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A BOUQUET THEOREM FOR THE MILNOR FIBRE

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ABSTRACT. Let $f:(X, x) \to (\mathbb{C}, 0)$ be an analytic function germ and (X, x) an analytic space. Assume that both (X, x) and f have an isolated singularity. Let $n = \dim X - 1 \neq 2$. Then the Milnor fibre F has the homotopy type of a bouquet $F_0 \vee S^n \vee \cdots \vee S^n$, where F_0 is the complex link of (X, x).

Introduction

In this paper we study analytic function germs $f:(X, x) \to (\mathbb{C}, 0)$ where both (X, x) and f have an isolated singularity. Our main statement is:

Theorem 0.1. *Let $n = \dim X - 1 \neq 2$. The Milnor fibre of F is homotopy equivalent to a bouquet*

$$F_0 \vee S^n \vee \ldots \vee S^n$$

where F_0 is the Milnor fibre of a general linear form on (X, x) (the complex link).

This statement generalizes work of Milnor [Mi-2] in case $X = \mathbb{C}^{n+1}$, Hamm [Ha] and Lê [Lê-2] in case X is an isolated complete intersection and [Lê-4] in case (X, x) has "Milnor's property". If (X, x) is irreducible, then the neighbourhood boundary of (X, x) is connected. If one deals with the case X reducible, then one can restrict f to each of the irreducible components. The spheres may be attached to different components of F_0, but we still use the notation $F \overset{h}{\cong} F_0 \vee S^n \vee \cdots \vee S^n$.

The paper is organized as follows: In §1 we recall properties of functions on singular spaces and the generic approximation of those functions. The main reference for this is [Lê-4]. In §2 we show the "additivity of the vanishing homology", more precisely

$$\tilde{H}_*(F) = \bigoplus \tilde{H}_*(F_i),$$

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where F is the Milnor fibre of f and the F_i's are the Milnor fibres of the singularities in an approximation of f. This shows that F is on the level of homology already a bouquet of spheres (also if $n = 2$). In §3 we define a geometric map

$$\rho : F_0 \vee S^n \vee \cdots \vee S^n \to F,$$

which induces an isomorphism of fundamental groups if $n \geq 3$. Moreover, we show that if $n \geq 3$, the fundamental groups of F and $K := f^{-1}(0) \cap X \cap S_\epsilon(x)$ depend only on the fundamental group of $X \cap S_\epsilon(x)$, the neighborhood boundary of (X, x). In §4 we prove our main theorem with the help of the (generalized) Whitehead theorem. In §5 we consider quotient spaces and show in some special cases the bouquet theorem if $n \geq 1$.

1. Preliminaries

Let (X, x) be a germ of a complex analytic space with isolated singularity at $x \in X$, and

$$f : (X, x) \to \mathbb{C}$$

a complex function germ, which has an isolated singularity in the following sense:

Definition 1.1. $f : (X, x) \to \mathbb{C}$ has an isolated singularity at x if f is regular in a punctured neighbourhood of x.

NB. Lê introduced in [Lê-2] the notion of a complex function with isolated singularity on a complex analytic space with a Whitney stratification $\mathcal{S}$. In case (X, x) is isolated, then the stratification $\mathcal{S} = \{\{x\}, X\backslash\{x\}\}$ is Whitney and Lê's definition reduces to the above definition.

The existence of a Milnor type fibration is guaranteed by the following result of Lê:

Theorem 1.2. *Let $f : X \to \mathbb{C}$ be a complex analytic function with an isolated singularity at x on X relative to the Whitney stratification $\mathcal{S}$. There is an embedding of a neighbourhood U of x in X into $\mathbb{C}^N$, such that, there is $\epsilon_0 > 0$, such that, for any ϵ, $\epsilon_0 \geq \epsilon > 0$, there is $\eta_\epsilon > 0$, such that, for any η, $\eta_\epsilon \geq \eta > 0$, the function f induces a map $\varphi_{\epsilon,\eta}$ from $B_\epsilon(x) \cap X \cap f^{-1}(D_\eta(f(x)))$ onto $D_\eta(f(x))$:*

$$\varphi_{\epsilon,\eta} : B_\epsilon(x) \cap X \cap f^{-1}(D_\eta(f(x))) \to D_\eta(f(x))$$

which is a locally trivial fibration over the punctured disk

$$D_\eta^*(f(x)) := D_\eta(f(x)) - \{f(x)\}.$$

Proof. [Lê-4, 1.3]. □

As stated, this theorem is a consequence of the fibration theorem in [Lê-1], but in this particular case of an isolated singularity we furthermore obtain a topological fibration, induced by f, of $S_\epsilon(x) \cap X \cap f^{-1}(D_\eta(f(x)))$ onto the disk $D_\eta(f(x))$.

