c_1)Pr(c_1)$, we get: $Pr(c_2|\neg e)(x) =$

$$\begin{aligned}
&= \frac{1 + xPr(e|c_1, \neg c_2) - x - Pr(e|c_1, \neg c_2) + \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)} - \frac{xPr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2)|c_1)Pr(c_1)}}{1 + xPr(e|c_1, \neg c_2) - x - Pr(e|c_1, \neg c_2) + \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)} - \frac{xPr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)} + \frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)} - Pr(e|c_1, \neg c_2)\frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)}} \\
&= \frac{x\Big(Pr(e|c_1, \neg c_2) - 1 - \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}\Big) + 1 + \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}{x\Big(Pr(e|c_1, \neg c_2) - 1 - \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}\Big) + 1 + \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)} + \frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)} - Pr(e|c_1, \neg c_2)\frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)}} \\
&= \frac{x + \frac{1 + \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}{Pr(e|c_1, \neg c_2) - 1 - \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}}{x + \frac{1 + \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)} + \frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)}\big(1 - Pr(e|c_1, \neg c_2)\big)}{Pr(e|c_1, \neg c_2) - 1 - \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}} \\
&= \frac{x + \frac{\frac{Pr(c_2|c_1)Pr(c_1) + Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}{Pr(e|c_1, \neg c_2) - 1 - \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}}{x + \frac{\frac{Pr(c_2|c_1)Pr(c_1) + Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)} + \frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)}\big(1 - Pr(e|c_1, \neg c_2)\big)}{Pr(e|c_1, \neg c_2) - 1 - \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}} \\
&= \frac{x + \frac{\frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)}}{Pr(e|c_1, \neg c_2) - 1 - \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}}{x + \frac{\frac{Pr(c_2)}{Pr(c_2|c_1)Pr(c_1)} + \frac{Pr(\neg c_2|c_1)}{Pr(c_2|c_1)}\big(1 - Pr(e|c_1, \neg c_2)\big)}{Pr(e|c_1, \neg c_2) - 1 - \frac{Pr(c_2|\neg c_1)Pr(\neg c_1)}{Pr(c_2|c_1)Pr(c_1)}}} \\
&= \frac{x + \frac{Pr(c_2)}{Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) - Pr(c_2)}}{x + \frac{Pr(c_2) + Pr(\neg c_2|c_1)Pr(c_1)\big(1 - Pr(e|c_1, \neg c_2)\big)}{Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) - Pr(c_2)}} \\
&= \frac{x + \gamma}{x + \gamma + \alpha}
\end{aligned}$$

where $\gamma = \frac{Pr(c_2)}{Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) - Pr(c_2)}$ and $\alpha = \frac{Pr(\neg c_2|c_1)Pr(c_1)\big(1 - Pr(e|c_1, \neg c_2)\big)}{Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) - Pr(c_2)}$. $\square$

Observation: Equation (25) is hyperbolic in the probability $x = Pr(e|\neg c_1, c_2)$. Building on the properties of hyperbolic functions described in Section 2.5, we find that the vertical asymptote equals $x = s = -(\gamma + \alpha)$. We have that $\gamma + \alpha < 0$, since $Pr(c_2) > Pr(c_2|c_1)Pr(c_1)$. We will show that $(\gamma + \alpha) < -1$:

$$\gamma + \alpha = \frac{Pr(c_2)}{Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) - Pr(c_2)} + \frac{Pr(\neg c_2|c_1)Pr(c_1)\big(1 - Pr(e|c_1, \neg c_2)\big)}{Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) - Pr(c_2)}$$
$$= \frac{Pr(c_2) + Pr(\neg c_2|c_1)Pr(c_1)\big(1 - Pr(e|c_1, \neg c_2)\big)}{Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) - Pr(c_2)}$$

Now we need to show that:
$Pr(c_2) + Pr(\neg c_2|c_1)Pr(c_1)\big(1 - Pr(e|c_1, \neg c_2)\big) > |Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) - Pr(c_2)|$

We multiply $\big(Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1) - Pr(c_2)\big)$ by -1 :
$Pr(c_2) - Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1)$.

We now have to show that: $Pr(c_2) + Pr(\neg c_2|c_1)Pr(c_1)\big(1 - Pr(e|c_1, \neg c_2)\big) > Pr(c_2) - Pr(e|c_1, \neg c_2)Pr(c_2|c_1)Pr(c_1)$
Subtract $Pr(c_2)$ from both sides, and then divide by $Pr(c_1)$. Finally, we end up with:
$Pr(\neg c_2|c_1)\Big(1 - Pr(e|c_1, \neg c_2)\Big) > -Pr(e|c_1, \neg c_2)Pr(c_2|c_1)$, and thus $(\gamma + \alpha) < -1$.

Since $s > 1$, we find that the vertical asymptote is located to the right of the unit window. We have that the horizontal asymptote t of Equation (25) equals $t = 1$ and we conclude that Equation (25) is a fragment of a third-quadrant hyperbola branch. We find that the closer the vertex of the third-quadrant hyperbola branch lies to the point $(1, 1)$, the larger the propagation effects. We note that Equation (25) has its vertex at:

$$(s - \sqrt{|r|}, t - \sqrt{|r|}) = \Big(-\gamma - \alpha - \sqrt{|\alpha|}, 1 - \sqrt{|\alpha|}\Big)$$

We first discover that $\sqrt{|\alpha|} \to 0$ in order to have the y-coordinate approach 1. It becomes theoretically quite complicated to state for which parameter settings $\alpha \to 0$. However, in order for the vertex to approach the upper-right corner of the unit window, that is the point $(1, 1)$, we then also need $\gamma \to -1$. We note that if $Pr(c_2|c_1)$ and/or $Pr(c_1)$ is/are small then γ will approach -1 since $\gamma \approx \frac{Pr(c_2)}{-Pr(c_2)} \approx -1$.

In order to clarify, and supplement the findings above, we make some plots of Equation (25) with different input parameters $Pr(e|c_1, \neg c_2), Pr(c_2|c_1)$, $Pr(c_2|\neg c_1)$, and $Pr(c_1)$. The value of $Pr(c_2)$ is determined by $Pr(c_2|c_1)$, $Pr(c_2|\neg c_1)$ and $Pr(c_1)$. See Figure 15.

We observe that function $Pr(c_2|\neg e)(x)$ demonstrates that the strongest effects on the output probability $Pr(c_2|\neg e)$ can be expected based on the following:

- $Pr(c_1)$ and $Pr(c_2|c_1)$ have small values and $Pr(c_2|\neg c_1)$ has a large value;
- The propagation effects in the entire interval $[0, 1]$ are larger when $Pr(e|\neg c_1, c_2)$ is small. This effect is conveyed by the solid versus light function for each colour.

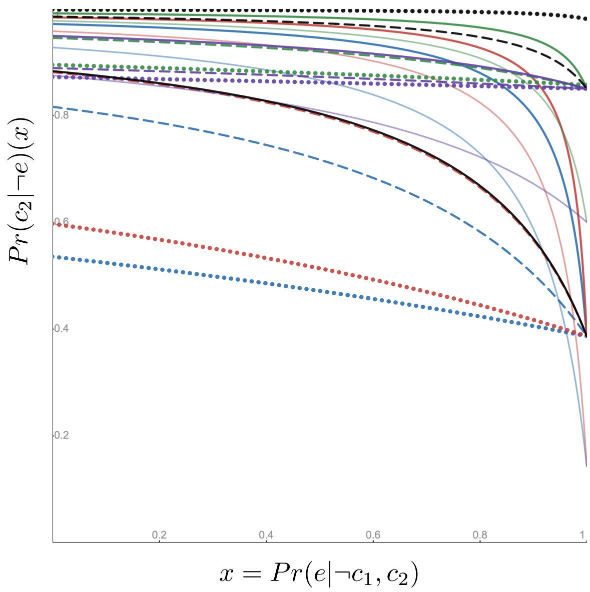

Figure 15: Several example sensitivity functions adhering to Theorem 4.11. (See Table 7 for parameter settings)

Parameter	Blue	Blue light	Blue dashed	Blue dotted	Red	Red light	Red dashed	Red dotted	Green	Green light
$Pr(e\|c_1, \neg c_2)$	0.85	0.6	0.85	0.85	0.85	0.6	0.85	0.85	0.85	0.6
$Pr(c_2\|c_1)$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.5	0.5
$Pr(c_2\|\neg c_1)$	0.5	0.5	0.5	0.5	0.9	0.9	0.9	0.9	0.9	0.9
$Pr(c_1)$	0.1	0.1	0.5	0.9	0.1	0.1	0.5	0.9	0.1	0.1
$Pr(c_2)$	0.46	0.46	0.3	0.14	0.82	0.82	0.5	0.18	0.86	0.86

Parameter	Green dashed	Green dotted	Purple	Purple light	Purple dashed	Purple dotted	Black	Black dashed	Black dotted
$Pr(e\|c_1, \neg c_2)$	0.85	0.85	0.85	0.6	0.85	0.85	0.85	0.85	0.85
$Pr(c_2\|c_1)$	0.5	0.5	0.5	0.5	0.5	0.5	0.1	0.5	0.9
$Pr(c_2\|\neg c_1)$	0.9	0.9	0.1	0.1	0.1	0.1	0.1	0.5	0.9
$Pr(c_1)$	0.53[2]	0.898[3]	0.1	0.1	0.5	0.9	0.098[4]	0.1	0.1
$Pr(c_2)$	0.7	0.54	0.14	0.14	0.3	0.46	0.1	0.5	0.9

Table 7: Parameter settings for sensitivity functions from Figure 15

We observe that the largest propagation effects happen when the noisy-OR parameter $x = Pr(e|\neg c_1, c_2)$ is large. Since one should especially focus on probabilities for $x = Pr(e|\neg c_1, c_2) \geq 0.6$, we note large propagation effects are possible for a wide range of parameter settings. To better investigate the sensitivity functions from Equation (25), we compute its first derivative and make some plots with different input parameters $Pr(e|c_1, \neg c_2), Pr(c_2|c_1), Pr(c_2|\neg c_1)$, and $Pr(c_1)$. The sensitivity functions corresponding to Equation (25) are a fragment of a third-quadrant hyperbola branch, and as a consequence, we know that the first derivative $\frac{d}{dx} Pr(c_2|\neg e)(x) < 0$ for all $x \in [0, 1]$.

[2] We choose 0.53 because otherwise this function would overlap with the purple function.
[3] We choose 0.898 because otherwise this function would overlap with the purple dashed function.
[4] We choose 0.098 because otherwise this function would overlap with the red dashed function.

Corollary 4.11.1. *The first derivative of the sensitivity function from Equation (25) is:*

$$\frac{d}{dx}Pr(c_2|\neg e)(x) = \frac{\frac{Pr(\neg c_2|c_1)Pr(c_1)\left(1-Pr(e|c_1,\neg c_2)\right)}{Pr(e|c_1,\neg c_2)Pr(c_2|c_1)Pr(c_1)-Pr(c_2)}}{\left(x+\frac{Pr(c_2)+Pr(\neg c_2|c_1)Pr(c_1)\left(1-Pr(e|c_1,\neg c_2)\right)}{Pr(e|c_1,\neg c_2)Pr(c_2|c_1)Pr(c_1)-Pr(c_2)}\right)^2}$$
$$= \frac{\alpha}{(x+\gamma+\alpha)^2} \tag{26}$$

where $\gamma = \frac{Pr(c_2)}{Pr(e|c_1,\neg c_2)Pr(c_2|c_1)Pr(c_1)-Pr(c_2)}$ *and* $\alpha = \frac{Pr(\neg c_2|c_1)Pr(c_1)\left(1-Pr(e|c_1,\neg c_2)\right)}{Pr(e|c_1,\neg c_2)Pr(c_2|c_1)Pr(c_1)-Pr(c_2)}$.

For specific parameter settings for some of the functions demonstrated demonstrated in Figure 15, namely the green (light), red (light) and blue (light) function, we plot the derivatives of Equation (26). See figure 16.

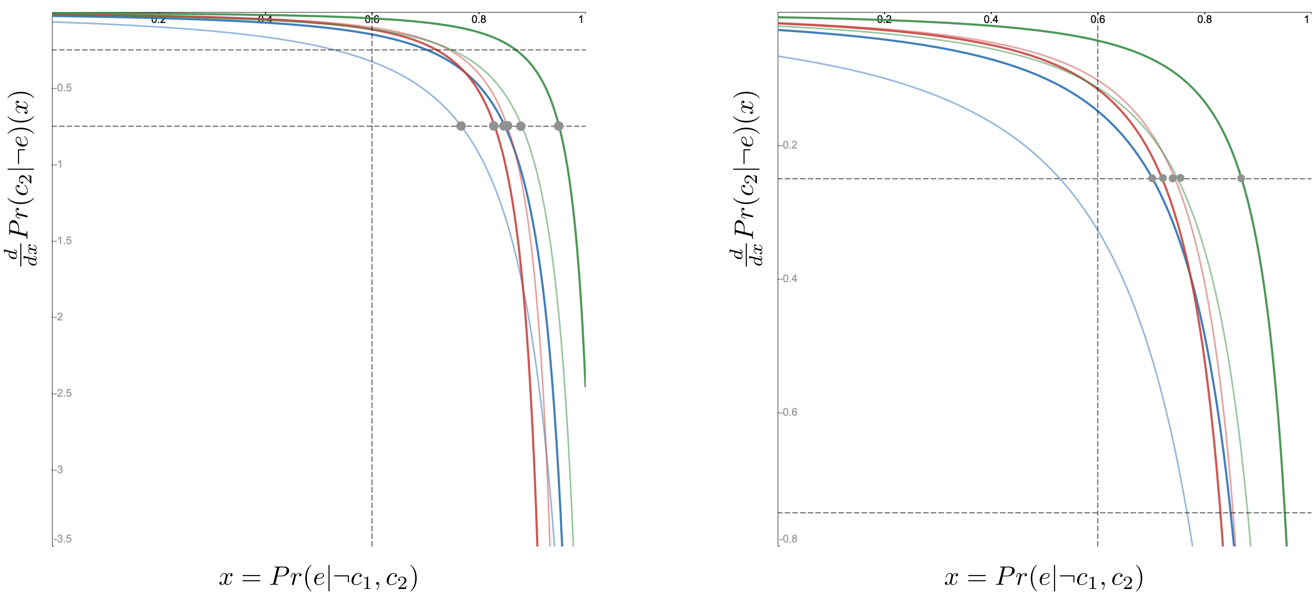

Figure 16: Several examples of Equation (26) restricted to the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_2|\neg e)(x) \in [-3.5, 0]$ (left) and the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_2|\neg e)(x) \in [-0.8, 0]$ (right). (See Table 7 for parameter settings)

We observe that the red (light), blue (light) and green (light) function all provide large propagation effects in the interval $x \in [0.6, 1]$. We conclude that the propagation effects in the interval $x \in [0.6, 1]$ become large for a wide range of parameter settings.

When we investigate the possible maximum value of $\frac{d}{dx}Pr(c_2|\neg e)(x)$ in the interval $x = Pr(e|\neg c_1, c_2) \in [0.6, 1]$, we find that the derivative of $\frac{d}{dx}Pr(c_2|\neg e)(x)$ can go to infinity for some parameter settings. Some functions of Equation (25) approach the vertical asymptote, and thus $\frac{d}{dx}Pr(c_2|\neg e)(x)$ can go to infinity. The propagation effects can become extremely large.

Studying the effect of possibly dependent cause variables on the propagation effects in the diagnostic direction lead to similar results as in Section 4.1.3. Namely, when examining $Pr(c_2|e)$ as a function of the probability $Pr(e|c_i, \neg c_j)$ where $i, j = 1, 2$, and keeping in mind the underlying properties of the noisy-OR model, we found that the propagation effects in the diagnostic direction due to inaccurate estimates of a noisy-OR parameter can become moderate at most. We discovered that the derivative can have *again* a maximum value of $|0.416666|$. On the other hand, the results in Section 4.2.2.3 show that the propagation effects in the diagnostic direction due to inaccurate estimates of a noisy-OR parameter can become large for a wide range of parameter settings when examining $Pr(c_2|\neg e)$ as a function of the probability $Pr(e|\neg c_1, c_2)$. These results align with the findings demonstrated in Section 4.1.3.3. We discovered again that the propagation effects in the diagnostic direction are highly dependent on which output probability we look at and which noisy-OR parameter we vary. We conclude that the presence of possibly dependent cause variables does not lead to remarkably different propagation effects in the diagnostic direction.

4.3 Propagation effects due to leaky noisy-OR parameter changes: independent causes

We will continue our research with the conditional probability tables for the three variables of the basic mechanism from Figure 3 again. However, we will now assume that the property of accountability is not satisfied, meaning that $Pr(e|\neg c_1, \neg c_2) = p$, where $p > 0$. We will use the leaky noisy-OR model described in Section 2.4. When evaluating the corresponding propagation effects, we assume the following:

- The prior probability distributions for cause variables C_1 and C_2 are non-degenerate, that is $Pr(c_i) \neq 0$ and $Pr(\neg c_i) \neq 0$ for $i = 1, 2$;
- A leak-probability typically attains a small value in practise [1]. Because of this, we mainly focus for $p \in (0, 0.2]$ in our research;
- Because leaky noisy-OR parameters are assumed to be large [5], we mainly focus on $Pr(e|c_1, \neg c_2), Pr(e|\neg c_1, c_2) \in [0.6, 1]$ in our research. We specifically use this constraint when evaluating the propagation effects.

We will carry on with the gradation of the gradient described in Section 4.1.1 of a sensitivity function under study. We consider the gradient ∇ to be small when $|\nabla| \leq 0.25$, moderate when $|\nabla| \in (0.25, 0.75)$, and large when $|\nabla| \geq 0.75$.

4.3.1 Propagation effects in the causal direction

We begin by investigating the possible effects on the probability $Pr(e)$ due to changes in a leaky noisy-OR parameter.

Theorem 4.12. *Consider the causal mechanism in Figure 3 and assume it models a leaky noisy-OR. Let $x = Pr(e|\neg c_1, c_2)$ be the leaky noisy-OR parameter associated with cause C_2. Then the sensitivity function $Pr(e)(x)$ has the following form:*

$$\begin{aligned} Pr(e)(x) = {} & xPr(c_2)\Big(\frac{Pr(c_1)(1 - Pr(e|c_1, \neg c_2))}{1-p} + Pr(\neg c_1)\Big) + \\ & \frac{Pr(c_1)Pr(c_2)\Big(Pr(e|c_1, \neg c_2) - p\Big)}{1-p} + Pr(e|c_1, \neg c_2)Pr(c_1)Pr(\neg c_2) + pPr(\neg c_1)Pr(\neg c_2)) \end{aligned} \tag{27}$$

Proof: Using the leaky noisy-OR model, we compute that:

$$\begin{aligned} Pr(e|c_1, c_2) &= 1 - (1-p)\Big(\frac{(1 - Pr(e|\neg c_1, c_2)}{1-p}\frac{(1 - Pr(e|c_1, \neg c_2)}{1-p}\Big) \\ &= 1 - (1-p)\Big(\frac{(1-x)(1 - Pr(e|c_1, \neg c_2)}{(1-p)^2}\Big) \\ &= 1 - \left(\frac{1 - Pr(e|c_1, \neg c_2) - x + xPr(e|c_1, \neg c_2)}{1-p}\right) \\ &= \frac{1-p}{1-p} - \frac{1 - Pr(e|c_1, \neg c_2) - x + xPr(e|c_1, \neg c_2)}{1-p} \\ &= \frac{Pr(e|c_1, \neg c_2) + x - xPr(e|c_1, \neg c_2) - p}{1-p} \end{aligned} \tag{28}$$

Thus we have $Pr(e)(x) =$

$$= \left(\frac{Pr(e|c_1,\neg c_2) + x - xPr(e|c_1,\neg c_2) - p}{1-p}\right)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2) + Pr(e|c_1,\neg c_2)Pr(c_1)Pr(\neg c_2)$$
$$+ pPr(\neg c_1)Pr(\neg c_2)$$
$$= \frac{Pr(e|c_1,\neg c_2)Pr(c_1)Pr(c_2)}{1-p} + \frac{xPr(c_1)Pr(c_2)}{1-p} - \frac{xPr(e|c_1,\neg c_2)Pr(c_1)Pr(c_2)}{1-p} - \frac{pPr(c_1)Pr(c_2)}{1-p} + xPr(\neg c_1)Pr(c_2)$$
$$+ Pr(e|c_1,\neg c_2)Pr(c_1)Pr(\neg c_2) + pPr(\neg c_1)Pr(\neg c_2)$$
$$= x\left(\frac{Pr(c_1)Pr(c_2)}{1-p} - \frac{Pr(e|c_1,\neg c_2)Pr(c_1)Pr(c_2)}{1-p} + Pr(\neg c_1)Pr(c_2)\right) + \frac{Pr(e|c_1,\neg c_2)Pr(c_1)Pr(c_2)}{1-p} - \frac{pPr(c_1)Pr(c_2)}{1-p}$$
$$+ Pr(e|c_1,\neg c_2)Pr(c_1)Pr(\neg c_2) + pPr(\neg c_1)Pr(\neg c_2)$$
$$= xPr(c_2)\left(\frac{Pr(c_1)}{1-p} - \frac{Pr(e|c_1,\neg c_2)Pr(c_1)}{1-p} + Pr(\neg c_1)\right) + \frac{Pr(c_1)Pr(c_2)\Big(Pr(e|c_1,\neg c_2) - p\Big)}{1-p}$$
$$+ Pr(e|c_1,\neg c_2)Pr(c_1)Pr(\neg c_2) + pPr(\neg c_1)Pr(\neg c_2)$$
$$= xPr(c_2)\left(\frac{Pr(c_1)(1 - Pr(e|c_1,\neg c_2))}{1-p} + Pr(\neg c_1)\right) + \frac{Pr(c_1)Pr(c_2)\Big(Pr(e|c_1,\neg c_2) - p\Big)}{1-p} + Pr(e|c_1,\neg c_2)Pr(c_1)Pr(\neg c_2)$$
$$+ pPr(\neg c_1)Pr(\neg c_2). \quad \square$$

Observation: Equation (27) shows that strong effects on the output probability $Pr(e)$ can be expected only if both $Pr(c_2)$ and $Pr(\neg c_1)$ have a large value, that is, cause C_2 is likely to be present and C_1 absent. Furthermore, one can observe that substituting $p = 0$ in Equation (27) will result in Equation (7) again.
Note that the parameter $Pr(e|c_1,\neg c_2)$ is a leaky noisy-OR parameter, which is assumed to be large. In addition, the leak probability p is small, thus we establish that the term $\frac{Pr(c_1)(1-Pr(e|c_1,\neg c_2))}{1-p}$ will be small in any case.
Since $Pr(e)$ is a probability and Equation (27) is a linear function we note that the gradient cannot become larger than 1. Also, since the size of the gradient is mainly determined by $Pr(\neg c_1)$ and $Pr(c_2)$, the leak-probability p possesses a weak additional influence on the propagation effects compared to the noisy-OR model (without the leak-probability). However, if we must say whether the addition of a leak-probability has a positive/negative influence on the propagation effects, we can say that the propagation effects are *slightly* stronger since we have that the larger the leak-probability p, the larger the the gradient.