Morsification. Let $\overline{f} : \mathscr{U} \to \mathbb{C}$ be defined on an open neighbourhood of 0 in $\mathbb{C}^N$ and let X be a closed analytic space in $\mathscr{U}$ with isolated singularity in $x = 0$. We call f the restriction of $\overline{f}$ to X.

A general definition of [Lê-4, 2.1] reduces in this case to

Definition 1.3. $f : (X, x) \to \mathbb{C}$ is of *Morse type* if

(1) $d\overline{f}(0)$ has rank 1 and the kernel Ker $d\overline{f}(0)$ is not a limit of tangent hyperplanes to $X - \{x\}$,
(2) $f|_{X-\{x\}}$ has only nondegenerate critical points (ordinary quadratic points, A_1-points).

Using the notations as above, we state the following result (cf. [Lê-3, 2.2]:

Proposition 1.4. *There is an open dense subset* Ω *of the space of complex linear forms of* $\mathbb{C}^N$, *such that, for any linear form* $\ell \in \Omega$, *there is* $\epsilon_0 > 0$ *such that, for any* ϵ, $\epsilon_0 \geq \epsilon > 0$, *there is* $\tau_\epsilon > 0$, *such that, for any* t, $\tau_\epsilon \geq |t| > 0$, *the restriction to* X *of the function* $\overline{f} + t\ell$ *is of Morse type in* $B_\epsilon(0) \cap X$.

Denote by $D_\tau(0)$ the closed disk of the complex line $\mathbb{C}$ centered at 0 with radius τ sufficiently small. We can define the following unfolding

$$\Phi : (X \cap \mathscr{U}) \times D_\tau(0) \to \mathbb{C} \times D_\tau(0)$$

of f by

$$\Phi(z, t) := (\overline{f}(z) + t\ell(z), t)$$

and denote

$$f_t(z) = \overline{f}(z) + t\ell(z).$$

The next theorems of Lê are by him mostly stated for open balls; but his proofs are (at least in our context) also valid for closed balls.

Proposition 1.5. *There exist* ϵ, η, *satisfying the conditions from* 1.2 *and* $\tau \leq \tau_\epsilon$ *with the property*: *The unfolding* Φ *of* f *induces a map* $\Phi(\epsilon, \eta, \tau)$ *of* $B_\epsilon(0) \times D_\tau(0)$ *onto* $D_\eta(f(0)) \times D_\tau(0)$, *which is a locally* *trivial topological fibration over the complement in* $D_\eta(f(0)) \times D_\tau(0)$ *of a curve* Δ.

Proof. [Lê-4, 3.2], where also details are given about the curve Δ. $\square$

Corollary 1.6. *Let* ϵ *and* τ *be fixed as above. The fibres*

$$\Phi(\epsilon, \eta, \tau)^{-1}(u, 0) \quad \text{and} \quad \Phi(\epsilon, \eta, \tau)^{-1}(u, t)$$

are homeomorphic whenever $(u, 0)$ *and* (u, t) *are in* $D_\eta(f(0)) \times D_\tau(0) - \Delta$.

Proof. [Lê-4, 3.3]. $\square$

Proposition 1.7. *There exist* $\epsilon, \eta, \tau_\epsilon, \tau$, *satisfying the conditions from 1.5 with the property: For any* $t, \tau > |t|$, *the space* $f_t^{-1}(D_\eta(f(0))) \cap B_\epsilon(0)$ *is homeomorphic to* $f^{-1}(D_\eta(f(0))) \cap B_\epsilon(0)$ *and, therefore, is contractible; moreover* $f_t^{-1}(D_\eta(f(0))) \cap S_\epsilon(0)$ *is homeomorphic to* $f^{-1}(D_\eta(f(0))) \cap S_\epsilon(0)$ *and* $K_t := f_t^{-1}(f(0)) \cap S_\epsilon(0)$ *is homeomorphic to* $K_0 := f^{-1}(f(0)) \cap S_\epsilon(0)$.

Proof. [Lê-4, 3.4]. Choose τ sufficiently small such that Δ does not meet the set $\partial D_\eta(f(0)) \times D_\tau(0)$ $\square$

2. The relative homology of a holomorphic map

Let as before (X, x) be a space with isolated singularity, embedded in $(\mathbb{C}^N, 0)$. Let $B = B_\epsilon(0)$ be the closed ϵ-ball in $\mathbb{C}^N$ around 0 and let

$$h : X \cap B \to \mathbb{C}$$

be a holomorphic function with isolated singularities and critical values $b_0 = h(x), b_1, \ldots, b_\sigma$ contained in a closed disc $D := D_\eta(f(x)) \subset \mathbb{C}$. We make the following assumption:

$$h^{-1}(w) \pitchfork (X \cap \partial B) \quad \text{for all } w \in D.$$

We choose

(i) a system of disjoint closed discs D_i around every b_i,
(ii) points $s_i \in \partial D_i$ and $\tilde{s} \in \partial D$,
(iii) a system of paths $\gamma_0, \gamma_1, \ldots, \gamma_\sigma$ from $\tilde{s}$ to s_i (in the usual way, see Figure 1).

We use the following notations for any $A \subset \mathbb{C}$ and $w \in \mathbb{C}$

$$X_A = h^{-1}(A), \qquad X_w = h^{-1}(w),$$
$$D' = \bigcup D_i, \qquad \Gamma = \bigcup \gamma_i .$$

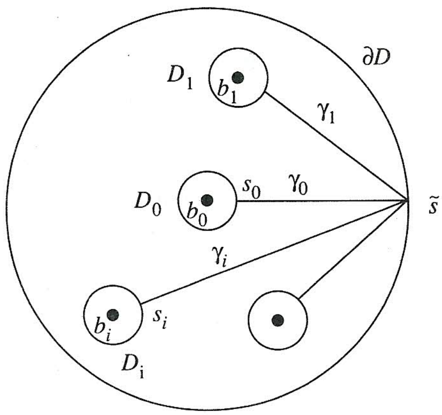

FIGURE 1

Lemma 2.1. $H_*(X_D, X_{\tilde{s}}) \cong \bigoplus_{i=0}^{\sigma} H_*(X_{D_i}, X_{s_i})$

Proof. Since h is locally trivial over the complement of D, the pair $(X_D, X_{\tilde{s}})$ is homotopy equivalent to $(X_{D'\cup\Gamma}, X_\Gamma)$. So

$$H_*(X_D, X_{\tilde{s}}) = H_*(X_{D'\cup\Gamma}, X_\Gamma) = H_*(X_{D'}, X_{D'\cap\Gamma}) = \bigoplus_{i=0}^{\sigma} H_*(D_i, X_{s_i})$$

(by excision). □

Let now $x = x_0, x_1, \ldots, x_\sigma$ be the critical points of h. For notational convenience we assume that the critical values $h(x_0) = b_0, \ldots, h(x_\sigma) = b_\sigma$ are all different. This does not influence the result. Next apply the local Milnor construction at $x_0, \ldots., x_\sigma$ and suppose that the sets $B_i = B_{\epsilon_i}(x_i)$ and $D_{\eta_i}(h(x_i)) = D_i$ are all different and inside B or D. Next write:

$$E_i = B_i \cap X_{D_i}, \qquad E = X_D,$$
$$F_i = B_i \cap X_{s_i}, \qquad F = X_{\tilde{s}}.$$

Proposition 2.2. $H_*(E, F) = \bigoplus_{i=0}^{\sigma} H_*(E_i, F_i)$.

Proof. First remark that (X_{D_i}, X_{s_i}) and $(E_i \cup X_{s_i}, X_{s_i})$ are homotopy equivalent relative X_{s_i} (For details cf. [Lo].) Next

$$H_*(X_{D_i}, X_{s_i}) = H_*(E_i \cup X_{s_i}, X_{s_i}) = H_*(E_i, F_i)$$

(by excision). Apply Lemma 2.1. □

Remark 2.3. The Main Example. We apply Proposition 2.2 to the situation before, $f : (X, x) \to \mathbb{C}$ a function with isolated singularity and $h = f_t$ an approximation (from §1). Then F is diffeomorphic to the Milnor fibre of f and E is contractible. The local singularities at $x_1, \dots, x_\sigma$ are all of Morse type, so their Milnor fibres are homotopy equivalent to a sphere S^n. Although h itself is not linear the fibre F_0 is diffeomorphic to the complex link of X at x (the Milnor fibre of a generic linear function) since $dh(x)$ is a general linear map (cf. [Go-Ma]). It follows that

$$\tilde{H}_k(F) = H_{k+1}(E, F) = \bigoplus_{i=0}^{\sigma} H_{k+1}(E_i, F_i) = \begin{cases} \tilde{H}_k(F_0) \oplus \mathbb{Z}^\sigma, & k = n, \\ \tilde{H}_k(F_0), & k \neq n. \end{cases}$$

Especially: $\tilde{H}_k(F)$ depends for $k \neq n$ only on the complex link F_0.