Analogous observations hold for the function obtained for the probability of interest $Pr(e)$ if $x = Pr(e|c_1,\neg c_2)$.

4.3.2 Propagation effects in the diagnostic direction

Now we investigate the consequences of changes in a leaky noisy-OR parameter upon propagation in the diagnostic direction. If we write $Pr(c_2|e)$ as a function of the probability $Pr(e|\neg c_1, c_2)$, the result is a hyperbolic function in x. The corresponding sensitivity function is:

$$Pr(c_2|e)(x) = \frac{Pr(c_2,e)(x)}{Pr(c_2,e)(x) + Pr(\neg c_2,e)(x)}$$
$$= \frac{Pr(e|c_1,c_2)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2)}{Pr(c_2,e)(x) + Pr(\neg c_2,e)(x)} \text{ (where } Pr(e|c_1,c_2) \text{ is dependent of } x)$$
$$= \frac{Pr(e|c_1,c_2)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2)}{Pr(c_2,e)(x) + Pr(e|c_1,\neg c_2)Pr(c_1)(Pr(\neg c_2) + Pr(e|\neg c_1,\neg c_2)Pr(\neg c_1)Pr(\neg c_2)} \;(Pr(e|c_1,c_2) \text{ is dependent of } x)$$
$$= \frac{Pr(e|c_1,c_2)Pr(c_1)Pr(c_2) + xPr(\neg c_1)Pr(c_2)}{Pr(c_2,e)(x) + Pr(e|c_1,\neg c_2)Pr(c_1)(Pr(\neg c_2) + pPr(\neg c_1)Pr(\neg c_2)} \text{ (where } Pr(e|c_1,c_2) \text{ is dependent of } x)$$

We note an important difference from function (13) (obtained with the noisy-OR model), namely, the denominator now includes the extra term $pPr(\neg c_1)Pr(\neg c_2)$.

Theorem 4.13. *Consider the causal mechanism in Figure 3 and assume it models a leaky noisy-OR. Let $x = Pr(e|\neg c_1, c_2)$ be the leaky noisy-OR parameter associated with cause C_2. Then the sensitivity function $Pr(c_2|e)(x)$ has the following form:*

$$Pr(c_2|e)(x) = \frac{x+\gamma}{x+\gamma+\zeta} \tag{29}$$

where $\gamma = \frac{Pr(e|c_1,\neg c_2)-p}{\frac{1}{Pr(c_1)}\left(1-pPr(\neg c_1)\right)-Pr(e|c_1,\neg c_2)}$ *and* $\zeta = \frac{\frac{Pr(\neg c_2)}{Pr(c_2)}\left(Pr(e|c_1,\neg c_2)+p\frac{Pr(\neg c_1)}{Pr(c_1)}\right)}{\frac{Pr(\neg c_1)}{Pr(c_1)}+\frac{1-Pr(e|c_1,\neg c_2)}{(1-p)}}$.

Proof: We have:

$$\begin{aligned}
Pr(c_2|e)(x) &= \frac{\left(\frac{Pr(e|c_1,\neg c_2)+x-xPr(e|c_1,\neg c_2)-p)}{1-p}\right)Pr(c_1)Pr(c_2)+xPr(\neg c_1)Pr(c_2)}{Pr(e)(x)}\\
&= \frac{\frac{Pr(e|c_1,\neg c_2)Pr(c_1)Pr(c_2)+xPr(c_1)Pr(c_2)-xPr(e|c_1,\neg c_2)Pr(c_1)Pr(c_2)-pPr(c_1)Pr(c_2)}{1-p}+xPr(\neg c_1)Pr(c_2)}{Pr(e)(x)}
\end{aligned}$$

We have that $Pr(e)(x) = Pr(e|c_1, c_2)Pr(c_1)Pr(c_2)+xPr(\neg c_1)Pr(c_2)+Pr(e|c_1, \neg c_2)Pr(c_1)Pr(\neg c_2)+pPr(\neg c_1)Pr(\neg c_2)$. We substitute $Pr(e|c_1, c_2)$ with $\frac{Pr(e|c_1,\neg c_2)+x-xPr(e|c_1,\neg c_2)-p}{1-p}$ and divide the denominator and numerator by $Pr(c_1)Pr(c_2)$ and get:

$$\begin{aligned}
Pr(c_2|e)(x) &= \frac{x\left(\frac{1-Pr(e|c_1,\neg c_2)}{1-p}+\frac{Pr(\neg c_1)}{Pr(c_1)}\right)+\frac{Pr(e|c_1,\neg c_2)-p}{1-p}}{x\left(\frac{1-Pr(e|c_1,\neg c_2)}{1-p}+\frac{Pr(\neg c_1)}{Pr(c_1)}\right)+\frac{Pr(e|c_1,\neg c_2)-p}{1-p}+Pr(e|c_1,\neg c_2)\frac{Pr(\neg c_2)}{Pr(c_2)}+p\frac{Pr(\neg c_1)Pr(\neg c_2)}{Pr(c_1)Pr(c_2)}}\\
&= \frac{x+\frac{Pr(e|c_1,\neg c_2)-p}{\frac{1}{Pr(c_1)}-Pr(e|c_1,\neg c_2)-p\frac{Pr(\neg c_1)}{Pr(c_1)}}}{x+\frac{Pr(e|c_1,\neg c_2)-p}{\frac{1}{Pr(c_1)}-Pr(e|c_1,\neg c_2)-p\frac{Pr(\neg c_1)}{Pr(c_1)}}+\frac{\frac{Pr(\neg c_2)}{Pr(c_2)}\left(Pr(e|c_1,\neg c_2)+p\frac{Pr(\neg c_1)}{Pr(c_1)}\right)}{\frac{Pr(\neg c_1)}{Pr(c_1)}+\frac{1-Pr(e|c_1,\neg c_2)}{(1-p)}}}\\
&\text{by dividing by } \left(\frac{1-Pr(e|c_1,\neg c_2)}{1-p}+\frac{Pr(\neg c_1)}{Pr(c_1)}\right) \text{ and rearrangement. We finally obtain:}\\
&= \frac{x+\gamma}{x+\gamma+\zeta}
\end{aligned}$$

where $\gamma = \frac{Pr(e|c_1,\neg c_2)-p}{\frac{1}{Pr(c_1)}\left(1-pPr(\neg c_1)\right)-Pr(e|c_1,\neg c_2)}$ and $\zeta = \frac{\frac{Pr(\neg c_2)}{Pr(c_2)}\left(Pr(e|c_1,\neg c_2)+p\frac{Pr(\neg c_1)}{Pr(c_1)}\right)}{\frac{Pr(\neg c_1)}{Pr(c_1)}+\frac{1-Pr(e|c_1,\neg c_2)}{(1-p)}}$. □

Observation: Equation (29) is hyperbolic in the probability $x = Pr(e|\neg c_1, c_2)$. We use the properties of hyperbolic functions described in Section 2.5, and discover that the vertical asymptote of Equation (29) lies at $x = s = -(\gamma+\zeta)$. Since $\gamma + \zeta > 0$, the asymptote is located to the left of the unit window and the horizontal asymptote lies at $t = 1$. As a result, we find that Equation (29) is a fragment of a fourth-quadrant hyperbola branch.

The effect of deviations in the $x-$value on the output probability of interest mainly depend on the location of the vertex of the corresponding hyperbola branch. We have that the closer the vertex of the fourth-quadrant hyperbola branch lies to the upper-left corner of the unit window, that is the point $(0, 1)$, the larger the propagation effects. We find that Equation (29) has its vertex at:

$$(s+\sqrt{|r|}, t-\sqrt{|r|}) = \Big(-(\gamma+\zeta)+\sqrt{|\zeta|}, 1-\sqrt{|\zeta|}\Big)$$

The vertex is located within the unit window for some values of γ, ζ with $(\gamma+\zeta) < \sqrt{|\zeta|} < 1$. To obtain $(\gamma+\zeta) < \sqrt{|\zeta|}$ given that $(\gamma + \zeta) > \zeta$, we discover that rather small values of ζ will result in a vertex with a positive $x-$coordinate.

The vertex only approaches the upper-left corner of the unit window, if in addition, the difference $(\gamma+\zeta)-\gamma=\zeta$ is rather small. For $\zeta \to 0$, we observe that $Pr(c_1)$ should be small and $Pr(c_2)$ large. For $\gamma \to 0$ we need that $Pr(c_1)$ should be small as well. Unfortunately, it becomes theoretically quite difficult to say something about the effect of p on the location of the vertex within the unit window. In order to get a better idea of what the influence is of the value of p on the output probability $Pr(c_2|e)$, we make some plots of Equation (29) with different input parameters $Pr(c_1), Pr(c_2)$, and p. We set $Pr(e|c_1, \neg c_2)$ equal to 0.8. See Figure 17.

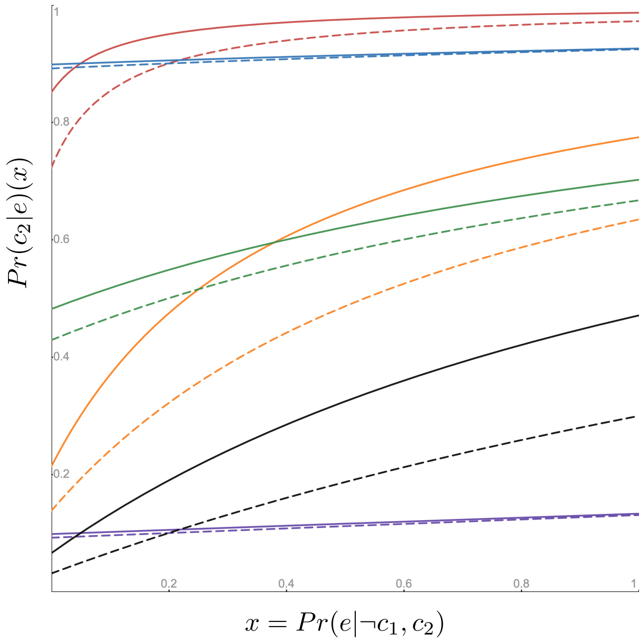

Figure 17: Several example sensitivity functions adhering to Theorem 4.13. (See Table 8 for parameter settings)

Parameter	Red	Red dashed	Green	Green dashed	Purple	Purple dashed	Orange	Orange dashed	Black	Black dashed	Blue	Blue dashed
$Pr(e|c_1, \neg c_2)$	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
$Pr(c_1)$	0.1	0.1	0.5	0.5	0.9	0.9	0.1	0.1	0.1	0.1	0.9	0.9
$Pr(c_2)$	0.9	0.9	0.5	0.5	0.1	0.1	0.3	0.3	0.1	0.1	0.9	0.9
p	0.05	0.2	0.05	0.2	0.05	0.2	0.05	0.2	0.05	0.2	0.05	0.2

Table 8: Parameter settings for sensitivity functions from Figure 17

We observe that Equation (29) demonstrates that the largest propagation effects in the entire interval $x \in [0,1]$ on the output probability $Pr(c_2|e)$ can be expected based on the following information:

- The influence of the leak-probability p on the propagation effects is dependent of the other parameters. However, this influence is relatively small;
- The *strongest* propagation effects occur when $Pr(c_1)$ is small and $Pr(c_2)$ is small/moderate.

We indeed observe that the vertex only approaches the upper-left corner of the unit window if $Pr(c_1)$ is small and $Pr(c_2)$ large, see red (dashed) function. However, as one can see in Figure 17, this parameter setting will not lead to the largest propagation effects in the entire interval $[0,1]$. For example, the orange (dashed) functions leads to larger propagation effects in the entire interval $[0,1]$ than, for example, the red (dashed) function. In addition, we note that the leak-probability p exercises only a minor influence on the propagation effects on the output probability $Pr(c_2|e)$. The propagation effects are in line with the results obtained in Section 4.1.3.1, where the leak-probability

p is absent.

To gain better insight into the behaviour of Equation (29) in the interval $x = Pr(e|\neg c_1, c_2) \geq 0.6$, we compute its first derivative. The sensitivity functions corresponding to Equation (29) are a fragment of a fourth-quadrant hyperbola branch indicating that the first derivative $\frac{d}{dx}Pr(c_2|e)(x) > 0$ for all $x \in [0,1]$.

Corollary 4.13.1. *The first derivative of the sensitivity function from Equation (29) is:*

$$\frac{d}{dx}Pr(c_2|e)(x) = \frac{\frac{\frac{Pr(\neg c_2)}{Pr(c_2)}\left(Pr(e|c_1,\neg c_2)+p\frac{Pr(\neg c_1)}{Pr(c_1)}\right)}{\frac{Pr(\neg c_1)}{Pr(c_1)}+\frac{1-Pr(e|c_1,\neg c_2)}{(1-p)}}}{\left(x+\frac{Pr(e|c_1,\neg c_2)-p}{\frac{1}{Pr(c_1)}\left(1-pPr(\neg c_1)\right)-Pr(e|c_1,\neg c_2)}+\frac{\frac{Pr(\neg c_2)}{Pr(c_2)}\left(Pr(e|c_1,\neg c_2)+p\frac{Pr(\neg c_1)}{Pr(c_1)}\right)}{\frac{Pr(\neg c_1)}{Pr(c_1)}+\frac{1-Pr(e|c_1,\neg c_2)}{(1-p)}}\right)^2}$$
$$= \frac{\zeta}{(x+\gamma+\zeta)^2} \tag{30}$$

where $\gamma = \frac{Pr(e|c_1,\neg c_2)-p}{\frac{1}{Pr(c_1)}\left(1-pPr(\neg c_1)\right)-Pr(e|c_1,\neg c_2)}$ *and* $\zeta = \frac{\frac{Pr(\neg c_2)}{Pr(c_2)}\left(Pr(e|c_1,\neg c_2)+p\frac{Pr(\neg c_1)}{Pr(c_1)}\right)}{\frac{Pr(\neg c_1)}{Pr(c_1)}+\frac{1-Pr(e|c_1,\neg c_2)}{(1-p)}}$.

For specific parameter settings for some of the functions demonstrated demonstrated in Figure 17, namely the orange (dashed), black (dashed) and green (dashed) function, we plot the derivatives of Equation (30). See figure 18.

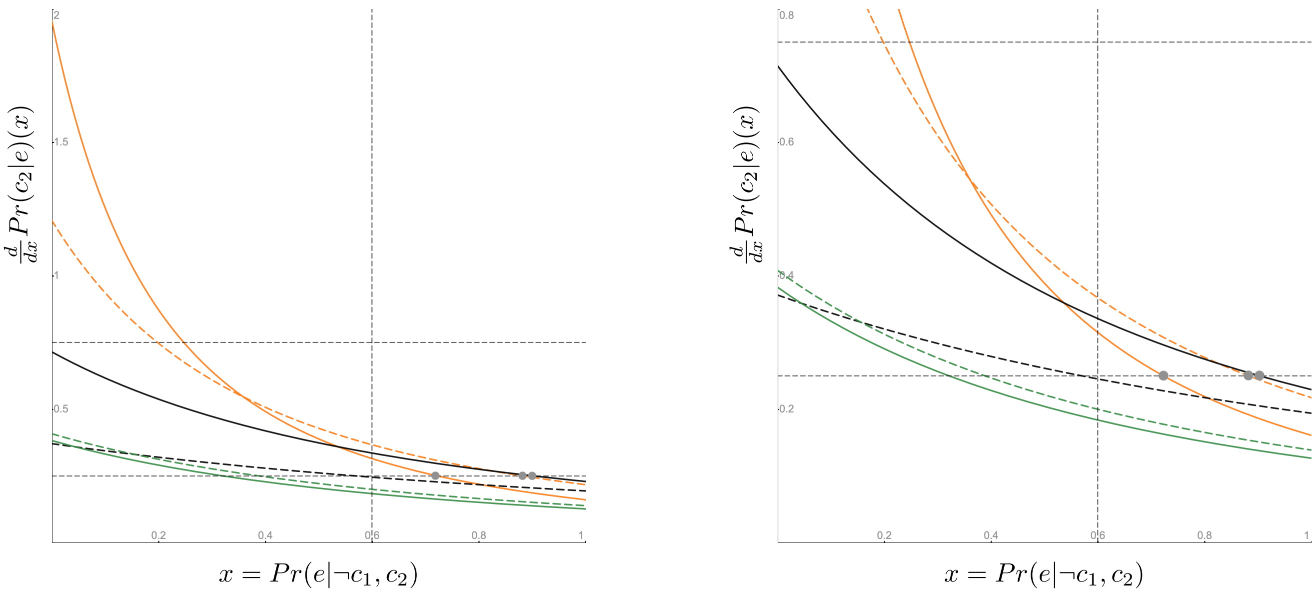

Figure 18: Several examples of Equation (30) restricted to the window $x \in [0,1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [0,2]$ (left) and the window $x \in [0,1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [0,0.8]$ (right). (See Table 8 for parameter settings)

In Figure 18, we observe moderate and small propagation effects for the orange (dashed) and black dashed function. We again observe, when only concentrating on the interval $x \in [0.6, 1]$, that the effect of the leak is small.

With the help of WOLFRAM MATHEMATICA, we find a maximum of max $\frac{d}{dx}Pr(c_2|e)(x) = 0.416666$ in the interval $x = Pr(e|\neg c_1, c_2) \in [0.6, 1]$ of Equation (30) with several parameter settings (see Appendix C.1). For example, one of the parameter settings is:

$$Pr(e|c_1,\neg c_2) = 0.823118, Pr(c_1) = 2.08182 \cdot 10^{-7}, Pr(c_2) = 0.140771 \text{ and } p = 0.140771.$$

This maximum lies at $x = 0.6$ and is the same as the result obtained in Section 4.1.3.1. We again discover that our findings concerning the propagation effects of Equation (29) in the interval $x = Pr(e|\neg c_1, c_2) \in [0.6, 1]$ are moderate or small, which are in line with the results derived in Section 4.1.3.1.

We have examined the consequences of the addition of a leak-probability p on a deviating noisy-OR parameter upon the causal and diagnostic propagation through the basic mechanism from Figure 3. We observed that the influence of the leak-probability p, where $p \in (0, 0.2]$, in both cases is very little. The propagation effects in the causal direction are *slightly* larger when the property of accountability is not satisfied, since we have that the larger the leak probability p, the larger the gradient. The effect of the leak is negligible when examining the propagation effects in the diagnostic direction; the propagation effects are again mainly determined by which output probability we look at and which noisy-OR parameter we vary. We conclude that the overall propagation effects are in line with the results obtained in Section 4.1.

4.4 Causal mechanisms with multiple cause variables: independent causes

Up to this point, we have focused on two-cause mechanisms only. We will now continue our research to mechanisms involving three cause variables C_1, C_2, C_3 and their common effect variable E, see Figure 19. Given these three cause variables, the conditional probability table for E now contains eight probabilities. Using the noisy-OR model for the variable E, we now have that the values for three of these probabilities must be explicitly specified, namely the noisy-OR parameters associated with causes C_1, C_2, C_3. These are the conditional probabilities for the occurrence of the effect e arising in the presence of only one of the three causes.

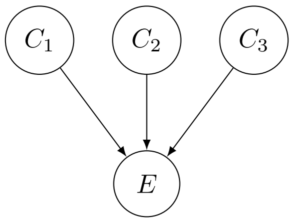

Figure 19: A causal mechanism with the effect variable E and cause variables C_1, C_2, C_3.

We first assume the following:

- The prior probability distributions for cause variables C_1, C_2 and C_3 are non-degenerate, that is $Pr(c_i) \neq 0$ and $Pr(\neg c_i) \neq 0$ for $i = 1, 2, 3$;
- $Pr(e|\neg c_1, \neg c_2, \neg c_3) = 0$, by the property of accountability [1];
- Because noisy-OR parameters are assumed to be large [5], we mainly focus on $Pr(e|c_1, \neg c_2, \neg c_3)$, $Pr(e|\neg c_1, c_2, \neg c_3)$ $Pr(e|\neg c_1, \neg c_2, c_3) \in [0.6, 1]$ in our research. We specifically use this constraint when evaluating the propagation effects.

We have that probability $Pr(e)$ equals:

$$\begin{aligned} Pr(e) = {} & Pr(e|c_1, c_2, c_3)Pr(c_1)Pr(c_2)Pr(c_3) + Pr(e|\neg c_1, c_2, c_3)Pr(\neg c_1)Pr(c_2)Pr(c_3) + Pr(e|c_1, \neg c_2, c_3)Pr(c_1)Pr(\neg c_2)Pr(c_3) \\ & + Pr(e|c_1, c_2, \neg c_3)Pr(c_1)Pr(c_2)Pr(\neg c_3) + Pr(e|\neg c_1, \neg c_2, c_3)Pr(\neg c_1)Pr(\neg c_2)Pr(c_3) \\ & + Pr(e|\neg c_1, c_2, \neg c_3)Pr(\neg c_1)Pr(c_2)Pr(\neg c_3) + Pr(e|c_1, \neg c_2, \neg c_3)Pr(c_1)Pr(\neg c_2)Pr(\neg c_3) \\ & + Pr(e|\neg c_1, \neg c_2, \neg c_3)Pr(\neg c_1)Pr(\neg c_2)Pr(\neg c_3) \qquad (31) \end{aligned}$$

Note that the last term $Pr(e|\neg c_1, \neg c_2, \neg c_3)Pr(\neg c_1)Pr(\neg c_2)Pr(\neg c_3) = 0$ since $Pr(e|\neg c_1, \neg c_2, \neg c_3) = 0$.

Furthermore, we carry on with the gradation of the gradient described in Section 4.1.1 of a sensitivity function under study. We consider the gradient ∇ to be small when $|\nabla| \leq 0.25$, moderate when $|\nabla| \in (0.25, 0.75)$, and large when $|\nabla| \geq 0.75$.