Remark 2.4. The same reasoning applies to other deformations of f (having more complicated singularities) as long as E and F are homeomorphic to the space $f^{-1}(D_\eta(f(0))) \cap B_\epsilon(0)$ and respectively $f^{-1}(s) \cap B_\epsilon(0)$, the Milnor fibre of f. We call the formula

$$H_*(E, F) = \bigoplus_{i=0}^{\sigma} H_*(E_i, F_i)$$

the *additivity of the vanishing homology*. In the case of certain nonisolated singularities and $X = \mathbb{C}^{n+1}$ the formula is stated in [Si].

Notation 2.5. From now on we use the notations

$$E^* = X_{D' \cup \Gamma} \overset{h}{\cong} X_D, \qquad F^* = X_\Gamma \overset{h}{\cong} X_{\tilde{s}},$$

so we work with homotopy models of the Milnor construction. Remark that F^* is homeomorphic to the product $\Gamma \times F$. The model of the Milnor fibre contains the vanishing cycles $\partial e_1, \dots, \partial e_\sigma$ for each Morse point $x_1, \dots, x_\sigma$. The corresponding $(n+1)$-cells $e_1, \dots, e_\sigma$ are called the "thimbles" of the vanishing cycles. Denote $e = e_1 \cup \cdots \cup e_\sigma$. The notation $F^* \cup e$ is now clear, and so is the inclusion $F_0 \hookrightarrow F^* \cup e$.

Proposition 2.6. *The inclusion $F_0 \hookrightarrow F^* \cup e$ induces isomorphisms of all homology groups.*

Proof. We have the following inclusions:

$$F_0 \hookrightarrow F^* \hookrightarrow F^* \cup e, \qquad E_0 \hookrightarrow E^* = E^*.$$

For homology we have

$$\begin{tikzcd}
H_q(E_0, F_0) \arrow[r] \arrow[dr, "\alpha"'] & H_q(E^*, F^*) \arrow[r] \arrow[d, equal] & H_q(E^*, F^* \cup e) \arrow[d, equal] \\
& \bigoplus_{i=0}^{\sigma} H_q(E_i, F_i) \arrow[r, "\beta"] & H_q(E_0, F_0)
\end{tikzcd}$$

The vertical mappings are induced by deformation retractions and excision; α = inclusion onto the first factor, β = projection onto the first factor. Since E_0 and E^* are contractible, this implies, that

$$H_q(F_0) \longrightarrow H_q(F^* \cup e)$$

is an isomorphism for all q. □

Notation and Definition 2.7. We first make a homotopy model for the space $F_0 \vee S^n \vee \cdots \vee S^n$. Let p be a base point in F_0 and $p_1, \ldots, p_\sigma$ be base points in σ copies of the n-sphere S^n, denoted by $S_1^n, \ldots, S_\sigma^n$. Use intervals $I_1, \ldots, I_\sigma$ to connect (F_0, p) with each of the spheres (S_k^n, p_k). The resulting space

$$F_0 \cup (I_1 \cup S_1^n) \cup \cdots \cup (I_\sigma \cup S_\sigma^n)$$

is homotopy equivalent to

$$F_0 \vee S_1^n \vee \cdots \vee S_\sigma^n.$$

The space

$$F_0 \cup (I_1 \cup B_1^{n+1}) \cup \cdots \cup (I_\sigma \cup B_\sigma^{n+1})$$

which is defined similarly, is homotopy equivalent to F_0. Take for each k a path Γ_k in F^* from p to a point on ∂e_k (nonintersection with each other, no selfintersections, only one intersection point with ∂e_k) and map for each k an interval I_k homeomorphically onto that path Γ_k, each S_k^n to ∂e_k and each B_k^{n+1} to e_k. This construction results into mappings

$$\rho : F_0 \cup (I_1 \cup S_1^n) \cup \cdots \cup (I_\sigma \cup S_\sigma^n) \to F^*,$$

$$\rho' : F_0 \cup (I_1 \cup B_1^{n+1}) \cup \cdots \cup (I_\sigma \cup B_\sigma^{n+1}) \to F^* \cup e.$$