4.4.1 Propagation effects in the causal direction

We begin by investigating the possible effects on the probability $Pr(e)$ due to changes in a noisy-OR parameter.

Theorem 4.14. *Consider the causal mechanism in Figure 19 and assume it models a noisy-OR. Let* $x = Pr(e|c_1, \neg c_2, \neg c_3)$ *be the noisy-OR parameter associated with cause* C_1*. Then the sensitivity function* $Pr(e)(x)$ *has the following form:*

$$Pr(e)(x) = xPr(c_1)\Big(1 + \beta\gamma Pr(c_2)Pr(c_3) - \beta Pr(c_2) - \gamma Pr(c_3)\Big) + \beta Pr(c_2) + \gamma Pr(c_3) - \beta\gamma Pr(c_2)Pr(c_3) \quad (32)$$

where $\beta = Pr(e|\neg c_1, c_2, \neg c_3)$ *and* $\gamma = Pr(e|\neg c_1, \neg c_2, c_3)$.

Proof:
We first focus on the remaining four probabilities of the conditional probability table, that are: $Pr(e|c_1, c_2, c_3), Pr(e|\neg c_1, c_2, c_3), Pr(e|c_1, \neg c_2, c_3)$ and $Pr(e|c_1, c_2, \neg c_3)$. These probabilities can be computed from the noisy-OR model. For simplicity, we write the noisy-OR parameters associated with cause C_2 and C_3 as:

$Pr(e|\neg c_1, c_2, \neg c_3) = \beta$
$Pr(e|\neg c_1, \neg c_2, c_3) = \gamma$

Using the noisy-OR model we first compute $Pr(e|c_1, c_2, c_3)$:

$$\begin{aligned}
Pr(e|c_1, c_2, c_3) &= 1 - \big((1 - Pr(e|c_1, \neg c_2, \neg c_3))(1 - Pr(e|\neg c_1, c_2, \neg c_3))(1 - Pr(e|\neg c_1, \neg c_2, c_3))\big) \\
&= 1 - \big((1 - Pr(e|c_1, \neg c_2, \neg c_3))(1 - \beta)(1 - \gamma)\big) \\
&= 1 - \big((1 - x)(1 - \beta)(1 - \gamma)\big) \\
&= 1 - \big((1 - x)(1 - \beta - \gamma + \beta\gamma)\big) \\
&= 1 - (1 - \beta - \gamma + \beta\gamma - x + x\beta + x\gamma - x\beta\gamma) \\
&= \beta + \gamma - \beta\gamma + x - x\beta - x\gamma + x\beta\gamma) \\
&= x(1 - \gamma - \beta + \beta\gamma) + \beta + \gamma - \beta\gamma
\end{aligned} \quad (33)$$

Secondly, we compute $Pr(e|\neg c_1, c_2, c_3)$:

$$\begin{aligned}
Pr(e|\neg c_1, c_2, c_3) &= 1 - \big((1 - Pr(e|\neg c_1, c_2, \neg c_3))(1 - Pr(e|\neg c_1, \neg c_2, c_3)\big) \\
&= 1 - \big((1 - \beta)(1 - \gamma\big) \\
&= 1 - \big(1 - \beta - \gamma + \beta\gamma)\big) \\
&= \beta + \gamma - \beta\gamma \\
&= \beta(1 - \gamma) + \gamma
\end{aligned} \quad (34)$$

Thirdly, we compute $Pr(e|c_1, \neg c_2, c_3)$:

$$\begin{aligned}
Pr(e|c_1, \neg c_2, c_3) &= 1 - \big((1 - Pr(e|c_1, \neg c_2, \neg c_3))(1 - Pr(e|\neg c_1, \neg c_2, c_3)\big) \\
&= 1 - \big((1 - x)(1 - \gamma)\big) \\
&= 1 - \big(1 - \gamma - x + x\gamma)\big) \\
&= x - x\gamma + \gamma \\
&= x(1 - \gamma) + \gamma
\end{aligned} \quad (35)$$

Finally, we compute $Pr(e|c_1, c_2, \neg c_3)$:

$$\begin{aligned}
Pr(e|c_1, c_2, \neg c_3) &= 1 - \big((1 - Pr(e|c_1, \neg c_2, \neg c_3))(1 - Pr(e|\neg c_1, c_2, \neg c_3)\big) \\
&= 1 - \big((1 - x)(1 - \beta)\big) \\
&= 1 - \big(1 - \beta - x + x\beta)\big) \\
&= x(1 - \beta) + \beta
\end{aligned} \quad (36)$$

From Equation (31), we substitute $Pr(e|c_1, c_2, c_3), Pr(e|\neg c_1, c_2, c_3), Pr(e|c_1, \neg c_2, c_3)$ and $Pr(e|c_1, c_2, \neg c_3)$ with (33), (34), (35), and (36) respectively, which gives us:

$$\begin{aligned}Pr(e)(x) = &\Big(x(1-\gamma-\beta+\beta\gamma)+\beta+\gamma-\beta\gamma\Big)Pr(c_1)Pr(c_2)Pr(c_3) + \Big(\beta(1-\gamma)+\gamma\Big)Pr(\neg c_1)Pr(c_2)Pr(c_3)\\
&+\Big(x(1-\gamma)+\gamma\Big)Pr(c_1)Pr(\neg c_2)Pr(c_3) + \Big(x(1-\beta)+\beta\Big)Pr(c_1)Pr(c_2)Pr(\neg c_3) + \beta Pr(\neg c_1)Pr(c_2)Pr(\neg c_3)\\
&+\gamma Pr(\neg c_1)Pr(\neg c_2)Pr(c_3) + xPr(c_1)Pr(\neg c_2)Pr(\neg c_3)\end{aligned}$$

$$\begin{aligned}Pr(e)(x) = &\beta Pr(c_1)Pr(c_2)Pr(c_3) + \gamma Pr(c_1)Pr(c_2)Pr(c_3) - \beta\gamma Pr(c_1)Pr(c_2)Pr(c_3)\\
&+xPr(c_1)Pr(c_2)Pr(c_3) - \beta xPr(c_1)Pr(c_2)Pr(c_3) - \gamma xPr(c_1)Pr(c_2)Pr(c_3) + \beta\gamma xPr(c_1)Pr(c_2)Pr(c_3)\\
&+\beta Pr(\neg c_1)Pr(c_2)Pr(c_3) + \gamma Pr(\neg c_1)Pr(c_2)Pr(c_3) - \beta\gamma Pr(\neg c_1)Pr(c_2)Pr(c_3)\\
&+xPr(c_1)Pr(\neg c_2)Pr(c_3) - \gamma xPr(c_1)Pr(\neg c_2)Pr(c_3) + \gamma Pr(c_1)Pr(\neg c_2)Pr(c_3)\\
&+xPr(c_1)Pr(c_2)Pr(\neg c_3) - \beta xPr(c_1)Pr(c_2)Pr(\neg c_3) + \beta Pr(c_1)Pr(c_2)Pr(\neg c_3)\\
&+\beta Pr(\neg c_1)Pr(c_2)Pr(\neg c_3) + \gamma Pr(\neg c_1)Pr(\neg c_2)Pr(c_3) + xPr(c_1)Pr(\neg c_2)Pr(\neg c_3)\end{aligned}$$

We use that $Pr(i) + Pr(\neg i) = 1$ for $i = 1, 2, 3$, and simplify:

$$\begin{aligned}Pr(e)(x) = &\beta Pr(c_2)Pr(c_3) + \gamma Pr(c_2)Pr(c_3) - \beta\gamma Pr(c_2)Pr(c_3) + xPr(c_1)Pr(c_3) - \gamma xPr(c_1)Pr(c_3)\\
&+\gamma Pr(\neg c_2)Pr(c_3) + xPr(c_1)Pr(\neg c_3) + \beta Pr(c_2)Pr(\neg c_3) - \beta xPr(c_1)Pr(c_2) + \beta\gamma xPr(c_1)Pr(c_2)Pr(c_3)\end{aligned}$$

$$\begin{aligned}Pr(e)(x) = &\beta Pr(c_2) + \gamma Pr(c_3) - \beta\gamma Pr(c_2)Pr(c_3) + xPr(c_1) - \beta xPr(c_1)Pr(c_2) - \gamma xPr(c_1)Pr(c_3)\\
&+\beta\gamma xPr(c_1)Pr(c_2)Pr(c_3)\end{aligned}$$

And we finally obtain:

$$Pr(e)(x) = xPr(c_1)\Big(1 + \beta\gamma Pr(c_2)Pr(c_3) - \beta Pr(c_2) - \gamma Pr(c_3)\Big) + \beta Pr(c_2) + \gamma Pr(c_3) - \beta\gamma Pr(c_2)Pr(c_3)$$

where $\beta = Pr(e|\neg c_1, c_2, \neg c_3)$ and $\gamma = Pr(e|\neg c_1, \neg c_2, c_3)$. □

Observation: The gradient of Equation (32) equals $Pr(c_1)\Big(1+\beta\gamma Pr(c_2)Pr(c_3) - \beta Pr(c_2) - \gamma Pr(c_3)\Big)$ and is in the interval $(0, 1)$. The gradient is large when at least the probability $Pr(c_1)$ is large. Now to attain a large gradient, it is favourable that the probabilities $Pr(c_2)$ and $Pr(c_3)$ are small because we have that $\beta Pr(c_2) > \beta\gamma Pr(c_2)Pr(c_3)$ and $\gamma Pr(c_3) > \beta\gamma Pr(c_2)Pr(c_3)$. The reason for this is because we have assumed that $Pr(c_2), Pr(c_3) \in (0, 1)$, and thereby we have that the noisy-OR parameters β and γ are probabilities as well, meaning that they have a maximum value of 1. To express this more formally:

$0 < \beta\gamma Pr(c_2)Pr(c_3) < \beta Pr(c_2)$, and

$0 < \beta\gamma Pr(c_2)Pr(c_3) < \gamma Pr(c_3)$

then we trivially find that:

$\beta\gamma Pr(c_2)Pr(c_3) < \beta Pr(c_2) + \gamma Pr(c_3)$

Consequently, we find that larger probabilities for the noisy-OR parameters $Pr(e|\neg c_1, c_2, \neg c_3)$ and $Pr(e|\neg c_1, \neg c_2, c_3)$ provide a smaller gradient value. We conclude that a large gradient for Equation (32) will be obtained based on the following:

- $Pr(c_1)$ is large and $Pr(c_2), Pr(c_3)$ are small. In addition we have that the smaller the noisy-OR parameters associated with causes C_2 and C_3, the larger the gradient.

We discovered that the causal propagation effects with the three-cause mechanism (see Figure 19) are much in line with the effects found with the basic causal mechanism (see Section 4.1.1). However, we note that more parameters are present in the gradient of Equation (32) compared to the gradient of Equation (7). The gradient of Equation (7) consists of two prior probabilities and one noisy-OR parameter. The gradient of Equation (32) consists of three prior probabilities and two noisy-OR parameters. This means that the prior probabilities of the corresponding causes of the deviating noisy-OR parameter need to have more skewed prior probability distributions to attain the same propagation effects as for a causal mechanism involving two cause variables. We conclude that large propagation effects may still happen; however, an increase in the number of cause variables will possibly lead to less propagation effects.

4.4.2 Propagation effects in the diagnostic direction

We now examine the effects of a deviating noisy-OR value upon diagnostic propagation. We express the example posterior probability of interest $Pr(c_1|e)$ as a function of the value $x = Pr(e|c_1, \neg c_2, \neg c_3)$ for the probability under study.

Theorem 4.15. *Consider the causal mechanism in Figure 19 and assume it models a noisy-OR. Let $x = Pr(e|c_1, \neg c_2, \neg c_3)$ be the noisy-OR parameter associated with cause C_1. Then the sensitivity function $Pr(c_1|e)(x)$ has the following form:*

$$Pr(c_1|e)(x) = \frac{x + \frac{\zeta}{1-\zeta}}{x + \frac{\zeta}{Pr(c_1)(1-\zeta)}} \tag{37}$$

where $\zeta = \beta Pr(c_2) + \gamma Pr(c_3) - \beta\gamma Pr(c_2)Pr(c_3)$, and wherein $\beta = Pr(e|\neg c_1, c_2, \neg c_3)$ and $\gamma = Pr(e|\neg c_1, \neg c_2, c_3)$.

Proof:
We have for $x = Pr(e|c_1, \neg c_2, \neg c_3)$:

$$\begin{aligned} Pr(c_1|e)(x) &= \frac{Pr(c_1, e)(x)}{Pr(e)(x)} \\ &= \frac{Pr(e, c_1, c_2, c_3)(x) + Pr(e, c_1, \neg c_2, c_3)(x) + Pr(e, c_1, c_2, \neg c_3)(x) + Pr(e, c_1, \neg c_2, \neg c_3)(x)}{Pr(c_1, e)(x) + Pr(\neg c_1, e)} \\ &= \frac{Pr(e, c_1, c_2, c_3)(x) + Pr(e, c_1, \neg c_2, c_3)(x) + Pr(e, c_1, c_2, \neg c_3)(x) + Pr(e, c_1, \neg c_2, \neg c_3)(x)}{Pr(c_1, e)(x) + Pr(e, \neg c_1, c_2, c_3) + Pr(e, \neg c_1, \neg c_2, c_3) + Pr(e, \neg c_1, c_2, \neg c_3)} + 0 \end{aligned}$$

We use again that $\beta = Pr(e|\neg c_1, c_2, \neg c_3)$ and $\gamma = Pr(e|\neg c_1, \neg c_2, c_3)$, and substitute $Pr(e|c_1, c_2, c_3)$, $Pr(e|\neg c_1, c_2, c_3)$, $Pr(e|c_1, \neg c_2, c_3)$ and $Pr(e|c_1, c_2, \neg c_3)$ with (33), (34), (35), and (36) respectively.
For the numerator we have:

$$\begin{aligned} & xPr(c_1)Pr(c_2)Pr(c_3) - x\gamma Pr(c_1)Pr(c_2)Pr(c_3) - x\beta Pr(c_1)Pr(c_2)Pr(c_3) + x\beta\gamma Pr(c_1)Pr(c_2)Pr(c_3) + \beta Pr(c_1)Pr(c_2)Pr(c_3) \\ & + \gamma Pr(c_1)Pr(c_2)Pr(c_3) - \beta\gamma Pr(c_1)Pr(c_2)Pr(c_3) \\ & + xPr(c_1)Pr(\neg c_2)Pr(c_3) - x\gamma Pr(c_1)Pr(\neg c_2)Pr(c_3) + \gamma Pr(c_1)Pr(\neg c_2)Pr(c_3) \\ & + xPr(c_1)Pr(c_2)Pr(\neg c_3) - x\beta Pr(c_1)Pr(c_2)Pr(\neg c_3) + \beta Pr(c_1)Pr(c_2)Pr(\neg c_3) \\ & + xPr(c_1)Pr(\neg c_2)Pr(\neg c_3) \end{aligned}$$

And for the denominator we have:

$$\begin{aligned} & xPr(c_1)Pr(c_2)Pr(c_3) - x\gamma Pr(c_1)Pr(c_2)Pr(c_3) - x\beta Pr(c_1)Pr(c_2)Pr(c_3) + x\beta\gamma Pr(c_1)Pr(c_2)Pr(c_3) + \beta Pr(c_1)Pr(c_2)Pr(c_3) \\ & + \gamma Pr(c_1)Pr(c_2)Pr(c_3) - \beta\gamma Pr(c_1)Pr(c_2)Pr(c_3) \\ & + xPr(c_1)Pr(\neg c_2)Pr(c_3) - x\gamma Pr(c_1)Pr(\neg c_2)Pr(c_3) + \gamma Pr(c_1)Pr(\neg c_2)Pr(c_3) \\ & + xPr(c_1)Pr(c_2)Pr(\neg c_3) - x\beta Pr(c_1)Pr(c_2)Pr(\neg c_3) + \beta Pr(c_1)Pr(c_2)Pr(\neg c_3) \\ & + xPr(c_1)Pr(\neg c_2)Pr(\neg c_3) \\ & + \beta Pr(\neg c_1)Pr(c_2)Pr(c_3) - \beta\gamma Pr(\neg c_1)Pr(c_2)Pr(c_3) + \gamma Pr(\neg c_1)Pr(c_2)Pr(c_3) \\ & + \beta Pr(\neg c_1)Pr(c_2)Pr(\neg c_3) \\ & + \gamma Pr(\neg c_1)Pr(\neg c_2)Pr(c_3) \end{aligned}$$

Now we use that $Pr(i) + Pr(\neg i)$=1 for $i = 1, 2, 3$, and simplify:

We now obtain for the numerator:

$$\begin{aligned} & xPr(c_1)Pr(c_3) - x\gamma Pr(c_1)Pr(c_3) + \gamma Pr(c_1)Pr(c_3) + xPr(c_1)Pr(\neg c_3) - x\beta Pr(c_1)Pr(c_2) \\ & + \beta Pr(c_1)Pr(c_2) + x\beta\gamma Pr(c_1)Pr(c_2)Pr(c_3) - \beta\gamma Pr(c_1)Pr(c_2)Pr(c_3) \\ & = xPr(c_1) - x\gamma Pr(c_1)Pr(c_3) - x\beta Pr(c_1)Pr(c_2) + x\beta\gamma Pr(c_1)Pr(c_2)Pr(c_3) \\ & + \gamma Pr(c_1)Pr(c_3) + \beta Pr(c_1)Pr(c_2) - \beta\gamma Pr(c_1)Pr(c_2)Pr(c_3) \end{aligned}$$

and denominator:

$$\begin{aligned}
&xPr(c_1)Pr(c_3) - x\gamma Pr(c_1)Pr(c_3) + \gamma Pr(c_1)Pr(c_3) + xPr(c_1)Pr(\neg c_3) - x\beta Pr(c_1)Pr(c_2)\\
&+ \beta Pr(c_1)Pr(c_2) + x\beta\gamma Pr(c_1)Pr(c_2)Pr(c_3) - \beta\gamma Pr(c_1)Pr(c_2)Pr(c_3)\\
&+ \beta Pr(\neg c_1)Pr(c_2)Pr(c_3) - \beta\gamma Pr(\neg c_1)Pr(c_2)Pr(c_3) + \gamma Pr(\neg c_1)Pr(c_2)Pr(c_3)\\
&+ \beta Pr(\neg c_1)Pr(c_2)Pr(\neg c_3) + \gamma Pr(\neg c_1)Pr(\neg c_2)Pr(c_3)\\
&= xPr(c_1) - x\gamma Pr(c_1)Pr(c_3) - x\beta Pr(c_1)Pr(c_2) + x\beta\gamma Pr(c_1)Pr(c_2)Pr(c_3)\\
&+ \gamma Pr(c_1)Pr(c_3) + \beta Pr(c_1)Pr(c_2) + \beta Pr(\neg c_1)Pr(c_2) - \beta\gamma Pr(c_2)Pr(c_3) + \gamma(\neg c_1)Pr(c_3)\\
&= xPr(c_1) - x\gamma Pr(c_1)Pr(c_3) - x\beta Pr(c_1)Pr(c_2) + x\beta\gamma Pr(c_1)Pr(c_2)Pr(c_3) + \beta Pr(c_2) + \gamma Pr(c_3) - \beta\gamma Pr(c_2)Pr(c_3)
\end{aligned}$$

When dividing both numerator and denominator by $Pr(c_1)$, we obtain:

$$Pr(c_1|e)(x) = \frac{x\Big(1 - \beta Pr(c_2) - \gamma Pr(c_3) + \beta\gamma Pr(c_2)Pr(c_3)\Big) + \beta Pr(c_2) + \gamma Pr(c_3) - \beta\gamma Pr(c_2)Pr(c_3)}{x\big(1 - \beta Pr(c_2) - \gamma Pr(c_3) + \beta\gamma Pr(c_2)Pr(c_3)\Big) + \frac{1}{Pr(c_1)}\Big(\beta Pr(c_2) + \gamma Pr(c_3) - \beta\gamma Pr(c_2)Pr(c_3)\Big)}$$

Now we divide both the numerator and denominator by $\Big(1 - \beta Pr(c_2) - \gamma Pr(c_3) + \beta\gamma Pr(c_2)Pr(c_3)\Big)$, we get:

$$\begin{aligned}
Pr(c_1|e)(x) &= \frac{x + \frac{\beta Pr(c_2)+\gamma Pr(c_3)-\beta\gamma Pr(c_2)Pr(c_3)}{1-\big(\beta Pr(c_2)+\gamma Pr(c_3)-\beta\gamma Pr(c_2)Pr(c_3)\big)}}{x + \frac{\beta Pr(c_2)+\gamma Pr(c_3)-\beta\gamma Pr(c_2)Pr(c_3)}{Pr(c_1)\Big(1-\big(\beta Pr(c_2)+\gamma Pr(c_3)-\beta\gamma Pr(c_2)Pr(c_3)\big)\Big)}}\\
&= \frac{x + \frac{\zeta}{1-\zeta}}{x + \frac{\zeta}{Pr(c_1)(1-\zeta)}}.
\end{aligned}$$

where $\zeta = \beta Pr(c_2) + \gamma Pr(c_3) - \beta\gamma Pr(c_2)Pr(c_3)$, and wherein $\beta = Pr(e|\neg c_1, c_2, \neg c_3)$ and $\gamma = Pr(e|\neg c_1, \neg c_2, c_3)$. $\square$

Observation: We start by noticing that Equation (37) has a similar format as Equation (14), see Section 4.1.3.1. The vertical asymptote of Equation (37) lies at $x = s = -\frac{\zeta}{Pr(c_1)(1-\zeta)}$ and the horizontal asymptote lies at $t = 1$. Equation (37) is a fragment of a fourth-quadrant hyperbola branch, and has its vertex at:

$$(s + \sqrt{|r|}, 1 - \sqrt{|r|}) = \Big(-\frac{\zeta}{Pr(c_1)(1-\zeta)} + \sqrt{\Big(\frac{\zeta}{Pr(c_1)(1-\zeta)} - \frac{\zeta}{(1-\zeta)}\Big)}, 1 - \sqrt{\Big(\frac{\zeta}{Pr(c_1)(1-\zeta)} - \frac{\zeta}{(1-\zeta)}\Big)}\Big)$$

Making use of the results obtained in Section 4.1.3.1, we here find that rather small values of $\frac{\zeta}{(1-\zeta)}$ and $\frac{\zeta}{Pr(c_1)(1-\zeta)}$ in Equation (37) produce a vertex with a positive x-coordinate. Furthermore, we have that the vertex only approaches the upper-left corner of the unit window, if in addition, the difference $\frac{\zeta}{Pr(c_1)(1-\zeta)} - \frac{\zeta}{(1-\zeta)}$ is rather small. Hence, in order to acquire a vertex approaching the upper-left corner of the unit window, we find that $Pr(c_2), Pr(c_3), Pr(e|\neg c_1, c_2, \neg c_3)$ and $Pr(e|\neg c_1, \neg c_2, c_3)$ need to be small and $Pr(c_1)$ should be large.