After composing ρ and ρ' with the homotopy inverses of the obvious maps

$$F_0 \cup (I_1 \cup S_1^n) \cup \cdots \cup (I_\sigma \cup S_\sigma^n) \to F_0 \vee S_1^n \vee \cdots \vee S_\sigma^n,$$

$$F_0 \cup (I_1 \cup B_1^{n+1}) \cup \cdots \cup (I_\sigma \cup B_\sigma^{n+1}) \to F_0 \vee B_1^{n+1} \vee \cdots \vee B_\sigma^{n+1}$$

we get the following maps, which we still denote by ρ and ρ':

$$\rho : F_0 \vee S_1^n \vee \cdots \vee S_\sigma^n \to F^*,$$

$$\rho' : F_0 \vee B_1^{n+1} \vee \cdots \vee B_\sigma^{n+1} \to F^* \cup e.$$

Denote by ρ'' the obvious map

$$F_0 \to F_0 \vee B_1^{n+1} \vee \cdots \vee B_\sigma^{n+1} \to F^* \cup e.$$

Proposition 2.8.

$$\rho : F_0 \vee S_1^n \vee \cdots \vee S_\sigma^n \to F^*$$

induces isomorphisms on all homology groups.

Proof. The induced maps between the homology sequences of the pairs

$$(F_0 \vee B_1^{n+1} \vee \cdots \vee B_\sigma^{n+1}, F_0 \vee S_1^n \vee \cdots \vee S_\sigma^n) \quad \text{and} \quad (F^* \cup e, F^*)$$

are as follows:

$$\begin{array}{ccc}
H_{q+1}(\text{relative}) & \xrightarrow{\rho_*^{\text{rel}}} & H_{q+1}(F^* \cup e, F^*) \\
\downarrow & & \downarrow \\
H_q(F_0 \vee S_1^n \vee \cdots \vee S_\sigma^n) & \xrightarrow{\rho_*} & H_q(F^*) \\
\downarrow & & \downarrow \\
H_q(F_0) = H_q(F_0 \vee B_1^{n+1} \vee \cdots \vee B_\sigma^{n+1}) & \xrightarrow{\rho'_*} & H_q(F^* \cup e)
\end{array}$$

The map ρ_*^{rel} is nothing else than the "identity"

$$\bigoplus H_{q+1}(B_k^{n+1}, S_k^n) \longrightarrow \bigoplus H_{q+1}(e_k, \partial e_k)$$

(this follows by excision). The map ρ'_* is an isomorphism by Proposition 2.6. This implies that ρ_* is an isomorphism. □

3. Isomorphism of fundamental groups

Our goal is to show that

$$\rho : F_0 \vee S^n \vee \cdots \vee S^n \longrightarrow F^*$$

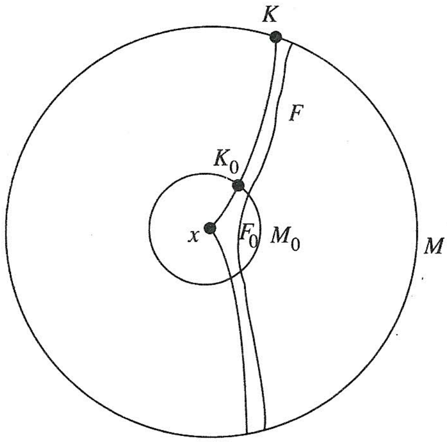

FIGURE 2

is a homotopy equivalence. The main tool from algebraic topology is the Whitehead theorem [Wh, Theorem 3]: *A continuous map between CW-complexes, inducing isomorphisms both on the fundamental groups and on all homology groups of the universal covers, is a homotopy equivalence.*

(NB. In many books the Whitehead theorem is mentioned only in the simply connected case.)

We study first the induced maps on the fundamental groups. We fix some notations

$M = \partial B$, the neighbourhood boundary of (X, x),
$K = f^{-1}(0) \cap M$, the link of the singularity f,
$F = f^{-1}(s) \cap B$, the Milnor fibre of f.