We gain more insight into the effects by looking at concrete parameter settings. See Figure 20. We see that Equation (37) shows that the strongest effects on the output probability $Pr(c_1|e)$ in the entire interval $[0, 1]$ can be expected based on the following parameter setting:

- The prior probabilities $Pr(c_2)$, $Pr(c_3)$ are small and the prior probability $Pr(c_1)$ is small/moderate. In addition we have that the smaller the noisy-OR parameters $Pr(e|\neg c_1, c_2, \neg c_3)$, $Pr(e|\neg c_1, \neg c_2, c_3)$, the larger the propagation effects.

The results obtained here are in line with the results shown in Section 4.1.3.1. Thereby, we want to emphasize that one should especially focus on the propagation effects for $x = Pr(e|c_1, \neg c_2, \neg c_3) \geq 0.6$, since the probabilities of

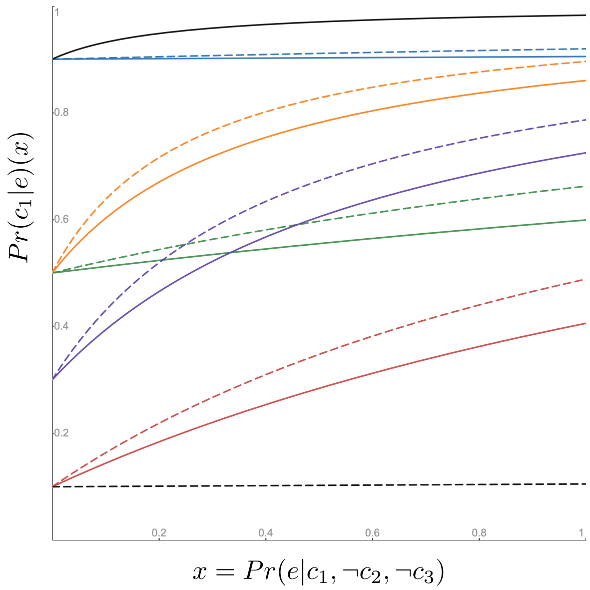

Figure 20: Several example sensitivity functions adhering to Theorem 4.15. (See Table 9 for parameter settings)

Parameter	Red	Red dashed	Blue	Blue dashed	Green	Green dashed	Purple	Purple dashed	Orange	Orange dashed	Black	Black dashed
$Pr(c_1)$	0.1	0.1	0.9	0.9	0.5	0.5	0.3	0.3	0.5	0.5	0.9	0.1
$Pr(c_2)$	0.1	0.1	0.9	0.9	0.5	0.5	0.1	0.1	0.1	0.1	0.1	0.9
$Pr(c_3)$	0.1	0.1	0.9	0.9	0.5	0.5	0.1	0.1	0.1	0.1	0.1	0.9
$\beta = Pr(e\|\neg c_1, c_2, \neg c_3)$	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.85
$\gamma = Pr(e\|\neg c_1, \neg c_2, c_3)$	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.85

Table 9: Parameter settings for sensitivity functions from Figure 20

noisy-OR parameters are assumed to be large [5]. As one can observe in Figure 20, strong propagation effects only occur, with a particular parameter settings, when x is smaller than 0.5, see orange (dashed), purple (dashed) and red (dashed) function.

To gain better insight into Equation's (37) propagation effects in the interval $x = Pr(e|c_1, \neg c_2, \neg c_3) \geq 0.6$, we compute its first derivative. As we have mentioned, the sensitivity functions corresponding to Equation (37) are a fragment of a fourth-quadrant hyperbola branch, and as a consequence, we know that the first derivative $\frac{d}{dx}Pr(c_1|e)(x) > 0$ for all $x \in [0, 1]$.

Corollary 4.15.1. *The first derivative of the sensitivity function from Equation (37) is:*

$$\begin{aligned}\frac{d}{dx}Pr(c_1|e)(x) &= \frac{\frac{(\beta Pr(c_2)+\gamma Pr(c_3)-\beta\gamma Pr(c_2)Pr(c_3))(1-Pr(c_1))}{Pr(c_1)(1-(\beta Pr(c_2)+\gamma Pr(c_3)-\beta\gamma Pr(c_2)Pr(c_3)))}}{\left(x+\frac{\beta Pr(c_2)+\gamma Pr(c_3)-\beta\gamma Pr(c_2)Pr(c_3)}{Pr(c_1)(1-(\beta Pr(c_2)+\gamma Pr(c_3)-\beta\gamma Pr(c_2)Pr(c_3)))}\right)^2}\\ &= \frac{\frac{\zeta(1-Pr(c_1))}{Pr(c_1)(1-\zeta)}}{\left(x+\frac{\zeta}{Pr(c_1)(1-\zeta)}\right)^2}.\end{aligned} \tag{38}$$

where $\zeta = \beta Pr(c_2) + \gamma Pr(c_3) - \beta\gamma Pr(c_2)Pr(c_3)$*, and wherein* $\beta = Pr(e|\neg c_1, c_2, \neg c_3)$ *and* $\gamma = Pr(e|\neg c_1, \neg c_2, c_3)$*.*

We plot the derivatives of Equation (38) of specific parameter settings for some functions demonstrated in Figure 20, namely the red (dashed), green (dashed), purple (dashed) and orange (dashed) function. See Figure 21.

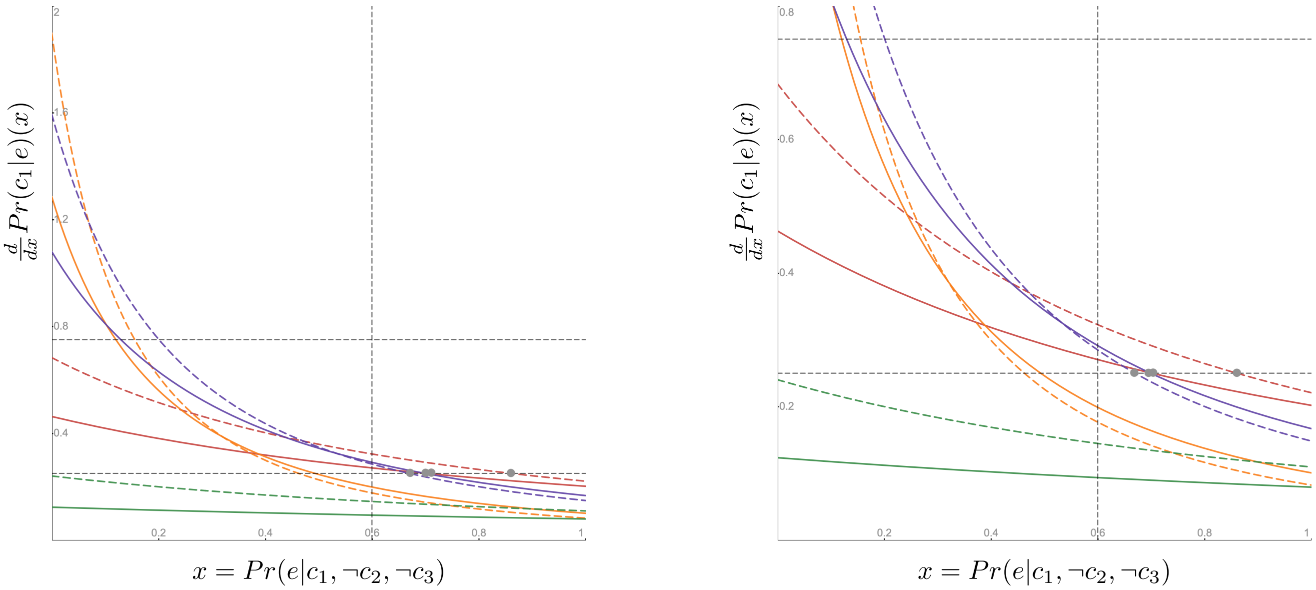

Figure 21: Several examples of Equation (38) restricted to the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_1|e)(x) \in [0, 2]$ (left) and the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [0, 0.8]$ (right). (See Table 9 for parameter settings)

In Figure 21, we observe that the derivatives of the exemplar sensitivity functions for $x \geq 0.6$, all have values smaller than 0.75. This indicates that the propagation effects considering these circumstances are moderate at most. Furthermore, we observe that the exemplar red dashed function, where the prior probability $Pr(c_1), Pr(c_2)$ and $Pr(c_3)$ are set to 0.1, possesses the largest value for the derivative in the interval $[0.6, 1]$.

With the help of WOLFRAM MATHEMATICA, we find a maximum of max $\frac{d}{dx}Pr(c_1|e)(x) = 0.416666$ in the interval $x = Pr(e|c_1, \neg c_2, \neg c_3) \in [0.6, 1]$ of Equation (38) with the following parameter setting (see Appendix D.1):

$Pr(c_1) = 6.34063 * 10^{-7}, Pr(c_2) = 2.45256 \cdot 10^{-7}, Pr(c_3) = 2.45256 \cdot 10^{-7}, Pr(e|\neg c_1, c_2, \neg c_3) = 0.77592$ and $Pr(e|\neg c_1, \neg c_2, c_3) = 0.77592$.

This maximum lies at $x = 0.6$ and we observe that the maximum propagation effects in the interval $x \in [0.6, 1]$ are the same as in Section 4.1.3.1. However, now three (instead of two) extremely small prior probabilities are needed to attain the same propagation effects as in Section 4.1.3.1, indicating more skewed prior probability distributions are needed to attain the same propagation effects as for the causal mechanism shown in Figure 3. Again, we note that the propagation effects in the interval $x = Pr(e|c_1, \neg c_2, \neg c_3) \in [0.6, 1]$ can become moderate *at most*. We discovered that the diagnostic propagation effects are again mainly determined by which output probability we look at and which noisy-OR parameter we vary, which are in line with the result found in Section 4.1.3.1.

We conclude that the propagation effects in both the causal and diagnostic direction of the causal mechanism shown in Figure 22 involving three cause variables, are in line with the results of the causal mechanism shown in Figure 3, involving two cause variables, see Section 4.1.1. However, more skewed prior probabilities are needed to attain the same propagation effects as for the causal mechanism shown in Figure 3. We conclude that the same propagation effects may still happen; however, the size of the propagation effects will possibly decrease when the number of cause variables increases.

4.5 Causal mechanism with leak as third cause variable: independent causes

In Section 4.3, the leaky noisy-OR model was introduced wherein the leak-probability p represents the probability that the effect e will occur as a result of unmodelled causes. Because the leak probability p attains a small value in practice [1, 2], we assumed $p \in (0, 0.2)$. However, since it is (almost) impossible to model all the cause variables, we can also argue that the leak is *always* present. As mentioned earlier, in Section 4.3 we considered the leak p as the probability of the effect e occurring when all modelled causes are absent. If we now explicitly include the cause of the leak as a cause in our model, the noisy-OR parameter of this cause becomes equal to p (so still small). The prior probability of the leak cause is typically set to 1.

In this section, we therefore let the leak represent an additional cause variable, namely C_L. The noisy-OR parameter associated with cause C_L, that is $Pr(e|\neg c_1, \neg c_2, c_L)$, will be very small since it represents the case that the effect will happen spontaneously.

We will examine the propagation effects on a probability of interest due to deviations in a noisy-OR parameter if we express the leak-variable as an explicit cause variable. Thereby, we assume that the noisy-OR parameter associated with this cause variable possesses a small value. The property of accountability is still satisfied. See Figure 22.

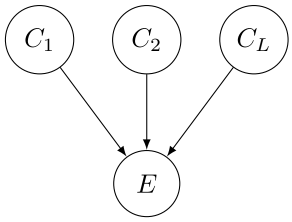

Figure 22: A causal mechanism with the effect variable E and cause variables C_1, C_2, C_L.

We assume the following:

- The prior probability distributions for cause variables C_1 and C_2 are non-degenerate, that is $Pr(c_i) \neq 0$ and $Pr(\neg c_i) \neq 0$ for $i = 1, 2$;
- $Pr(c_L) = 1$;
- Because regular noisy-OR parameters are assumed to be large [5], we mainly focus on $Pr(e|c_1, \neg c_2, \neg c_L)$ and $Pr(e|\neg c_1, c_2, \neg c_L) \in [0.6, 1]$ in our research. We specifically use this constraint when evaluating the propagation effects;
- The noisy-OR parameter $Pr(e|\neg c_1, \neg c_2, c_L)$ associated with cause C_L is in fact the leak probability p from Section 4.3, and thus has a small probability. We let $Pr(e|\neg c_1, \neg c_2, c_L) \in (0, 0.2]$.
- We assume that the property of accountability is again satisfied, we have $Pr(e|\neg c_1, \neg c_2, \neg c_L) = 0$.

We carry on with the gradation of the gradient described in Section 4.1.1 of a sensitivity function under study. We consider the gradient ∇ to be small when $|\nabla| \leq 0.25$, moderate when $|\nabla| \in (0.25, 0.75)$, and large when $|\nabla| \geq 0.75$.

4.5.1 Propagation effects in the causal direction

First we examine the propagation effects in the causal direction, that is, the possible effects on the probability $Pr(e)$ due to changes in a noisy-OR parameter.

Theorem 4.16. *Consider the causal mechanism in Figure 22 and assume it models a noisy-OR. Let $x = Pr(e|c_1, \neg c_2, \neg c_L)$ be the noisy-OR parameter associated with cause C_1. Then the sensitivity function $Pr(e)(x)$ has the following form:*

$$Pr(e)(x) = xPr(c_1)\Big(1 + \beta\gamma Pr(c_2) - \beta Pr(c_2) - \gamma\Big) + \beta Pr(c_2) + \gamma - \beta\gamma Pr(c_2) \tag{39}$$

where $\beta = Pr(e|\neg c_1, c_2, \neg c_L)$ and $\gamma = Pr(e|\neg c_1, \neg c_2, c_L)$.

Proof:
The result follows directly from Equation (32) taking into account that $C_L = C_3$ and $Pr(c_3) = 1$. □

Observation: The gradient of Equation (39) equals $Pr(c_1)\Big(1 + \beta\gamma Pr(c_2) - \beta Pr(c_2) - \gamma\Big)$ and is in the interval $(0, 1)$. In Section 4.4.1 we found that a large gradient of Equation (32) will be obtained based on the following:

- $Pr(c_1)$ is large and the prior probabilities of the other causes are small. In addition we have that the smaller the noisy-OR parameters associated with causes other than C_1, the larger the gradient.

In contrast with the current situation, we now have that the prior probability of the leak cause is 1. We will now analyse if this affects our earlier conclusions. We want to examine whether:

$$\Big(Pr(c_1)\big(1+\beta\gamma Pr(c_2)Pr(c_3)-\beta Pr(c_2)-\gamma Pr(c_3)\big)\Big)_{Eq.(32)} < \Big(Pr(c_1)\big(1+\beta\gamma Pr(c_2)-\beta Pr(c_2)-\gamma\big)\Big)_{Eq.39}$$

that is, whether the propagation effects of Equation (39) are larger than of Equation (32). Thus, we need to find out whether:

$$\Big(\gamma Pr(c_3)\big(\beta Pr(c_2)-1\big)\Big)_{Eq.(32)} < \Big(\gamma\big(\beta Pr(c_2)-1\big)\Big)_{Eq.39} \tag{40}$$

Since $\beta Pr(c_2)-1<0$ in (40), we find that both terms are less than zero. Consequently, we want to examine when:

$$\big(\gamma Pr(c_3)\big)_{Eq.(32)} > (\gamma)_{Eq.39}.$$

Now recall that we assumed that $\gamma \in (0,0.2]$ in Equation (39) and $\gamma \in [0.6,1]$ in Equation (32). We find that if $Pr(c_3) > \frac{1}{3}$ inequality (40) always holds. Therefore, we carefully claim that the propagation effects of Equation (39) are *generally slightly* larger than of Equation (32). However, note that the size of the gradient is mainly determined by $Pr(c_1)$: a large gradient can only be obtained if the prior probability $Pr(c_1)$ is large.

We conclude that the propagation effects of Equation (39) are in line with Equation (32).

4.5.2 Propagation effects in the diagnostic direction

In this section, we will examine the effects of a deviating noisy-OR value upon diagnostic propagation. We will express the example posterior probability of interest $Pr(c_1|e)$ as a function of the value $x = Pr(e|c_1,\neg c_2,\neg c_L)$ for the probability under study.

Theorem 4.17. *Consider the causal mechanism in Figure 22 and assume it models a noisy-OR. Let $x = Pr(e|c_1,\neg c_2,\neg c_L)$ be the noisy-OR parameter associated with cause C_1. Then the sensitivity function $Pr(c_1|e)(x)$ has the following form:*

$$Pr(c_1|e)(x) = \frac{x + \frac{\zeta}{1-\zeta}}{x + \frac{\zeta}{Pr(c_1)(1-\zeta)}} \tag{41}$$

where $\zeta = \beta Pr(c_2) + \gamma - \beta\gamma Pr(c_2)$, and wherein $\beta = Pr(e|\neg c_1, c_2,\neg c_L)$ and $\gamma = Pr(e|\neg c_1,\neg c_2, c_L)$.

Proof:
We have again that the result follows directly from Equation (37) taking into account that $C_L = C_3$ and $Pr(c_3) = 1$. $\square$

Observation: Making use of the results obtained in Section 4.4.2, we likewise find that rather small values of $\frac{\zeta}{(1-\zeta)}$ and $\frac{\zeta}{Pr(c_1)(1-\zeta)}$ in Equation (41) produce a vertex with a positive x-coordinate. Furthermore, we have that the vertex only approaches the upper-left corner of the unit window, if in addition, the difference $\frac{\zeta}{Pr(c_1)(1-\zeta)} - \frac{\zeta}{(1-\zeta)}$ is rather small. Hence, in order to acquire a vertex approaching the upper-left corner of the unit window, we find that $Pr(c_2), Pr(e|\neg c_1, c_2,\neg c_L)$ and $Pr(e|\neg c_1,\neg c_2, c_L)$ need to be small and $Pr(c_1)$ should be large. Note that we now assume that the parameter $Pr(e|\neg c_1,\neg c_2, c_L)$ is indeed small, in contrast to Section 4.4. If the probability $Pr(c_1)$ approaches 1, and hence the value $\frac{\zeta}{Pr(c_1)(1-\zeta)} - \frac{\zeta}{(1-\zeta)}$ becomes very small, the x-coordinate of the vertex will indeed approach 0. However, this might not result in the largest propagation effects in the entire interval $x \in [0,1]$. This is because $Pr(c_1|e)(1)$ will then approach 1 and since we have that horizontal asymptote lies at $t = 1$ and $\frac{d}{dx}Pr(c_1|e)(x) > 0$ for $x \in [0,1]$, the propagation effects will be limited.

To support the above mentioned findings and gain more insight into the effects, we consider concrete parameter settings, see Figure 23. We see, when focusing on the entire interval $x = Pr(e|c_1,\neg c_2,\neg c_L) \in [0,1]$, that the strongest effects on the output probability $Pr(c_1|e)$ can be expected based on the following parameter setting:

- The prior probabilities $Pr(c_2)$ and $Pr(c_1)$ are small. In addition we have that the smaller the noisy-OR parameters $Pr(e|\neg c_1, c_2, \neg c_L)$, $Pr(e|\neg c_1, \neg c_2, c_L)$, the larger the propagation effects.

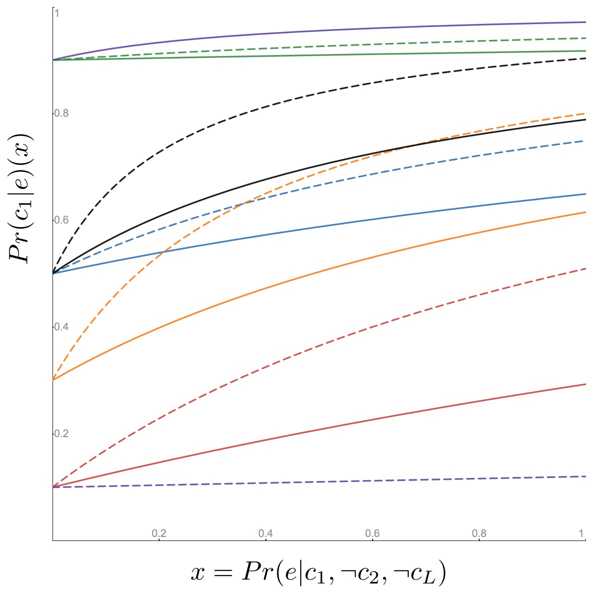

Figure 23: Several example sensitivity functions adhering to Theorem 4.17. (See Table 10 for parameter settings)

Parameter	Red	Red dashed	Green	Green dashed	Blue	Blue dashed	Orange	Orange dashed	Black	Black dashed	Purple	Purple dashed
$Pr(c_1)$	0.1	0.1	0.9	0.9	0.5	0.5	0.3	0.3	0.5	0.5	0.9	0.1
$Pr(c_2)$	0.1	0.1	0.9	0.9	0.5	0.5	0.1	0.1	0.1	0.1	0.1	0.9
$\beta = Pr(e|\neg c_1, c_2, \neg c_L)$	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.6	0.85	0.85
$\gamma = Pr(e|\neg c_1, \neg c_2, c_L)$	0.2	0.05	0.2	0.05	0.2	0.05	0.2	0.05	0.2	0.05	0.2	0.2

Table 10: Parameter settings for sensitivity functions from Figure 23

To gain better insight into Equation's (41) behaviour in the interval $x = Pr(e|c_1, \neg c_2, \neg c_L) \geq 0.6$, we compute its first derivative. We can build upon the results derived in Section 4.4.2. As before, we have that the result follows directly from Equation (38), taking into account that $C_L = C_3$ and $Pr(c_3) = 1$.