Consider again the generic approximation g of f (Figure 2). Identify M, K and F with their homeomorphic images after this deformation. For the local singularity of g at x a similar construction gives us spaces B_0, M_0, K_0, and F_0. We can assume $F_0 \subset F$ by taking

$$F_0 = g^{-1}(s) \cap B_0, \qquad F = g^{-1}(s) \cap B$$

with a proper choice of s. Moreover we define

$$F_+ = \overline{F - F_0}, \qquad M_+ = \overline{B - B_0}.$$

Consider next the following diagram (all mappings are induced by inclusions):

$$\begin{array}{ccccc}
\pi_1(M_0) & \longrightarrow & \pi_1(M_+) & \longleftarrow & \pi_1(M) \\
\uparrow & & \uparrow & & \uparrow \\
\pi_1(K_0) & \longrightarrow & \pi_1(F_+) & \longleftarrow & \pi_1(K) \\
\downarrow & & \downarrow & & \\
\pi_1(F_0) & \longrightarrow & \pi_1(F) & &
\end{array}$$

The relation between the homotopy types of M, K, and F are discussed in the book of Milnor [Mi-2] if $(X, x) = (\mathbb{C}^{n+1}, 0)$ and in the nonsmooth case in the paper of Hamm [Ha, Satz 2.9].

Lemma 3.1. *The pairs* (M, K) *and* (M_0, K_0) *are* $(n-2)$*-connected.*

Proof. This is shown in Hamm [Ha, Satz 2.9]. He uses the function

$$\| f \|^2 : M \setminus K \to \mathbb{R}$$

and applies Morse theory: M is constructed from K by adding cells of dimension $\geq n$. The same argument applies to the pair (M_0, K_0). $\square$

NB. Our notation (M, K) corresponds to Hamm's notation (Σ^*, Σ).

Lemma 3.2. *The pairs* (F, K), (F_+, K), *and* (F, F_+) *are* $(n-2)$*-connected.*

Proof. Let X be embedded in $\mathbb{C}^N$ as before. Consider a point $p \in \mathbb{C}^N$ and the distance square function

$$D_p : \mathbb{C}^N \to \mathbb{R} \text{ defined by } D_p(x) = \| x - p \|^2 .$$

In Milnor's book on Morse Theory [Mi-1] is shown: Let Z be an n-dimensional complex analytic submanifold of $\mathbb{C}^N$, then D_p, restricted to Z is a Morse function for almost all $p \in \mathbb{C}^N$. Moreover the indices of all critical points are $\leq n$. If the restriction of D_p to $f^{-1}(s)$ is Morse for $p = 0$ then the function $-D_p$ has only Morse points of index $\geq n$ on $F = f^{-1}(s) \cap B$. Otherwise take p near enough to the origin and such that the restriction of D_p to $f^{-1}(s)$ is Morse. Use the homotopy $\{D_{tp}\}_{t\in[0,1]}$ to change B into $B' = \{x \in \mathbb{C}^N \mid D_p(x) \leq \epsilon^2\}$, F into $F' = f^{-1}(s) \cap B'$, and similarly B_0 into B_0', F into F_0', F_+ into F_+'. Since we do not change the diffeomorphism types of the spaces, it is sufficient to consider as well $-D_p$ on F', and to apply Morse theory. $\square$

Since M_0, M_+, and M are clearly homotopy equivalent, it follows that for $n \geq 3$ all arrows in the above diagrams correspond to isomorphisms.

We conclude:

Proposition 3.3. *The mappings*

$$\begin{aligned} \rho''_{\#} &: \pi_1(F_0) \longrightarrow \pi_1(F^*), \\ \rho_{\#} &: \pi_1(F_0 \vee S^n \vee \cdots \vee S^n) \longrightarrow \pi_1(F^*) \end{aligned}$$

are isomorphisms if $n \geq 3$.

Remark 3.4. If $n \geq 3$ we have shown above that for any isolated singularity $f : (X, x) \to (\mathbb{C}, 0)$ holds:

$$\pi_1(K) \cong \pi_1(F) \cong \pi_1(M).$$

So these groups depend only on the link of X in x.

For $n = 2$ this is no longer true. If $(X, x) = (\mathbb{C}^3, 0)$ is and f has an isolated singularity, then according to Mumford [Mu]: $\pi_1(K) \neq 0$. But $\pi_1(M) = 0$ obviously and $\pi_1(F) = 0$ by Milnor [Mi-2].