Corollary 4.17.1. *The first derivative of the sensitivity function from Equation (41) is:*

$$\frac{d}{dx} Pr(c_1|e)(x) = \frac{\frac{\zeta(1-Pr(c_1))}{Pr(c_1)(1-\zeta)}}{\left(x + \frac{\zeta}{Pr(c_1)(1-\zeta)}\right)^2}. \tag{42}$$

where $\zeta = \beta Pr(c_2) + \gamma - \beta\gamma Pr(c_2)$, and wherein $\beta = Pr(e|\neg c_1, c_2, \neg c_L)$ and $\gamma = Pr(e|\neg c_1, \neg c_2, c_L)$.

For specific parameter settings for some of the functions demonstrated in Figure 23, namely the red (dashed), blue (dashed), black (dashed) and orange (dashed) function, we plot the derivatives of Equation (42). See Figure 24.

In Figure 24, we observe that small prior probabilities for $Pr(c_1)$ and $Pr(c_2)$ and, in addition, small probabilities for the noisy-OR parameters $Pr(e|\neg c_1, c_2, \neg c_L)$ and $Pr(e|\neg c_1, \neg c_2, c_L)$, lead to the largest propagation effects in the interval $x \in [0.6, 1]$, see red dashed. Recall that by assumption $Pr(e|\neg c_1, c_2, \neg c_L) \in [0.6, 1]$ and $Pr(e|\neg c_1, \neg c_2, c_L) \in (0, 0.2]$.

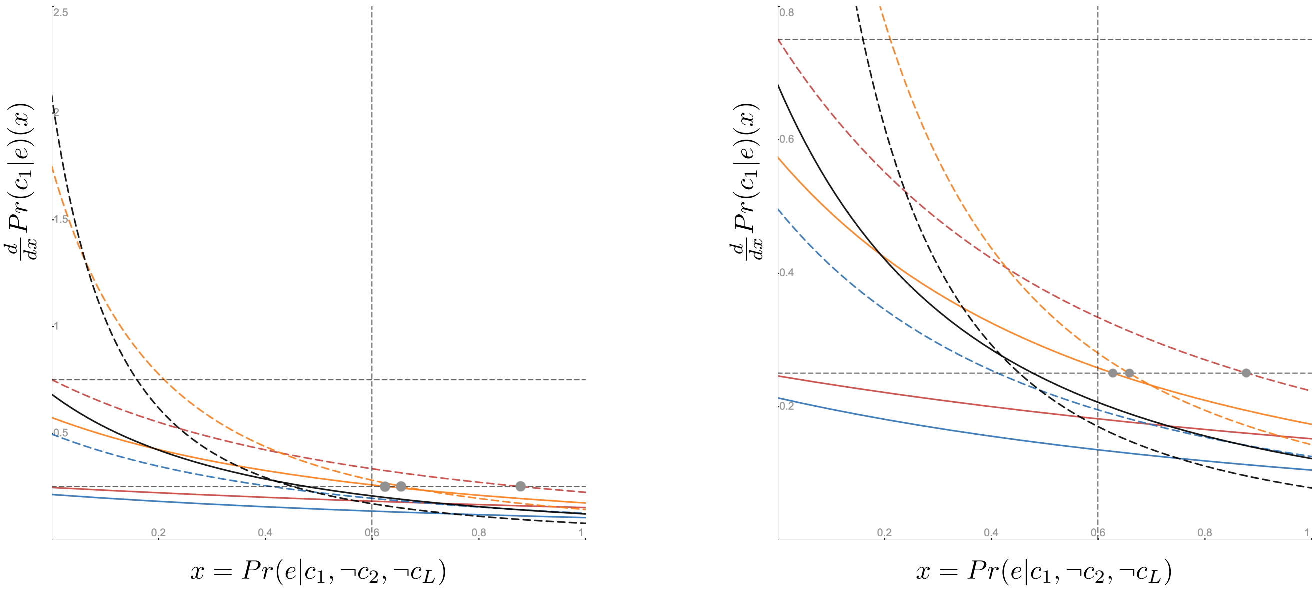

Figure 24: Several examples of Equation (42) restricted to the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_1|e)(x) \in [0, 2.5]$ (left) and the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [0, 0.8]$ (right). (See Table 10 for parameter settings)

With the help of WOLFRAM MATHEMATICA, we find a maximum of max $\frac{d}{dx}Pr(c_1|e)(x) = 0.416666$ in the interval $x = Pr(e|c_1, \neg c_2, \neg c_L) \in [0.6, 1]$ of Equation (41) with the following parameter setting (see Appendix E.1):

$Pr(c_1) = 6.31615 \cdot 10^{-7}, Pr(c_2) = 2.45757 \cdot 10^{-7}, Pr(e|\neg c_1, c_2, \neg c_L) = 0.775591$ and $Pr(e|\neg c_1, \neg c_2, c_L) = 1.88362 \cdot 10^{-7}$.

This maximum lies at $x = 0.6$. We note that the maximum value of $\frac{d}{dx}Pr(c_1|e)(x)$, for $x \in [0, 1]$, is the same as the result obtained in Section 4.4.2. Apparently, the small value of $\gamma = Pr(e|\neg c_1, \neg c_2, c_L) \in (0, 0.2]$ is compensated by the large value of the prior probability $Pr(c_L) = 1$. We note, in order to reach this maximum value 0.416666, that the prior probabilities $Pr(c_1)$ and $Pr(c_2)$ are extremely small. Furthermore, the noisy-OR parameter $Pr(e|\neg c_1, c_2, \neg c_L)$ is approximately 0.77, which is the same value as in Section 4.4.2. We find that the propagation effects can become moderate *at most*.

The overall results obtained in Section 4.5 demonstrate that explicitly modelling leak causes, and thereby restricting the noisy-OR parameter associated with that cause variable to a value in the interval $(0, 0.2]$, lead to similar propagation effects, see Section 4.4. The propagation effects in the causal and diagnostic direction are mainly in line with the results obtained in Section 4.4.

5 Propagation effects due to noisy-MAX parameter changes: independent causes

So far, we have examined the propagation effects of the (leaky) noisy-OR model. However, since the (leaky) noisy-OR model involves binary variables only, its ability to model real-world Bayesian networks is limited to a certain level. To better model real-world Bayesian networks, the noisy-MAX model came into practice. The purpose of this interaction model is to extend the noisy-OR model to non-binary discrete variables. The noisy-MAX model still meets the property of accountability. We have that c_i^0 denotes the absence of a given cause C_i and the other values of variable C_i, that is c_i^j where $j > 0$, capture different levels of manifestation of cause C_i. The noisy-MAX model takes in the probabilities of the parameters that represent for each cause variable individually the influence of its different manifestation levels for the effect e to arise; that is the following parameter probabilities: $Pr(e^i|c_1^0, ..., c_{j-1}^0, c_j^k, c_{j+1}^0, ..., c_m^0)$ for all values $c_j^k, k > 0$, of the cause variable C_j and all values $e^i, i \geq 0$, of the effect variable E. The remaining probabilities for the CPT are defined through [1, 2]

$$Pr(e^i|\mathbf{c}) = \begin{cases} Pr(E \leq e^i|\mathbf{c}) - Pr(E \leq e^{i-1}|\mathbf{c}) & \text{for } i > 0 \\ Pr(E \leq e^0|\mathbf{c}) & \text{for } i = 0 \end{cases}$$

with

$$Pr(E \leq e^i|\mathbf{c}) = \prod_{j \in J} \sum_{l=0,\dots,i} Pr(e^i|c_1^0, ..., c_{j-1}^0, c_j^k, c_{j+1}^0, ..., c_m^0)$$

where J is the set of indices of the cause variables C_j that are marked as having a value c_j^k with $k > 0$ in the joint value combination $\mathbf{c}$.

Woudenberg and van der Gaag examined the propagation effects of the noisy-MAX model by investigating the possible effects of propagating deviating *model-calculated* probabilities [1]. They studied $Pr(e)(x, y)$ and $Pr(c_1^1|e)(x, y)$ where $x = Pr(e|c_1^1, c_2)$ and $y = Pr(e|c_1^2, c_2)$. We will, as done through this entire thesis, examine the propagation effects of noisy-MAX parameter changes. We will again use the basic mechanism in Figure 3, and for simplicity reasons, assume that cause variable C_1 is ternary and the cause variable C_2 and effect variable E are binary, as Woudenberg and van der Gaag assumed in [1]. We have:

$C_1 = \{c_1^0, c_1^1, c_1^2\}$
$C_2 = \{\neg c_2, c_2\}$
$E = \{\neg e, e\}$

For the basic mechanism from Figure 3 we now assume the following:

- The prior probability distributions for cause variables C_1 and C_2 are non-degenerate, that is $Pr(c_i^j) \neq 0$ for $i = 1, 2$ and $j = 0, 1, 2$;
- $Pr(e|c_1^0, \neg c_2) = 0$, by the property of accountability;
- For our experimental analysis, the values of the noisy-MAX parameter probabilities are ordered. With this we mean that $Pr(e|c_1^2, \neg c_2) > Pr(e|c_1^1, \neg c_2) > Pr(e|c_1^0, \neg c_2) = 0$. This indicates that we assume that "higher" levels of a cause variable will be more likely to cause the effect e. We will focus on $Pr(e|c_1^2, \neg c_2) \in [0.7, 1]$ and $Pr(e|c_1^1, \neg c_2) \in [0.4, 0.7]$ in our research, where we ensure that the constraint $Pr(e|c_1^2, \neg c_2) > Pr(e|c_1^1, \neg c_2)$ always holds.
- Since cause variable C_2 is binary, we assume, as for the noisy-OR parameters, that the presence of this single factor is likely to trigger the effect e [5]. Therefore, we will focus on $Pr(e|c_1^0, c_2) \in [0.6, 1]$ in our research. We specifically use this constraint when evaluating the propagation effects.

We will carry on with the gradation of the gradient described in Section 4.1.1 of a sensitivity function under study. We consider the gradient ∇ to be small when $|\nabla| \leq 0.25$, moderate when $|\nabla| \in (0.25, 0.75)$, and large when $|\nabla| \geq 0.75$.

5.1 Propagation effects in the causal direction

First, we examine the possible effects on the probability $Pr(e)$ due to changes in a noisy-MAX parameter. Since we particularly want to investigate the influence of involving ternary variables in our model, we will start with examining the possible effects due to deviations in the noisy-MAX parameter $Pr(e|c_1^1, \neg c_2)$; that is the noisy-MAX parameter associated with value c_1^1 of the ternary cause C_1.

Theorem 5.1. *Consider the causal mechanism in Figure 3 and assume it models a noisy-MAX. Let $x = Pr(e|c_1^1, \neg c_2)$ be the noisy-MAX parameter associated with value c_1^1 of cause C_1. Then the sensitivity function $Pr(e)(x)$ has the following form:*

$$Pr(e)(x) = xPr(c_1^1)\Big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\Big) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2) + Pr(e|c_1^0, c_2)Pr(c_2) \quad (43)$$

Proof:
We have that probability $Pr(e)$ is equal to:

$$Pr(e) = Pr(e|c_1^1, c_2)Pr(c_1^1)Pr(c_2) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(\neg c_2) + Pr(e|c_1^2, c_2)Pr(c_1^2)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2) + Pr(e|c_1^0, c_2)Pr(c_1^0)Pr(c_2) + Pr(e|c_1^0, \neg c_2)Pr(c_1^0)Pr(\neg c_2) \quad (44)$$

Note that $Pr(e|c_1^0, \neg c_2)Pr(c_1^0)Pr(\neg c_2) = 0$ since $Pr(e|c_1^0, \neg c_2) = 0$ by the property of accountability.

We set $x = Pr(e|c_1^1, \neg c_2)$ and probabilities have been set for the other 2 noisy-MAX parameters $Pr(e|c_1^2, \neg c_2)$ and $Pr(e|c_1^0, c_2)$. We compute the values of $Pr(e|c_1^1, c_2)$ and $Pr(e|c_1^2, c_2)$ by the noisy-MAX model:

1. First, we compute $Pr(e|c_1^1, c_2)$, this probability is dependent of x:

$$\begin{aligned} Pr(e|c_1^1, c_2) &= Pr(E \leq e|\mathbf{c}) - Pr(E \leq \neg e|\mathbf{c}) \\ &= Pr(E \leq e|c_1^1, c_2) - Pr(E \leq \neg e|c_1^1, c_2) \end{aligned}$$

We compute the first term $Pr(E \leq e|c_1^1, c_2)$:

$$\begin{aligned} Pr(E \leq e|c_1^1, c_2) &= \Big(Pr(\neg e|c_1^1, \neg c_2) + Pr(e|c_1^1, \neg c_2)\Big) \cdot \Big(Pr(\neg e|c_1^0, c_2) + Pr(e|c_1^0, c_2)\Big) \\ &= 1 \cdot 1 = 1 \end{aligned}$$

And the second term $Pr(E \leq \neg e|c_1^1, c_2)$:

$$\begin{aligned} Pr(E \leq \neg e|c_1^1, c_2) &= Pr(\neg e|c_1^1, \neg c_2) \cdot Pr(\neg e|c_1^0, c_2) \\ &= (1 - x)(1 - Pr(e|c_1^0, c_2)) \\ &= 1 - Pr(e|c_1^0, c_2) - x + xPr(e|c_1^0, c_2) \end{aligned}$$

We have for $Pr(e|c_1^1, c_2)$:

$$\begin{aligned} Pr(e|c_1^1, c_2) &= 1 - \Big(1 - Pr(e|c_1^0, c_2) - x + xPr(e|c_1^0, c_2)\Big) \\ &= Pr(e|c_1^0, c_2) + x - xPr(e|c_1^0, c_2) \end{aligned} \quad (45)$$

2. Now, we compute $Pr(e|c_1^2, c_2)$:

$$\begin{aligned} Pr(e|c_1^2, c_2) &= Pr(E \leq e|\mathbf{c}) - Pr(E \leq \neg e|\mathbf{c}) \\ &= Pr(E \leq e|c_1^2, c_2) - Pr(E \leq \neg e|c_1^2, c_2) \end{aligned}$$

We compute the first term $Pr(E \leq e|c_1^2, c_2)$:

$$\begin{aligned} Pr(E \leq e|c_1^2, c_2) &= \Big(Pr(\neg e|c_1^2, \neg c_2) + Pr(e|c_1^2, \neg c_2)\Big) \cdot \Big(Pr(\neg e|c_1^0, c_2) + Pr(e|c_1^0, c_2)\Big) \\ &= 1 \cdot 1 = 1 \end{aligned}$$

And the second term $Pr(E \leq \neg e|c_1^2, c_2)$:

$$\begin{aligned}Pr(E \leq \neg e|c_1^2, c_2) &= Pr(\neg e|c_1^2, \neg c_2) \cdot Pr(\neg e|c_1^0, c_2) \\ &= (1 - Pr(e|c_1^2, \neg c_2))(1 - Pr(e|c_1^0, c_2)) \\ &= 1 - Pr(e|c_1^0, c_2) - Pr(e|c_1^2, \neg c_2) + Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)\end{aligned}$$

We have for $Pr(e|c_1^2, c_2)$:

$$\begin{aligned}Pr(e|c_1^2, c_2) &= 1 - \Big(1 - Pr(e|c_1^0, c_2) - Pr(e|c_1^2, \neg c_2) + Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)\Big) \\ &= Pr(e|c_1^0, c_2) + Pr(e|c_1^2, \neg c_2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2) \qquad (46)\end{aligned}$$

Now we substitute $Pr(e|c_1^1, c_2)$ and $Pr(e|c_1^2, c_2)$ with the obtained values in Equations (45) and (46) respectively:

$$\begin{aligned}Pr(e)(x) &= \Big(Pr(e|c_1^0, c_2) + x - xPr(e|c_1^0, c_2)\Big)Pr(c_1^1)Pr(c_2) + xPr(c_1^1)Pr(\neg c_2) \\ &+ \Big(Pr(e|c_1^0, c_2) + Pr(e|c_1^2, \neg c_2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)\Big)Pr(c_1^2)Pr(c_2) \\ &+ Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2) + Pr(e|c_1^0, c_2)Pr(c_1^0)Pr(c_2) \\ &= Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2) + xPr(c_1^1)Pr(c_2) - xPr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2) + xPr(c_1^1)Pr(\neg c_2) \\ &+ \Big(Pr(e|c_1^0, c_2) + Pr(e|c_1^2, \neg c_2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)\Big)Pr(c_1^2)Pr(c_2) \\ &+ Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2) + Pr(e|c_1^0, c_2)Pr(c_1^0)Pr(c_2) \\ &= x\Big(Pr(c_1^1)Pr(c_2) - Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2) + Pr(c_1^1)Pr(\neg c_2)\Big) + Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2) \\ &+ \Big(Pr(e|c_1^0, c_2) + Pr(e|c_1^2, \neg c_2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)\Big)Pr(c_1^2)Pr(c_2) \\ &+ Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2) + Pr(e|c_1^0, c_2)Pr(c_1^0)Pr(c_2) \\ &= x\Big(Pr(c_1^1) - Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2)\Big) + Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2) \\ &+ \Big(Pr(e|c_1^0, c_2) + Pr(e|c_1^2, \neg c_2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)\Big)Pr(c_1^2)Pr(c_2) \\ &+ Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2) + Pr(e|c_1^0, c_2)Pr(c_1^0)Pr(c_2) \\ &= xPr(c_1^1)\Big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\Big) + Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2) \\ &+ Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(c_2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2) \\ &+ Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2) + Pr(e|c_1^0, c_2)Pr(c_1^0)Pr(c_2)\end{aligned}$$

Finally, we obtain:

$$\begin{aligned}Pr(e)(x) &= xPr(c_1^1)\Big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\Big) + Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2) + Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2) \\ &+ Pr(e|c_1^2, \neg c_2)Pr(c_1^2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2) + Pr(e|c_1^0, c_2)Pr(c_1^0)Pr(c_2) \\ &= xPr(c_1^1)\Big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\Big) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2) \\ &+ Pr(e|c_1^0, c_2)Pr(c_2)\end{aligned}$$

since we have that $Pr(c_1^0) + Pr(c_1^1) + Pr(c_1^2) = 1$. □

Observation: The gradient of the linear function from Equation (43) is large when at least the prior probability $Pr(c_1^1)$ is large and $Pr(c_2)$ and/or $Pr(e|c_1^0, c_2)$ is/are small. Since $Pr(e|c_1^0, c_2)$ is a noisy-MAX parameter, and we assume that this noisy-MAX parameter has a probability in the interval $[0.6, 1]$, we find that the gradient of Equation (43) is large when the prior probability $Pr(c_1^1)$ is large and $Pr(c_2)$ is small. In addition we have that the smaller the noisy-MAX parameter $Pr(e|c_1^0, c_2)$, the larger the gradient.
Analogous observations hold for the sensitivity function obtained for the probability of interest $Pr(e)$ when $x =$

$Pr(e|c_1^2, \neg c_2)$; c_1^2 and c_1^1 merely exchange roles.

We now refer to Equation (7), where the possible effects on the probability $Pr(e)$ due to changes in a noisy-OR parameter $x = Pr(e|\neg c_1, c_2)$ were examined, and all involved variables are binary. The gradient of Equation (7) is $Pr(c_2)\big(1 - Pr(e|c_1, \neg c_2)Pr(c_1)\big)$. Since we assumed that both $Pr(e|c_1^0, c_2)$ (in Eq. (43)) and $Pr(e|c_1, \neg c_2)$ (in Eq. (7)) are in the interval $[0.6, 1]$, we now obtain comparable propagation effects as Equation (7). We conclude that the propagation effects are in line with the propagation effects of the noisy-OR model involving binary variables only, see Section 4.1.1. We found similar results since the noisy-MAX parameters associated with the non-binary cause variable C_1 are absent in the gradient.

We will now examine the propagation effects on the prior probability $Pr(e)$ due to changes in the noisy-MAX parameter $Pr(e|c_1^0, c_2)$. Note that we here will study the possible effects due to deviations in the noisy-MAX parameter associated with the *binary* cause variable C_2.

Theorem 5.2. *Consider the causal mechanism in Figure 3 and assume it models a noisy-MAX. Let* $x = Pr(e|c_1^0, c_2)$ *be the noisy-MAX parameter associated with cause* C_2*. Then the sensitivity function* $Pr(e)(x)$ *has the following form:*

$$Pr(e)(x) = xPr(c_2)\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2) \tag{47}$$

Proof:
We have that probability $Pr(e)$ is equal to Equation (44).
We set $x = Pr(e|c_1^0, c_2)$ and probabilities have been set for the other 2 noisy-MAX parameters $Pr(e|c_1^1, \neg c_2)$ and $Pr(e|c_1^2, \neg c_2)$. The values of $Pr(e|c_1^1, c_2)$ and $Pr(e|c_1^2, c_2)$ are defined through the noisy-MAX model:

1. First, we compute $Pr(e|c_1^1, c_2)$, this probability is dependent of x:

$$\begin{aligned} Pr(e|c_1^1, c_2) &= Pr(E \leq e|\mathbf{c}) - Pr(E \leq \neg e|\mathbf{c}) \\ &= Pr(E \leq e|c_1^1, c_2) - Pr(E \leq \neg e|c_1^1, c_2) \end{aligned}$$

We compute the first term $Pr(E \leq e|c_1^1, c_2)$:

$$\begin{aligned} Pr(E \leq e|c_1^1, c_2) &= \Big(Pr(\neg e|c_1^1, \neg c_2) + Pr(e|c_1^1, \neg c_2)\Big) \cdot \Big(Pr(\neg e|c_1^0, c_2) + Pr(e|c_1^0, c_2)\Big) \\ &= 1 \cdot ((1 - x) + x) \\ &= 1 \cdot 1 = 1 \end{aligned}$$

And the second term $Pr(E \leq \neg e|c_1^1, c_2)$:

$$\begin{aligned} Pr(E \leq \neg e|c_1^1, c_2) &= Pr(\neg e|c_1^1, \neg c_2) \cdot Pr(\neg e|c_1^0, c_2) \\ &= (1 - Pr(e|c_1^1, \neg c_2))(1 - x) \\ &= 1 - x - Pr(e|c_1^1, \neg c_2) + Pr(e|c_1^1, \neg c_2)x \end{aligned}$$

We have for $Pr(e|c_1^1, c_2)$:

$$\begin{aligned} Pr(e|c_1^1, c_2) &= 1 - \Big(1 - x - Pr(e|c_1^1, c_2) + Pr(e|c_1^1, \neg c_2)x\Big) \\ &= x + Pr(e|c_1^1, \neg c_2) - xPr(e|c_1^1, \neg c_2) \end{aligned} \tag{48}$$

2. Now, we compute $Pr(e|c_1^2, c_2)$, which is also dependent of x:

$$\begin{aligned} Pr(e|c_1^2, c_2) &= Pr(E \leq e|\mathbf{c}) - Pr(E \leq \neg e|\mathbf{c}) \\ &= Pr(E \leq e|c_1^2, c_2) - Pr(E \leq \neg e|c_1^2, c_2) \end{aligned}$$

We compute the first term $Pr(E \leq e|c_1^2, c_2)$:

$$\begin{aligned}Pr(E \leq e|c_1^2, c_2) &= \Big(Pr(\neg e|c_1^2, \neg c_2) + Pr(e|c_1^2, \neg c_2)\Big) \cdot \Big(Pr(\neg e|c_1^0, c_2) + Pr(e|c_1^0, c_2)\Big) \\ &= 1 \cdot ((1-x) + x) \\ &= 1 \cdot 1 = 1\end{aligned}$$

And the second term $Pr(E \leq \neg e|c_1^2, c_2)$:

$$\begin{aligned}Pr(E \leq \neg e|c_1^2, c_2) &= Pr(\neg e|c_1^2, \neg c_2) \cdot Pr(\neg e|c_1^0, c_2) \\ &= (1 - Pr(e|c_1^2, \neg c_2))(1-x) \\ &= 1 - x - Pr(e|c_1^2, \neg c_2) + Pr(e|c_1^2, \neg c_2)x\end{aligned}$$

We have for $Pr(e|c_1^2, c_2)$:

$$\begin{aligned}Pr(e|c_1^2, c_2) &= 1 - \Big(1 - x - Pr(e|c_1^2, \neg c_2) + Pr(e|c_1^2, \neg c_2)x\Big) \\ &= x + Pr(e|c_1^2, \neg c_2) - Pr(e|c_1^2, \neg c_2)x \qquad (49)\end{aligned}$$

Now we substitute $Pr(e|c_1^1, c_2)$ and $Pr(e|c_1^2, c_2)$ with the obtained values in Equations (48) and (49) respectively:

$$\begin{aligned}Pr(e)(x) = &\Big(x + Pr(e|c_1^1, \neg c_2) - xPr(e|c_1^1, \neg c_2)\Big)Pr(c_1^1)Pr(c_2) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(\neg c_2) \\ &+ \Big(x + Pr(e|c_1^2, \neg c_2) - xPr(e|c_1^2, \neg c_2)\Big)Pr(c_1^2)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2) + xPr(c_1^0)Pr(c_2) \\ = &\, xPr(c_2)\Big(Pr(c_1^1) - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(c_1^2) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2) + Pr(c_1^0)\Big) \\ &+ Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(c_2) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(\neg c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(c_2) \\ &+ Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2)\end{aligned}$$

Since we have that $Pr(c_1^0) + Pr(c_1^1) + Pr(c_1^2) = 1$ and $Pr(c_2) + Pr(\neg c_2) = 1$ we finally obtain:

$$Pr(e)(x) = xPr(c_2)\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2). \quad \square$$

Observation: The gradient of Equation (47) is equal to $Pr(c_2)\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big)$. This gradient will obtain a large value when the prior probabilities of the cause variables that correspond with the values in the noisy-MAX parameter $Pr(e|c_1^0, c_2)$ are large; meaning the probabilities $Pr(c_1^0)$ and $Pr(c_2)$. In addition we have that the smaller the probabilities for the noisy-MAX parameters $Pr(e|c_1^2, \neg c_2)$ and $Pr(e|c_1^2, \neg c_2)$, the larger the propagation effects.
We now observe that both noisy-MAX parameters associated with cause C_1 are present in the gradient. By assumption we have that $Pr(e|c_1^2, \neg c_2) \in [0.7, 1]$ and $Pr(e|c_1^1, \neg c_2) \in [0.4, 0.7]$. For the term $Pr(e|c_1^2, \neg c_2)Pr(c_1^2)$ we find that if the prior probability $Pr(c_1^2)$ is large, the term $Pr(e|c_1^2, \neg c_2)Pr(c_1^2)$ presumably will be large, and thus, will decrease the gradient's value. For the term $Pr(e|c_1^1, \neg c_2)Pr(c_1^1)$ we have that this term will presumably be smaller since $Pr(e|c_1^1, \neg c_2) \in [0.4, 0.7]$. This means that this term will probably not decrease the value of the gradient as much as the term $Pr(e|c_1^2, \neg c_2)Pr(c_1^2)$. However, note that the above argument is not complete. If for example $Pr(c_1^2)$ is large, than $Pr(c_1^1) = 1 - Pr(c_1^2) - Pr(c_1^0)$ will probably be smaller than $Pr(c_1^2)$, and thus $Pr(e|c_1^1, \neg c_2)Pr(c_1^1) < Pr(e|c_1^2, \neg c_2)Pr(c_1^2)$, indicating that the gradient's value is more decreased by $Pr(e|c_1^2, \neg c_2)Pr(c_1^2)$.

The value of the gradient is largely determined by the prior probabilities $Pr(c_2)$ and $Pr(c_1^0)$. This indicates that the propagation effects in the causal direction when using the noisy-MAX model, are in line when using the noisy-OR model. However, by assumption we have that the lower the value c_i^j of the noisy-MAX parameters associated with cause C_i, the smaller the probability of the corresponding noisy-MAX parameter. For Equation (47), we find that the smaller the probabilities for the noisy-MAX parameters $Pr(e|c_1^2, \neg c_2)$ and $Pr(e|c_1^2, \neg c_2)$, the larger the gradient. Therefore, we conclude that the propagation effects in the causal direction are possibly higher when using the noisy-MAX model than for the noisy-OR model involving binary variables only.

5.2 Propagation effects in the diagnostic direction

In this section, we will study the propagation effects of a deviating noisy-MAX parameter in the diagnostic direction. We will first study the effects on $Pr(c_1^1|e)(x)$ where $x = Pr(e|c_1^1, \neg c_2)$.

Theorem 5.3. *Consider the causal mechanism in Figure 3 and assume it models a noisy-MAX. Let* $x = Pr(e|c_1^1, \neg c_2)$ *be the noisy-MAX parameter associated with value* c_1^1 *of cause* C_1*. Then the sensitivity function* $Pr(c_1^1|e)(x)$ *has the following form:*

$$Pr(c_1^1|e)(x) = \frac{x + \frac{\alpha}{\gamma}}{x + \frac{\beta}{\gamma}}. \qquad (50)$$

where $\gamma = Pr(c_1^1)\Big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\Big)$, $\alpha = Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2)$ *and* $\beta = Pr(e|c_1^0, c_2)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2)$.

Proof:
We have for $x = Pr(e|c_1^1, \neg c_2)$:

$$\begin{aligned} Pr(c_1^1|e)(x) &= \frac{Pr(c_1^1, e)(x)}{Pr(e)(x)} \\ &= \frac{Pr(e, c_1^1, c_2)(x) + Pr(e, c_1^1, \neg c_2)(x)}{Pr(e)(x)} \\ &= \frac{Pr(e|c_1^1, c_2)Pr(c_1^1)Pr(c_2) + xPr(c_1^1)Pr(\neg c_2)}{Pr(e)(x)} \quad \text{where } Pr(e|c_1^1, c_2) \text{ depends on } x \text{ as captured in Equation (45)} \end{aligned}$$

Furthermore, we have that $Pr(e)(x) = Pr(c_1^0, e)(x) + Pr(c_1^1, e)(x) + Pr(c_1^2, e)(x)$. $Pr(c_1^0, e)(x)$ is equal to:

$$\begin{aligned} Pr(c_1^0, e)(x) &= Pr(e, c_1^0, c_2)(x) + Pr(e, c_1^0, \neg c_2)(x) \\ &= Pr(e|c_1^0, c_2)Pr(c_1^0)Pr(c_2) + Pr(e|c_1^0, \neg c_2)Pr(c_1^0)Pr(\neg c_2) \\ &= Pr(e|c_1^0, c_2)Pr(c_1^0)Pr(c_2) + 0 \cdot Pr(c_1^0)Pr(\neg c_2) \\ &= Pr(e|c_1^0, c_2)Pr(c_1^0)Pr(c_2) \end{aligned}$$

and $Pr(c_1^2, e)(x)$ is equal to:

$$\begin{aligned} Pr(c_1^2, e)(x) &= Pr(e, c_1^2, c_2)(x) + Pr(e, c_1^2, \neg c_2)(x) \\ &= Pr(e|c_1^2, c_2)Pr(c_1^2)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2) \\ &\quad \text{where } Pr(e|c_1^2, c_2) \text{ depends on } x \text{ as captured in Equation (46)} \end{aligned}$$

We substitute $Pr(e|c_1^1, c_2)$ and $Pr(e|c_1^2, c_2)$ with the values From Equation (45) & (46), respectively, and obtain:

For the numerator $Pr(c_1^1|e)(x)$ we have:

$\Big(Pr(e|c_1^0, c_2) + x - xPr(e|c_1^0, c_2)\Big)Pr(c_1^1)Pr(c_2) + xPr(c_1^1)Pr(\neg c_2)$, working out the parentheses gives:
$Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2) + xPr(c_1^1)Pr(c_2) - xPr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2) + xPr(c_1^1)Pr(\neg c_2)$

Since $Pr(c_2) + Pr(\neg c_2) = 1$, we find for the numerator:

$xPr(c_1^1)\Big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\Big) + Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2)$.

For the denominator $Pr(e)(x)$ we have:

$Pr(e|c_1^0, c_2)Pr(c_1^0)Pr(c_2) + \Big(Pr(e|c_1^0, c_2) + x - xPr(e|c_1^0, c_2)\Big)Pr(c_1^1)Pr(c_2) + xPr(c_1^1)Pr(\neg c_2) + \Big(Pr(e|c_1^0, c_2) + Pr(e|c_1^2, \neg c_2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)\Big)Pr(c_1^2)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2)$

Because $Pr(c_1^0) + Pr(c_1^1) + Pr(c_1^2) = 1$ and $Pr(c_2) + Pr(\neg c_2) = 1$ we obtain for the denominator:

$$xPr(c_1^1)\Big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\Big) + Pr(e|c_1^0, c_2)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2).$$

Now, if we divide both the numerator and denominator by $Pr(c_1^1)\Big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\Big)$, we find:

$$Pr(c_1^1|e)(x) = \frac{x + \frac{Pr(e|c_1^0,c_2)Pr(c_1^1)Pr(c_2)}{Pr(c_1^1)\Big(1-Pr(e|c_1^0,c_2)Pr(c_2)\Big)}}{x + \frac{Pr(e|c_1^0,c_2)Pr(c_2)+Pr(e|c_1^2,\neg c_2)Pr(c_1^2)-Pr(e|c_1^2,\neg c_2)Pr(e|c_1^0,c_2)Pr(c_1^2)Pr(c_2)}{Pr(c_1^1)\Big(1-Pr(e|c_1^0,c_2)Pr(c_2)\Big)}} = \frac{x + \frac{\alpha}{\gamma}}{x + \frac{\beta}{\gamma}}$$

where $\gamma = Pr(c_1^1)\Big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\Big)$, $\alpha = Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2)$ and
$\beta = Pr(e|c_1^0, c_2)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2)$. □

Observation: Since Equation (50) is a hyperbolic function, we use the properties of hyperbolic functions described in Section 2.5, and discover that the vertical asymptote of Equation (50) lies at $x = s = -\frac{\beta}{\gamma}$. Because $\frac{\beta}{\gamma} > 0$, the asymptote is located to the left of the unit window and the horizontal asymptote lies at $t = 1$. As a result, we find that Equation (50) is a fragment of a fourth-quadrant hyperbola branch.

From generic research of sensitivity functions from Bayesian networks, we find that the effect of deviations in the x-value on the output probability of interest mainly depends on the location of the vertex of the corresponding hyperbola branch [1]. Generally, we have that the closer the vertex of the fourth-quadrant hyperbola branch lies to the upper-left corner of the unit window, the larger the propagation effects. We find that Equation (50) has its vertex at:

$$(s + \sqrt{|r|}, 1 - \sqrt{|r|}) = \Bigg(-\frac{\beta}{\gamma} + \sqrt{\Big|\frac{\alpha - \beta}{\gamma}\Big|}, 1 - \sqrt{\Big|\frac{\alpha - \beta}{\gamma}\Big|}\Bigg)$$

The vertex is located within the unit window for values of $\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}$ for which $\frac{\beta}{\gamma} < \sqrt{\Big|\frac{\alpha-\beta}{\gamma}\Big|} < 1$. To obtain $\frac{\beta}{\gamma} < \sqrt{\Big|\frac{\alpha-\beta}{\gamma}\Big|}$ given that $\frac{\beta}{\gamma} \geq \frac{\alpha}{\gamma}$, we discover that merely rather small values of $\frac{\alpha}{\gamma}$ produce a vertex with an x-coordinate in the unit range. Moreover, the vertex only approaches the upper-left corner of the unit window, if in addition the difference $\frac{\beta}{\gamma} - \frac{\alpha}{\gamma} = \frac{\beta-\alpha}{\gamma}$ is rather small, thus γ should be large and $\beta - \alpha$ small. For γ to be large, $Pr(c_1^1)$ should be large and $Pr(e|c_1^0, c_2)$ and $Pr(c_2)$ should be small. For $\beta - \alpha$ we have:

$$\begin{aligned}\beta - \alpha &= Pr(e|c_1^0, c_2)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2) - Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2)\\ &= Pr(e|c_1^0, c_2)Pr(c_2)\big(1 - Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\big) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\end{aligned}$$

Thus, for $\beta - \alpha$ to be small, $Pr(e|c_1^0, c_2)$ and $Pr(c_2)$ should be small and $Pr(c_1^1)$ large. Since $Pr(e|c_1^2, \neg c_2)Pr(c_1^2) > Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2)$ it is favourable that $Pr(e|c_1^2, \neg c_2)$ and $Pr(c_1^2)$ are small as well.

Hence, in order to acquire a vertex approaching the upper-left corner of the unit window, we find that $Pr(c_1^1)$ and $Pr(\neg c_2)$ need to be large and $Pr(c_1^0), Pr(c_1^2), Pr(e|c_1^0, c_2)$ and $Pr(e|c_1^2, \neg c_2)$ small.

To support the above mentioned findings and gain more insight into influence of the parameter settings on Equation (50), we consider concrete parameter settings, see Figure 25. When focusing on the entire interval $[0, 1]$ for $x = Pr(e|c_1^1, \neg c_2)$, we observe that the largest propagation effects indeed occur when the prior probabilities $Pr(c_1^1)$ and $Pr(\neg c_2)$ are large (purple & red). Moreover, we observe that smaller probabilities for the noisy-MAX parameters $Pr(e|c_1^0, c_2)$ and $Pr(e|c_1^2, \neg c_2)$ will lead to a minor increase in the propagation effects, this effect is conveyed by the solid versus dashed function for each colour.

Since, by assumption $Pr(e|c_1^1, \neg c_2) \in [0.4, 0.7]$, we should especially focus on propagation effects for $x =$

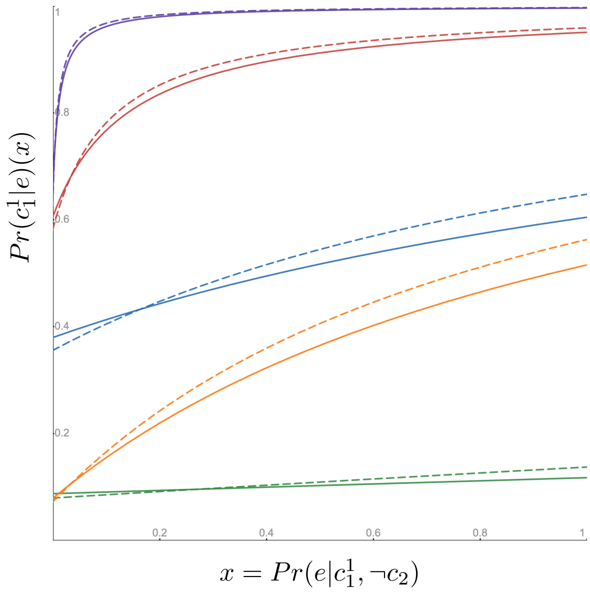

Figure 25: Several example sensitivity functions adhering to Theorem 5.3. (See Table 11 for parameter settings)

Parameter	Red	Red dashed	Blue	Blue dashed	Green	Green dashed	Purple	Purple dashed	Orange	Orange dashed
$Pr(c_1^0)$	0.05	0.05	0.25	0.25	0.45	0.45	0.005	0.005	0.3333	0.3333
$Pr(c_1^1)$	0.9	0.9	0.5	0.5	0.1	0.1	0.99	0.99	0.3333	0.3333
$Pr(c_1^2)$	0.05	0.05	0.25	0.25	0.45	0.45	0.005	0.005	0.3333	0.3333
$Pr(c_2)$	0.1	0.1	0.5	0.5	0.9	0.9	0.01	0.01	0.1	0.1
$Pr(e\|c_1^0, c_2)$	0.8	0.6	0.8	0.6	0.8	0.6	0.8	0.6	0.8	0.6
$Pr(e\|c_1^2, \neg c_2)$	0.85	0.7	0.85	0.7	0.85	0.7	0.85	0.7	0.85	0.7

Table 11: Parameter settings for sensitivity functions from Figure 25

$Pr(e|c_1^1, \neg c_2) \in [0.4, 0.7]$. In Figure 25 it is shown that large propagation effects only occur when $x = Pr(e|c_1^1, \neg c_2)$ is particularly small. To acquire better insight into Equation's (50) behaviour in the interval $x = Pr(e|c_1^1, \neg c_2) \in [0.4, 0.7]$, we compute its first derivative. Since the sensitivity functions corresponding to Equation (50) are a fragment of a fourth-quadrant hyperbola branch, we realize that $\frac{d}{dx}Pr(c_1^1|e)(x) > 0$ for all $x \in [0, 1]$.

Corollary 5.3.1. *The first derivative of the sensitivity function from Equation (50) is:*
$\frac{d}{dx}Pr(c_1^1|e)(x) =$

$$= \frac{Pr(e|c_1^0, c_2)Pr(c_2)\big(1 - Pr(c_1^1)\big) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2)}{Pr(c_1^1)\big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\big)\Big(x + \frac{Pr(e|c_1^0, c_2)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2)}{Pr(c_1^1)\big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\big)}\Big)^2}$$

$$= \frac{\beta - \alpha}{\gamma\big(x + \frac{\beta}{\gamma}\big)^2} \tag{51}$$

where $\gamma = Pr(c_1^1)\Big(1 - Pr(e|c_1^0, c_2)Pr(c_2)\Big)$, $\alpha = Pr(e|c_1^0, c_2)Pr(c_1^1)Pr(c_2)$ *and*
$\beta = Pr(e|c_1^0, c_2)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2) - Pr(e|c_1^2, \neg c_2)Pr(e|c_1^0, c_2)Pr(c_1^2)Pr(c_2)$.

For specific parameter settings for some of the functions demonstrated in Figure 25, namely the red (dashed), blue (dashed) and orange (dashed) function, we plot the derivatives of Equation (44). See Figure 26.

In Figure 26, the horizontal lines at $y = 0.25$ and $y = 0.75$ again indicate the boundaries between what we consider to be a small, moderate or large gradient. The vertical line at $x = 0.4$ indicates for which $x-$values we assume that the noisy-MAX parameter $x = Pr(e|c_1^1, \neg c_2)$ takes on plausible probabilities. As mentioned before, we focus on the propagation effects for $x \in [0.4, 0.7]$.

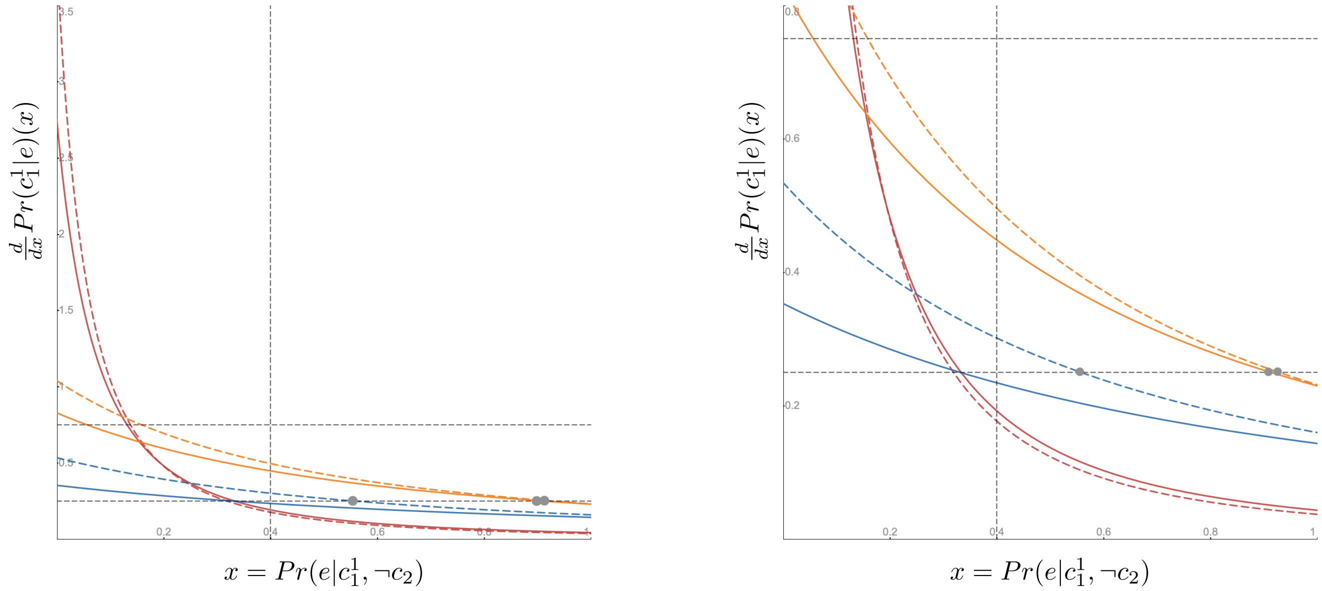

Figure 26: Several examples of Equation (51) restricted to the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_1^1|e)(x) \in [0, 3.5]$ (left) and the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_1^1|e)(x) \in [0, 0.8]$ (right). (See Table 11 for parameter settings)

We observe, when now focusing on the interval $x \in [0.4, 0.7]$, that for the sensitivity functions shown in Figure 25 the largest propagation effects occur for the orange (dashed) function. For the orange (dashed) function the prior probability $Pr(c_2)$ is small and the prior distribution for C_1 takes on an uniform distribution. Furthermore, we find similar results compared to Section 4.1.3.1: smaller noisy-MAX parameters don't lead to larger propagation effects anymore, this effect is conveyed by the solid versus dashed function for each colour. For the exemplar red (dashed) function we even observe the opposite effect: larger noisy-MAX parameters lead to larger propagation effects.

With the help of WOLFRAM MATHEMATICA, we find a maximum of $\max \frac{d}{dx}Pr(c_1^1|e)(x) = 0.625$ in the interval $x = Pr(e|c_1^1, \neg c_2) \in [0.4, 1]$ of Equation (51) with several parameter settings (see Appendix F.1). For example, one of the parameter settings is:

$Pr(c_1^0) = 0.205785, Pr(c_1^1) = 0.536938, Pr(c_1^2) = 0.257277, Pr(c_2) = 6.74835 \cdot 10^{-8}, Pr(e|c_1^0, c_2) = 0.750893$ and $Pr(e|c_1^2, \neg c_2) = 0.834803$.

This maximum lies at $x = 0.4$.

We conclude that Equation (50) shows that the strongest effects on the output probability $Pr(c_1^1|e)$ in the interval $x = Pr(e|c_1^1, \neg c_2) \in [0.4, 0.7]$ can be expected, based on the following:

- The probabilities $Pr(c_1^0)$, $Pr(c_1^2)$ are small, $Pr(c_2)$ is extremely small, and $Pr(c_1^1)$ is moderate.

We have found that the propagation effects again can become moderate at most. However, in the noisy-OR model we assumed that the noisy-OR parameters can't attain a value smaller than 0.6. Now, by assumption $x = Pr(e|c_1^1, \neg c_2) \in [0.4, 0.7]$. As a result, we find that the maximum gradient of $\frac{d}{dx}Pr(c_1^1|e)(x)$ can attain a larger value than we have found before: a gradient of 0.625 compared to 0.416666.

Now we will examine the propagation effects due to changes in the noisy-MAX parameter associated with the binary cause C_2. We continue our research of the noisy-MAX model by studying the effects on $Pr(c_2|e)(x)$ where $x = Pr(e|c_1^0, c_2)$.

Theorem 5.4. *Consider the causal mechanism in Figure 3 and assume it models a noisy-MAX. Let* $x = Pr(e|c_1^0, c_2)$ *be the noisy-MAX parameter associated with cause* C_2*. Then the sensitivity function* $Pr(c_2|e)(x)$ *has the following form:*

$$Pr(c_2|e)(x) = \frac{x + \frac{\beta}{1-\beta}}{x + \frac{\beta}{Pr(c_2)(1-\beta)}} \tag{52}$$

where $\beta = Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)$.

Note that the ternary cause variable C_1 is absent in both $x = Pr(e|c_1^0, c_2)$ and the output probability of interest $Pr(c_2|e)$. We expect that the results will be quite similar to Theorem 4.5

Proof:
We have for $x = Pr(e|c_1^0, c_2)$:

$$\begin{aligned}
Pr(c_2|e)(x) &= \frac{Pr(c_2, e)(x)}{Pr(e)(x)} \\
&= \frac{Pr(e, c_1^0, c_2)(x) + Pr(e, c_1^1, c_2)(x) + Pr(e, c_1^2, c_2)(x)}{Pr(e)(x)} \\
&= \frac{xPr(c_1^0)Pr(c_2) + Pr(e|c_1^1, c_2)Pr(c_1^1)Pr(c_2) + Pr(e|c_1^2, c_2)Pr(c_1^2)Pr(c_2)}{Pr(e)(x)}
\end{aligned}$$

Since $Pr(e|c_1^1, c_2)$ and $Pr(e|c_1^2, c_2)$ are dependent of x we compute these values with the noisy-MAX model. We find $Pr(e|c_1^1, c_2) = x + Pr(e|c_1^1, \neg c_2) - xPr(e|c_1^1, \neg c_2)$ (by Eq. 48) and $Pr(e|c_1^2, c_2) = x + Pr(e|c_1^2, \neg c_2) - Pr(e|c_1^2, \neg c_2)x$ (by Eq. 49)

In addition, we have that $Pr(e)(x) = Pr(c_2, e)(x) + Pr(\neg c_2, e)(x)$ and $Pr(\neg c_2, e)(x)$ is equal to:

$$\begin{aligned}
Pr(\neg c_2, e)(x) &= Pr(e, c_1^0, \neg c_2)(x) + Pr(e, c_1^1, \neg c_2)(x) + Pr(e, c_1^2, \neg c_2)(x) \\
&= Pr(e|c_1^0, \neg c_2)Pr(c_1^0)Pr(\neg c_2) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(\neg c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2) \\
&= Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(\neg c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(\neg c_2) \ \ (\text{since } Pr(e|c_1^0, \neg c_2) = 0)
\end{aligned}$$

For the numerator we find:

$$\begin{aligned}
&= xPr(c_1^0)Pr(c_2) + \Big(x + Pr(e|c_1^1, \neg c_2) - xPr(e|c_1^1, \neg c_2)\Big)Pr(c_1^1)Pr(c_2) + \Big(xPr(e|c_1^2, \neg c_2) - xPr(e|c_1^2, \neg c_2)\Big)Pr(c_1^2)Pr(c_2) \\
&= xPr(c_1^0)Pr(c_2) + xPr(c_1^1)Pr(c_2) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(c_2) - xPr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(c_2) + xPr(c_1^2)Pr(c_2) \\
&+ Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(c_2) - xPr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(c_2) \\
&= xPr(c_2) - xPr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(c_2) - xPr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(c_2) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(c_2) \\
&+ Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(c_2) \\
&= xPr(c_2)\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(c_2)
\end{aligned}$$

For the denominator we find:

$$\begin{aligned}
&= xPr(c_2)\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(c_2) \\
&+ Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(\neg c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(c_2) \\
&= xPr(c_2)\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)
\end{aligned}$$

Thus we find $Pr(c_2|e)(x) =$

$$= \frac{xPr(c_2)\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(c_2)}{xPr(c_2)\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big) + Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)}$$

$$= \frac{x + \frac{Pr(e|c_1^1, \neg c_2)Pr(c_1^1)Pr(c_2) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)Pr(c_2)}{Pr(c_2)\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big)}}{x + \frac{Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)}{Pr(c_2)\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big)}}$$

$$= \frac{x + \frac{Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)}{\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big)}}{x + \frac{Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)}{Pr(c_2)\Big(1 - Pr(e|c_1^1, \neg c_2)Pr(c_1^1) - Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\Big)}}$$

$$= \frac{x + \frac{\beta}{1-\beta}}{x + \frac{\beta}{Pr(c_2)(1-\beta)}}$$

where $\beta = Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)$. □

Observation: Since Equation (52) has exactly the same format as Equation (14), we refer to Section 4.1.3.1. We find that the vertex of Equation (52) approaches the upper-left corner of the unit window only if $Pr(c_1^0)$ and $Pr(c_2)$ are large and $Pr(c_1^1), Pr(c_1^2), Pr(e|c_1^1, \neg c_2)$ and $Pr(e|c_1^2, \neg c_2)$ are small.

In Figure 27, some sensitivity functions of Equation (52) are plotted. We indeed observe that the vertex of Equation (52) approaches the upper-left corner of the unit window if $Pr(c_1^0)$ and $Pr(c_2)$ are large and $Pr(c_1^1), Pr(c_1^2), Pr(e|c_1^1, \neg c_2)$ and $Pr(e|c_1^2, \neg c_2)$ are small, see red (dashed) function.

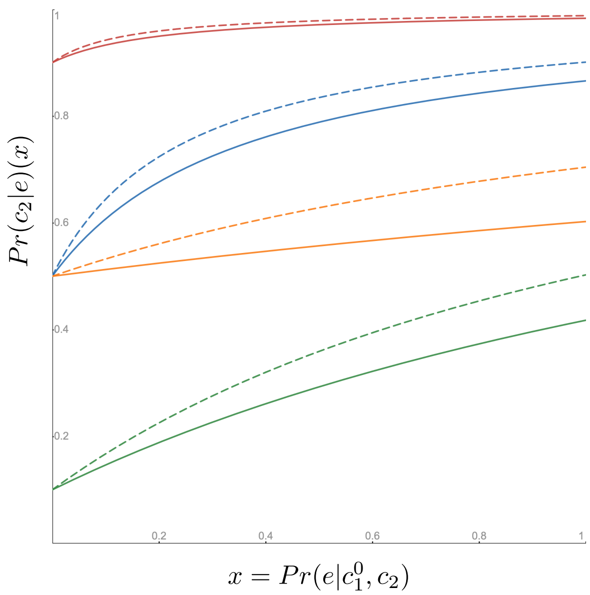

Figure 27: Several example sensitivity functions adhering to Theorem 5.4. (See Table 12 for parameter settings)

Parameter	Red	Red dashed	Blue	Blue dashed	Green	Green dashed	Orange	Orange dashed
$Pr(c_1^0)$	0.8	0.8	0.8	0.8	0.8	0.8	0.1	0.1
$Pr(c_1^1)$	0.1	0.1	0.1	0.1	0.1	0.1	0.7	0.7
$Pr(c_1^2)$	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2
$Pr(c_2)$	0.9	0.9	0.5	0.5	0.1	0.1	0.5	0.5
$Pr(e\|c_1^1, \neg c_2)$	0.7	0.4	0.7	0.4	0.7	0.4	0.7	0.4
$Pr(e\|c_1^2, \neg c_2)$	0.85	0.7	0.85	0.7	0.85	0.7	0.85	0.7

Table 12: Parameter settings for sensitivity functions from Figure 27

For better insight into the propagation effects for $x = Pr(e|c_1^0, c_2) \geq 0.6$, we compute its first derivative.

Corollary 5.4.1. *The first derivative of the sensitivity function from Equation (52) is:*
$\frac{d}{dx}Pr(c_2|e)(x) =$

$$= \frac{\big(Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\big)\big(1 - Pr(c_2)\big)}{Pr(c_2)\Big(1 - \big(Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)\big)\Big)\left(x + \frac{(Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2))}{Pr(c_2)\big(1 - (Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2))\big)}\right)^2}$$

$$= \frac{\beta\big(1 - Pr(c_2)\big)}{Pr(c_2)(1 - \beta)\left(x + \frac{\beta}{Pr(c_2)\big(1 - \beta\big)}\right)^2} \tag{53}$$

where $\beta = Pr(e|c_1^1, \neg c_2)Pr(c_1^1) + Pr(e|c_1^2, \neg c_2)Pr(c_1^2)$.

Now, we plot the derivatives for the red (dashed), blue (dashed) and green (dashed) function, see Figure 28. We observe that only for the green (dashed) exemplar sensitivity function the propagation effects are moderate for some $x \in [0.6, 1]$. The parameter settings for the green (dashed) function is shown in Table 12: $Pr(c_1^0)$ is large and $Pr(c_1^1), Pr(c_1^2)$ and $Pr(c_2)$ are small. Again we find that smaller noisy-MAX parameters $Pr(e|c_1^1, \neg c_2)$ and $Pr(e|c_1^2, \neg c_2)$ don't necessarily lead to larger propagation effect in the interval $x = Pr(e|c_1^0, c_2) \in [0.6, 1]$; for the blue (dashed) function we even observe the opposite effect.

With the help of WOLFRAM MATHEMATICA, we find a maximum of max $\frac{d}{dx}Pr(c_2|e)(x) = 0.41666$ in the interval $x = Pr(e|c_1^0, c_2) \in [0.6, 1]$ of Equation (53) with the following parameter settings (see Appendix F.2):

$Pr(c_1^0) = 0.9999...$, $Pr(c_1^1) = 3.0174 \cdot 10^{-7}$, $Pr(c_1^2) = 2.26303 \cdot 10^{-7}$, $Pr(c_2) = 6.3493 \cdot 10^{-7}$, $Pr(e|c_1^1, c_2) = 0.634626$ and $Pr(e|c_1^2, \neg c_2) = 0.837225$.

This maximum lies at $x = 0.6$.

We conclude that Equation (52) shows that the strongest effects on the output probability $Pr(c_2|e)$ in the interval $x = Pr(e|c_1^0, c_2) \in [0.6, 1]$ can be expected, based on the following:

- The probabilities $Pr(c_1^1)$, $Pr(c_1^2)$ and $Pr(c_2)$ are small and $Pr(c_1^0)$ is large. In addition we conclude that the influence of the noisy-MAX parameters $Pr(e|c_1^0, c_2)$ and $Pr(e|c_1^2, \neg c_2)$ is minor.

We discovered that the propagation effects in the diagnostic direction, when studying $Pr(c_2|e)(x)$ where $x = Pr(e|c_1^0, c_2)$ and $Pr(c_1^1|e)(x)$ where $x = Pr(e|c_1^1, \neg c_2)$, can again become moderate at most. The same maximum value of Equation (53) is found as the result corresponding to Equation (15) in Section 4.1.3.1, as expected. However, the maximum gradient's value of Equation (51) is higher, namely 0.625 instead of 0.41666. This means that the properties underlying the noisy-MAX model can have more impact on the propagation effects compared to the noisy-OR model, when studying the effects in the diagnostic direction. We discovered that the propagation effects in the diagnostic direction now mainly depend on which output probability we look at and *which value* of c_i^j of the noisy-MAX parameter associated with cause C_i we vary.

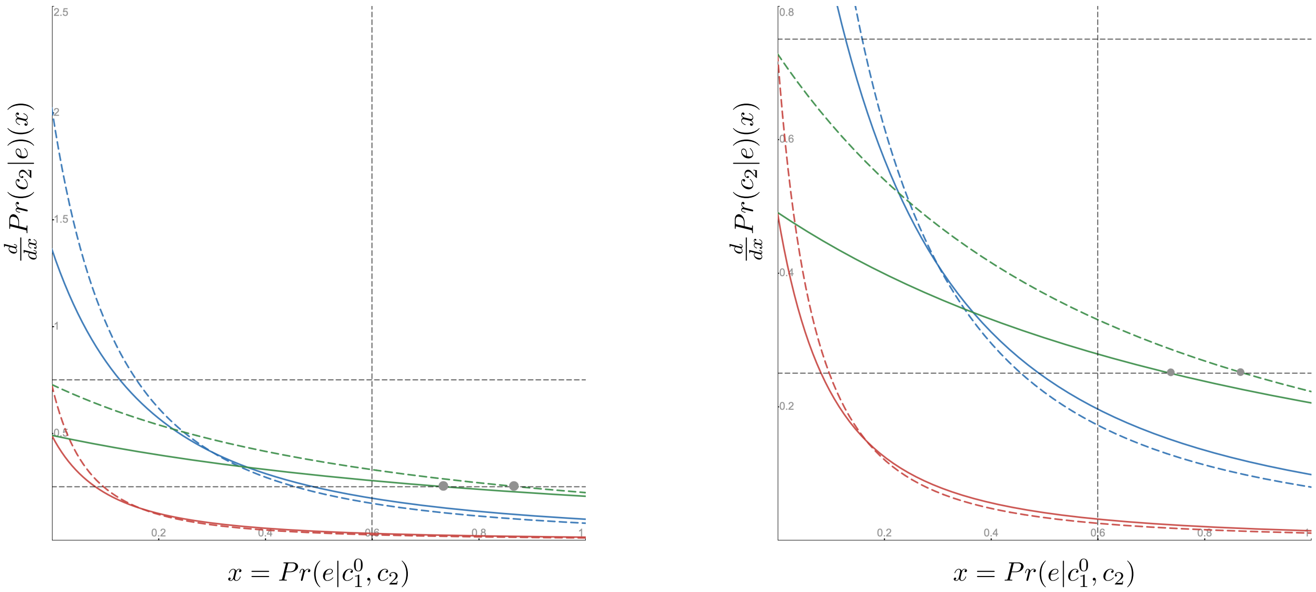

Figure 28: Several examples of Equation (53) restricted to the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [0, 2.5]$ (left) and the window $x \in [0, 1]$ and $\frac{d}{dx}Pr(c_2|e)(x) \in [0, 0.8]$ (right). (See Table 12 for parameter settings)

Altogether, we discovered that the overall propagation effects in both the causal and diagnostic direction possibly increase when using the noisy-MAX model compared to the noisy-OR model.

6 Summary and discussion

In this section, we will first summarise our findings and subsequently describe the differences and similarities of our findings with Woudenberg and van der Gaag [1].

We started our investigation with a basic causal mechanism with a common effect variable E and two independent cause variables, involving binary variables only. The conditional probability tables (CPT) of this causal mechanism involve eight probabilities, but using the noisy-OR model and thus assuming the property of accountability, only two parameters are required beforehand. We performed a sensitivity analysis wherein we examined the effect of deviations in one of the noisy-OR parameters on an output probability of interest. In addition, we also investigated the effect of possibly *dependent* cause variables in the basic causal mechanism. For both causal mechanisms, we discovered that large propagation effects in the causal direction on the probability of interest can only be expected if the prior probability of the cause associated with the noisy-OR parameter under study has a large probability of being present and the prior probability of the other cause is likely to be absent. Moreover, we found that a large probability for the other noisy-OR parameter involved, namely the noisy-OR parameter that is not set to x, will positively influence the robustness of the causal mechanism. In addition, we observed that by actually establishing the presence or absence of cause C_i, the propagation effects are no longer dependent of $Pr(c_i)$ or $Pr(\neg c_i)$, respectively. Furthermore, we found that the effect of possibly dependent cause variables is merely quite strong when the conditional probability $Pr(c_2|c_1)$, that is the probability capturing the strength of the dependency between the causes, has a completely different value than $Pr(c_2)$. The amount in which the propagation effects decrease/increase depends on the conditional probability $Pr(c_2|c_1)$ compared to the prior probability $Pr(c_2)$. We found that a small conditional probability $Pr(c_2|c_1)$ can possibly increase the propagation effects and a large value for $Pr(c_2|c_1)$ can possibly decrease the propagation effects. When examining the propagation effects in the diagnostic direction, we discovered that the propagation effects strongly depend on which output probability we look at and which noisy-OR parameter we vary.

When the accountability property is not satisfied/assumed, the leaky noisy-OR model can be used for the causal mechanism's elicitation task to complete the CPT. The same results generally hold for using the leaky noisy-OR model as for using the noisy-OR model. The leak probability increases the propagation effects *slightly* in the causal direction, since we discovered that the larger the leak probability, the larger the gradient of the corresponding sensitivity function. However, we again emphasize that this effect is minimal.

We demonstrated that including an additional cause variable does not lead to larger propagation effects. We observed that the prior probabilities of the corresponding causes of the deviating noisy-OR parameter need to have more skewed prior probability distributions to attain the same propagation effects as for a causal mechanism involving two cause variables, indicating possibly less propagation effects. However, a network engineer should keep in mind that the involvement of additional causes immediately enlarges the elicitation task, as the probabilities needed by an interaction model to complete the CPT increases linearly in the number of cause variables.

We also studied a situation wherein a causal mechanism modelled the leak as an explicit cause in our model. The prior probability of the leak cause was set to 1, indicating the presence of unavoidable unmodelled causes. The noisy-OR parameter associated with this cause represented the case that the effect occurred due to unmodelled causes. This parameter now attained a probability in the interval $(0, 0.2]$. Again, the causal and diagnostic propagation effects were in line with what we had discovered before.

Finally, we investigated a basic causal mechanism with a common effect variable E and two independent cause variables involving a ternary cause variable. Here, an interaction model called the noisy-MAX was used to compute the parameters involving the presence of more than one cause. Recall that the assumptions underlying the noisy-MAX model indicate that the lower the value of c_i^j of the noisy-MAX parameter associated with cause C_i, the lower the probability (interval) of the corresponding noisy-MAX parameter. In this case, we found that the propagation effects in the causal direction are possibly slightly higher than when using the noisy-OR model, involving binary variables only. The propagation effects in the diagnostic direction now mainly depend on which output probability we look at and *which value* of c_i^j of the noisy-MAX parameter associated with cause C_i we vary. The same result holds for the propagation effects in the diagnostic direction: the propagation effects are possibly higher than when using the noisy-OR model.

We conducted our research for different parameters than Woudenberg and van der Gaag since we considered a different research question. Woudenberg and van der Gaag examined the propagation effects due to deviating model-calculated probability values. We studied the propagation effects due to an inaccurate model's input parameter. We found that inaccurate estimates of the probabilities for the input parameters of the noisy-OR model and its variants can result in different output probabilities, and therefore, possibly harm the validity of a Bayesian network's output. However, we also discovered that inaccurate estimates for the input parameters in many cases will not lead to large propagation effects. These results agree with the findings of Woudenberg and van der Gaag. We will now clearly state the differences and similarities of our findings compared to [1].

For the causal propagation effects, when using the noisy-OR model, we obtained quite similar results as Woudenberg and van der Gaag in [1]. The results of both our studies indicate that the propagation effects in the causal direction can only be large if the yet unobserved cause variables corresponding to the varied parameter in the mechanism have quite skewed prior probability distributions. For clarification, when Woudenberg and van der Gaag set $x = Pr(e|c_1, c_2)$, they found that large propagation effects in the causal direction are possible if both causes C_1 and C_2 are likely to be present. When we set, e.g. $x = Pr(e|\neg c_1, c_2)$, then we discovered that large propagation effects are possible only if cause C_1 is likely to be absent and C_2 present. Contrary to [1], we found that the gradient of the corresponding sensitivity function, when studying the causal propagation effects, is *dependent* of the other noisy-OR parameter involved. This difference can be explained by the fact that Woudenberg and van der Gaag varied a parameter that is actually calculated from the input parameters by the model's rules; the other noisy-OR parameter is, in fact, *hidden* in the parameter they vary. In our study, the other noisy-OR parameter is in the gradient since the parameter we vary is not calculated from the input parameters but is an input parameter itself. However, the effect of this other noisy-OR parameter being present in the gradient in our study is rather small since the properties underlying the noisy-OR model assume that noisy-OR parameters attain high values in practice [5]. Since only a small value of the noisy-OR parameter involved in the gradient increases the propagation effects, we discovered that the effect of this parameter is rather small.

We now present the differences and similarities of our findings with Woudenberg and van der Gaag when studying the propagation effects in the diagnostic direction when using the noisy-OR model. Since the corresponding sensitivity functions are rectangular hyperbola functions when studying the propagation effects in the diagnostic direction, the effect of deviations in the x-value on the output probability of interest mainly depends on the location of the vertex of the corresponding hyperbola branch. We therefore started examining the influences of the parameters involved on the location of the vertex as done in [1]. We discovered, like Woudenberg and van der Gaag, that the propagation effects in the diagnostic direction can be large if the yet unobserved cause variables corresponding to the varied parameter in the mechanism have quite skewed prior probability distributions. In addition, we found that small values for the other noisy-OR parameter involved, that is the noisy-OR parameter which is not set to x, will increase the propagation effects. These results indeed agree with the findings in [1]. However, in contrast to Woudenberg and van der Gaag, we kept in mind the underlying assumptions of the noisy-OR model, and therefore our final observations are quite different. Contrary to [1], we found that the propagation effects in the diagnostic direction are mainly dependent on which output probability we studied and which noisy-OR parameter we varied.

Now we will indicate the most relevant similarities and differences, regarding the propagation effects in the causal direction, when studying the effect of possibly dependent cause variables, the leak probability, and the inclusion of (an) additional cause variable(s). When studying the effect of possibly dependent cause variables, we found similar results as Woudenberg and van der Gaag: possibly dependent cause variables can affect the propagation effects in the causal direction. The extent to which this could influence the propagation effects is dependent on the conditional probability $Pr(c_2|c_1)$, that is the probability capturing the strength of the dependency between the causes, compared to the prior probability $Pr(c_2)$. When studying the effect of the leak, Woudenberg and van der Gaag found that the leak probability was absent in the gradient when studying the causal propagation effects, and thus, the propagation effects were independent of the leak probability. The leak probability was absent in the gradient in their study, because again it is *hidden* in the parameter they vary. In our study, we discovered that the leak probability is indeed present in the gradient but can only *slightly* increase the propagation effects. Furthermore, we found similar results as Woudenberg and van der Gaag when including additional cause variable(s): the same propagation effects are possible as with mechanisms involving less cause variables; however, more skewed prior probabilities of the yet unobserved causes are needed to attain the same propagation effects.

Again, our most relevant findings when studying the effect of possibly dependent cause variables, the leak probability, and the inclusion of (an) additional cause variable(s) regarding the propagation effects in the diagnostic direction, are the same as before. The propagation effects in the diagnostic direction were mainly determined by which output probability we studied and which (leaky) noisy-OR parameter we varied. The effect of possibly dependent cause variables, the leak probability, and the inclusion of (an) additional cause variable(s) regarding the propagation effects in the diagnostic direction were all rather small and in line with our findings for the basic causal mechanism using the noisy-OR model.

Finally, we discovered that the overall propagation effects possibly increase when using the noisy-MAX model instead of the noisy-OR model. Woudenberg and van der Gaag concluded that similar results hold for the noisy-OR as for the noisy-MAX. Consequently, our findings differ from theirs since we found that the propagation effects for the models are different. The propagation effects corresponding to the noisy-MAX model are possibly larger since the model's underlying assumptions are different than for the noisy-OR. In contrast to Woudenberg and van der Gaag, we kept in mind the underlying assumptions to the corresponding interaction model, and therefore obtained different results.

Lastly, we want to share a remarkable finding that emerged from our study when examining the propagation effects in the diagnostic direction due to deviations in (leaky) noisy-OR/MAX parameters. This finding relates to the maximal value of the gradient of the sensitivity functions where the propagation effects could become moderate at most. When we focused on the interval $x \in [0.6, 1]$, the maximum value for the gradient was always equal to $|0.416666|$. The fact that this number at all times showed up raises questions. We acknowledge that there must be an explanation, but unfortunately, we don't have it yet.
Note that this observation *only* applies to the sensitivity functions where the propagation effects in the diagnostic direction can become small or moderate at most, *and* where we focus on the interval for $x \in [0.6, 1]$.

7 Conclusion

The elicitation of all the required probabilities in a Bayesian network is often the most demanding challenge. A network engineer can use interaction models like the noisy-OR model and its generalisations to ease this elicitation task. Recall that these interaction models require limited probability estimates since the remaining probabilities of the CPT are computed by the model's rules. The input parameters of the interaction model are the probability estimates taken in by the model to compute the remaining probabilities. However, obtaining accurate probability estimates for the input parameters may be a difficult job. Therefore, it can be helpful for a network engineer to determine beforehand how much effort he/she should put into acquiring accurate estimates for the input parameters in order to ensure the validity of the Bayesian network's output.

In this thesis, we examined the consequences of inaccurate estimates of (leaky) noisy-OR/MAX parameters on output probabilities. We studied generic sensitivity functions of different models and gained insight into the consequences of specific causal interaction models' input parameters being inaccurate. In our study, we assumed that the assumptions underlying the interaction models do indeed hold.

We discovered that the use of inaccurately estimated parameters of causal interaction models can result in different output probabilities, and therefore, possibly harm the validity of a Bayesian network's output. Since large propagation effects are possible under certain conditions, a network engineer is strongly advised to check in advance how accurate the probability estimates of the input parameters of the corresponding interaction model must be. Moreover, the results obtained in this study pertain to rather basic causal mechanisms; the results might deviate, for example, for larger network structures.

Our research found that the propagation effects in the causal direction can only be large if the prior probability of the cause associated with the noisy-OR parameter under study is large and the prior probability/probabilities of the other cause(s) small. For the diagnostic direction, we discovered that the propagation effects are highly dependent on which output probability we look at and which noisy-OR parameter we vary. We also found that the causal and diagnostic propagation effects are small or moderate in many cases. Since it is quite favourable for a network engineer to make use of causal interaction models, we highly recommend that he/she should beforehand verify whether large propagation effects may happen, using the insights from this thesis. A network engineer can then decide how much effort he/she has to put into obtaining accurate estimates for those parameters.

We have examined the propagation effects due to (leaky) noisy-OR/MAX parameters changes. For further research, the propagation effects of the interaction models such as the noisy-AND/MIN can be studied [2]. Furthermore, we only performed one-way sensitivity analyses. To acquire additional insight into the effects of inaccurate input parameters of interaction models, we advise performing higher-order sensitivity analyses as well. In a higher-order sensitivity analysis, multiple probabilities are varied at the same time, and therefore the joint combined effect of deviating input parameters is examined. Lastly, we leave the finding concerning the gradient's maximal value mentioned in the discussion in Section 6 for further research.

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Appendices

Appendix A 4.1.3

A.1 4.1.3.1

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```
In[22]:= f[a_, b_, c_, x_] := (a*b*(1 - c)) / (c*(1 - a*b)*(x + (a*b)/(c*(1 - a*b)))^2)

         FindMaximum[{f[a, b, c, x], 0.6 ≤ a ≤ 1, 0.6 ≤ x ≤ 1, 0 < b < 1, 0 < c < 1}, {{a, #[[1]]}, {b, #[[2]]}, {c, #[[3]]}, {x, #[[4]]}}] & /@ {{1, 0.001, 0.001, 0.6}, {0.6, 0.99, 0.99, 1}}
Out[23]= {{0.416666, {a → 0.775347, b → 3.28813×10^-7, c → 4.24907×10^-7, x → 0.6}}, {0.416666, {a → 0.775347, b → 3.28813×10^-7, c → 4.24907×10^-7, x → 0.6}}}

In[24]:= f[a_, b_, c_, x_] := (a*b*(1 - c)) / (c*(1 - a*b)*(x + (a*b)/(c*(1 - a*b)))^2)
         FindMaximum[{f[a, b, c, x], 0.6 ≤ a ≤ 1, 0.6 ≤ x ≤ 1, 0.01 < b < 1, 0.01 < c < 1}, {{a, #[[1]]}, {b, #[[2]]}, {c, #[[3]]}, {x, #[[4]]}}] & /@
          {{1, 0.001, 0.001, 0.6}, {0.6, 0.99, 0.99, 1}}
Out[25]= {{0.412496, {a → 0.600043, b → 0.0100007, c → 0.0100003, x → 0.6}}, {0.412496, {a → 0.600043, b → 0.0100007, c → 0.0100003, x → 0.6}}}

In[26]:= f[a_, b_, c_, x_] := (a*b*(1 - c)) / (c*(1 - a*b)*(x + (a*b)/(c*(1 - a*b)))^2)
         FindMaximum[{f[a, b, c, x], 0.6 ≤ a ≤ 1, 0.6 ≤ x ≤ 1, 0.05 < b < 1, 0.05 < c < 1}, {{a, #[[1]]}, {b, #[[2]]}, {c, #[[3]]}, {x, #[[4]]}}] & /@
          {{1, 0.001, 0.001, 0.6}, {0.6, 0.99, 0.99, 1}}
Out[27]= {{0.395741, {a → 0.600012, b → 0.050001, c → 0.0500003, x → 0.6}}, {0.395741, {a → 0.600012, b → 0.050001, c → 0.0500003, x → 0.6}}}
```

$a = Pr(e|c_1, \neg c_2), b = Pr(c_1)$ and $c = Pr(c_2)$.

A.2 4.1.3.2

```
In[28]:= g[a_, b_, c_, x_] := (a/(b*(1-a)) * (1 - (1-c*a)/(c*(1-a)))) / (x * (1-c*a)/(c*(1-a)) + a/(b*(1-a)))^2

In[34]:= FindMinimum[{g[a, b, c, x], 0.6 ≤ a ≤ 1, 0.6 ≤ x ≤ 1, 0 < b < 1, 0 < c < 1}, {{a, #[[1]]}, {b, #[[2]]}, {c, #[[3]]}, {x, #[[4]]}}] & /@
          {{0.99, 0.99, 0.001, 0.6}, {0.01, 0.5, 0.5, 0.3}, {0.4, 0.9, 0.1, 0.6}}
Out[34]= {{-0.416666, {a → 0.999999, b → 0.520073, c → 0.237831, x → 0.6}},
          {-0.416666, {a → 0.999999, b → 0.385073, c → 0.187681, x → 0.6}}, {-0.416666, {a → 0.999999, b → 0.385731, c → 0.187942, x → 0.6}}}
```

$a = Pr(e|\neg c_1, c_2), b = Pr(c_1)$ and $c = Pr(c_2)$.

Appendix B 4.2.2

B.1 4.2.2.1

WOLFRAM MATHEMATICA | STUDENT EDITION — Demonstrations | MathWorld | Wolfram Community | Help

```
In[16]:= m[a_, b_, d_, q_, x_] := (a (1 - b)) / (b (q/(b*d) - a) (x + a/(b*(q/(b*d) - a)))^2)

In[21]:= FindMaximum[{m[a, b, d, q, x], 0.6 ≤ a ≤ 1, 0.6 ≤ x ≤ 1, 0 < b < 1, 0 < d < 1, 0 < q < 1}, {{a, #[[1]]}, {b, #[[2]]}, {d, #[[3]]}, {q, #[[4]]}, {x, #[[5]]}}] & /@
          {{0.9, 0.5, 0.5, 0.5, 0.99}, {0.6, 0.5, 0.5, 0.5, 0.99}, {0.6, 0.1, 0.1, 0.9, 0.6}}

Out[21]= {{0.416666, {a → 0.84354, b → 2.08182×10^-7, d → 0.397759, q → 0.559209, x → 0.6}},
          {0.416666, {a → 0.725605, b → 2.08182×10^-7, d → 0.468729, q → 0.566853, x → 0.6}}, {0.416666, {a → 0.769538, b → 2.08182×10^-7, d → 0.446552, q → 0.572732, x → 0.6}}}
```

$a = Pr(e|c_1, \neg c_2), b = Pr(c_2|c_1)$, $d = Pr(c_1)$ and $q = Pr(c_2)$.

B.2 4.2.2.2

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```
In[•]:= n[a_, b_, d_, q_, x_] := ((a*q/d)/(b (1-a)) (1 - (1-a*b)/(b (1-a)))) / (x * (1-a*b)/(b (1-a)) + (a*q/d)/(b (1-a)))^2

In[•]:= FindMinimum[{n[a, b, d, q, x], 0.6 ≤ a ≤ 1, 0.6 ≤ x ≤ 1, 0 < b < 1, 0 < d < 1, 0 < q < 1}, {{a, #[[1]]}, {b, #[[2]]}, {d, #[[3]]}, {q, #[[4]]}, {x, #[[5]]}}] & /@
   {{0.9, 0.5, 0.5, 0.5, 0.99}, {0.6, 0.5, 0.5, 0.5, 0.99}, {0.6, 0.1, 0.1, 0.9, 0.6}}

Out[•]= {{-0.416666, {a → 0.999999, b → 0.184973, d → 0.560874, q → 0.274277, x → 0.6}},
   {-0.416666, {a → 0.999988, b → 0.0181928, d → 0.59435, q → 0.350127, x → 0.6}}, {-0.416667, {a → 0.999988, b → 0.0136418, d → 0.571685, q → 0.338336, x → 0.6}}}
```

$a = Pr(e|\neg c_1, c_2), b = Pr(c_2|c_1)$, $d = Pr(c_1)$ and $q = Pr(c_2)$.

Appendix C 4.3

C.1 4.3.2

WOLFRAM MATHEMATICA | STUDENT EDITION — Demonstrations | MathWorld | Wolfram Community | Help

```
In[24]:= q[a_, b_, c_, p_, x_] := ((1-c)/c (a + p * (1-b)/b) / ((1-b)/b + (1-a)/(1-p))) / (x + (a-p)/(1/b - a - p * (1-b)/b) + ((1-c)/c (a + p * (1-b)/b)) / ((1-b)/b + (1-a)/(1-p)))^2

In[25]:= FindMaximum[{m[a, b, c, p, x], 0.6 ≤ a ≤ 1, 0.6 ≤ x ≤ 1, 0 < b < 1, 0 < c < 1, 0 < p ≤ 0.2}, {{a, #[[1]]}, {b, #[[2]]}, {c, #[[3]]}, {p, #[[4]]}, {x, #[[5]]}}] & /@
   {{0.6, 0.5, 0.5, 0.05, 0.99}, {0.8, 0.5, 0.5, 0.2, 0.99}, {0.9, 0.1, 0.1, 0.2, 0.6}}

   ••• Power: Infinite expression 1/0. encountered.
   ••• Infinity: Indeterminate expression 0. ComplexInfinity encountered.
   ••• FindMaximum: The function value Indeterminate is not a real number at {a, b, c, p, x} = {0.8, 0.5, 0.5, 0.2, 0.99}.
   ••• IPOPTMinimize: Invalid objective function. The objective function doesn't evaluate to a real−valued numeric result at the initial point.
   ••• Power: Infinite expression 1/0. encountered.
   ••• Infinity: Indeterminate expression 0. ComplexInfinity encountered.
   ••• FindMaximum: The function value Indeterminate is not a real number at {a, b, c, p, x} = {0.8, 0.5, 0.5, 0.2, 0.989994}.
   ••• FindMaximum: Interior point method fails to converge.

Out[25]= {{0.416666, {a → 0.77184, b → 2.08191×10^-7, c → 0.102812, p → 0.132258, x → 0.6}},
   FindMaximum[{m[a, b, c, p, x], 0.6 ≤ a ≤ 1, 0.6 ≤ x ≤ 1, 0 < b < 1, 0 < c < 1, 0 < p ≤ 0.2}, {{a, {0.8, 0.5, 0.5, 0.2, 0.99}[[1]]},
     {b, {0.8, 0.5, 0.5, 0.2, 0.99}[[2]]}, {c, {0.8, 0.5, 0.5, 0.2, 0.99}[[3]]}, {p, {0.8, 0.5, 0.5, 0.2, 0.99}[[4]]}, {x, {0.8, 0.5, 0.5, 0.2, 0.99}[[5]]}}],
   {0.416666, {a → 0.823118, b → 2.08182×10^-7, c → 0.102613, p → 0.140771, x → 0.6}}}
```

$a = Pr(e|\neg c_1, c_2), b = Pr(c_1)$, $c = Pr(c_2)$ and $p = p$.

Appendix D 4.4

D.1 4.4.2

WOLFRAM MATHEMATICA | STUDENT EDITION — Demonstrations | MathWorld | Wolfram Community | Help

```
In[1]:= t[a_, b_, c_, d_, f_, x_] := ((d*b+f*c-d*f*b*c)*(1-a) / (a*(1-(d*b+f*c-d*f*b*c)))) / (x + (d*b+f*c-d*f*b*c) / (a*(1-(d*b+f*c-d*f*b*c))))^2

In[4]:= FindMaximum[{t[a, b, c, d, f, x], 0.6 ≤ d ≤ 1, 0.6 ≤ f ≤ 1, 0 < a < 1, 0 < b < 1, 0 < c < 1, 0.6 ≤ x ≤ 1}, {{a, #[[1]]}, {b, #[[2]]}, {c, #[[3]]}, {d, #[[4]]}, {f, #[[5]]}, {x, #[[6]]}}] & /@
   {{0.8, 0.8, 0.8, 0.6, 0.6, 0.6}, {0.2, 0.2, 0.2, 0.6, 0.6, 0.6}, {0.1, 0.1, 0.1, 0.7, 0.7, 0.7}}

Out[4]= {{0.416666, {a → 6.34063×10^-7, b → 2.45256×10^-7, c → 2.45256×10^-7, d → 0.775592, f → 0.775592, x → 0.6}},
   {0.416666, {a → 6.34063×10^-7, b → 2.45256×10^-7, c → 2.45256×10^-7, d → 0.775592, f → 0.775592, x → 0.6}},
   {0.416666, {a → 6.34063×10^-7, b → 2.45256×10^-7, c → 2.45256×10^-7, d → 0.775592, f → 0.775592, x → 0.6}}}

   Assuming a ragged array | Use as a list of pairs instead
   sublengths | flatten
```

$a = Pr(c_1), b = Pr(c_2)$, $c = Pr(c_3), d = Pr(e|\neg c_1, c_2, \neg c_3)$ and $f = Pr(e|\neg c_1, \neg c_2, c_3)$.

Appendix E 4.5

E.1 4.5.2

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```
In[10]:= r[a_, b_, d_, f_, x_] := ((d*b+f-d*f*b)*(1-a) / (a*(1-(d*b+f-d*f*b)))) / (x + (d*b+f-d*f*b)/(a*(1-(d*b+f-d*f*b))))^2

In[13]:= FindMaximum[{r[a, b, d, f, x], 0.6 ≤ d ≤ 1, 0 < f ≤ 0.2, 0 < a < 1, 0 < b < 1, 0.6 ≤ x ≤ 1}, {{a, #[[1]]}, {b, #[[2]]}, {d, #[[3]]}, {f, #[[4]]}, {x, #[[5]]}}] & /@
  {{0.8, 0.8, 0.6, 0.6, 0.01}, {0.2, 0.2, 0.6, 0.6, 0.1}, {0.1, 0.1, 0.7, 0.6, 0.2}}

Out[13]= {{0.416666, {a → 6.31615×10^-7, b → 2.45757×10^-7, d → 0.775591, f → 1.88362×10^-7, x → 0.6}},
  {0.416666, {a → 6.31615×10^-7, b → 2.45757×10^-7, d → 0.775591, f → 1.88362×10^-7, x → 0.6}},
  {0.416666, {a → 6.31615×10^-7, b → 2.45757×10^-7, d → 0.775591, f → 1.88362×10^-7, x → 0.6}}}
```

$a = Pr(c_1), b = Pr(c_2)$, $d = Pr(e|\neg c_1, c_2, \neg c_3)$ and $f = Pr(e|\neg c_1, \neg c_2, c_3)$.

Appendix F 5.2

F.1 5.2

WOLFRAM MATHEMATICA | STUDENT EDITION — Demonstrations | MathWorld | Wolfram Community | Help

```
In[3]:= y[b_, c_, d_, f_, g_, x_] := (f*d (1 - b) + g*c - g*f*c*d) / (b*(1 - f*d)*(x + (f*d+g*c-g*f*c*d)/(b*(1-f*d)))^2)

In[4]:= FindMaximum[{y[b, c, d, f, g, x], 0.6 ≤ f ≤ 1, 0.4 ≤ x ≤ 1, 0 < b < 1, 0 < c < 1, 0.7 ≤ g ≤ 1, 0 < d < 1}, {{b, #[[1]]}, {c, #[[2]]}, {d, #[[3]]}, {f, #[[4]]}, {g, #[[5]]}, {x, #[[6]]}}] & /@
  {{0.99, 0.001, 0.01, 0.6, 0.7, 0.4}, {0.9, 0.1, 0.1, 0.6, 0.7, 0.4}}

Out[4]= {{0.625, {b → 0.536938, c → 0.257277, d → 6.74835×10^-8, f → 0.750893, g → 0.834803, x → 0.4}},
  {0.625, {b → 0.555408, c → 0.264208, d → 6.49039×10^-8, f → 0.754339, g → 0.840866, x → 0.4}}}
```

$b = Pr(c_1^1), c = Pr(c_1^2), d = Pr(c_2), f = Pr(e|c_1^0, c_2)$ and $g = Pr(e|c_1^2, \neg c_2)$.

F.2 5.2

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```
In[5]:= q[a_, b_, c_, d_, f_, x_] := ((d*a + f*b)*(1 - c)) / (c*(1 - (d*a + f*b))*(x + (d*a+f*b)/(c*(1-(d*a+f*b))))^2)

In[6]:= FindMaximum[{q[a, b, c, d, f, x], 0.7 ≤ f ≤ 1, 0.6 ≤ x ≤ 1, 0 < a < 1, 0 < b < 1, 0.4 ≤ d ≤ 1, 0 < c < 1}, {{a, #[[1]]}, {b, #[[2]]}, {c, #[[3]]}, {d, #[[4]]}, {f, #[[5]]}, {x, #[[6]]}}] & /@
  {{0.99, 0.001, 0.001, 0.6, 0.7, 0.6}, {0.9, 0.1, 0.1, 0.6, 0.7, 0.6}}

Out[6]= {{0.416666, {a → 3.0174×10^-7, b → 2.26303×10^-7, c → 6.3493×10^-7, d → 0.634626, f → 0.837225, x → 0.6}},
  {0.416666, {a → 3.0174×10^-7, b → 2.26303×10^-7, c → 6.3493×10^-7, d → 0.634626, f → 0.837225, x → 0.6}}}
```

$a = Pr(c_1^1), b = Pr(c_1^2), c = Pr(c_2), d = Pr(e|c_1^1, \neg c_2)$ and $f = Pr(e|c_1^2, \neg c_2)$.