It is possible to relate the fundamental groups of F and M directly. This is Milnor's "second argument" (cf. [Mi-2, p. 57]), generalized by Hamm (cf. [Ha, p. 251, §(iv)]). Hamm shows that

$$\pi_1(F) \cong \pi_1(M) \ \text{if} \ n \geq 2.$$

As a corollary one knows that $\pi_1(F_0)$ and $\pi_1(F)$ are isomorphic groups for $n \geq 2$. We did not succeed in proving that $\rho''_{\#} : \pi_1(F_0) \longrightarrow \pi_1(F^*)$ is an isomorphism for $n = 2$.

Corollary 3.5. *Let* $n \geq 2$. *In case* X *has a simply connected neighbourhood boundary* M *we have*

$$F \overset{h}{\cong} F^* \overset{h}{\cong} F_0 \vee S^n \vee \cdots \vee S^n.$$

Proof. Apply Whitehead's theorem for simply connected spaces. □

4. The general case

Next we want to prove the theorem in the general case. We know that $(X, 0)$ has a cone-structure over $\partial B = M$. So we have a homeomorphism $B \cong CM$, the cone over M. M is smooth. Consider $\tilde{M}$, the universal cover of M. $\tilde{M}$ is smooth and simply connected. Let $\tilde{B} = C\tilde{M}$, the cone over $\tilde{M}$. The map

$$\pi : \tilde{B} \longrightarrow B$$

is defined as being compatible with the cone structure. $\tilde{B}$ is smooth outside the top $*$ of the cone. The restriction of π to $\tilde{B} \setminus \{0\}$ is a covering, which can be identified with $(0, 1] \times \tilde{M} \to (0, 1] \times M$. Given a function $f : (X, 0) \to \mathbb{C}$ this extends to

$$\begin{tikzcd} \tilde{M} \arrow[r, phantom, "\subset"] \arrow[d, "\pi"'] & \tilde{B} \arrow[d, "\pi"] \arrow[dr, "\tilde{f}=f_0\pi"] & \\ M \arrow[r, phantom, "\subset"] & B \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

Let

$$K = f^{-1}(0) \cap M, \qquad \tilde{K} = \tilde{f}^{-1}(0) \cap \tilde{M},$$
$$F = f^{-1}(s) \cap B, \qquad \tilde{F} = \tilde{f}^{-1}(s) \cap \tilde{B}.$$

Lemma 4.1. *Let $n \geq 3$. Then $\tilde{F}$ and $\tilde{K}$ are simply connected. Moreover the maps*

$$\pi : \tilde{F} \to F \quad \textit{and} \quad \pi : \tilde{K} \to K$$

are universal coverings.

Proof. It has been proved that the inclusions $K \hookrightarrow M$ and $K \hookrightarrow F$ induce isomorphisms on fundamental groups. Hence also $F \hookrightarrow B \setminus \{0\}$ induces an isomorphism for π_1. The statement in the lemma is now a consequence of the following fact on covering spaces: Let A be simply connected and locally path connected, locally simply connected, and let $\pi : \tilde{A} \to A$ be the universal covering. Let $B \subset A$ be connected and such that $B \hookrightarrow A$ induces an isomorphism of the fundamental groups. Let $\tilde{B} = \pi^{-1}(B)$. Then $\tilde{B} \to B$ is the universal covering of B. $\square$

Next we copy our earlier proof (§2) for the spaces $\tilde{B}$, $\tilde{M}$, $\tilde{F}$, $\tilde{K}$. First we consider a deformation $f_t : B \to \mathbb{C}$ and its lift $\tilde{f}_t : \tilde{B} \to \mathbb{C}$. We replace F and F_0 by their homotopy models as before, so we do with $\tilde{F}$ and $\tilde{F}_0$.

It is important that all our constructions avoid $0 \in B$ and $* \in \tilde{B}$, where we have no covering projection. The thimbles e_k of the vanishing cycles downstairs give rise to $\pi : \tilde{e}_k \to e_k$, where $\tilde{e}_k$ is a disjoint union of copies of e_k.

We have now

Lemma 4.2. F_0 *and* $F^* \cup e$ *are homotopy equivalent.*

Proof. The inclusions

$$\begin{array}{ccc} \tilde{F}_0 & \longrightarrow & \tilde{F}^* \cup \tilde{e} \\ \downarrow \pi & & \downarrow \pi \\ F_0 & \longrightarrow & F^* \cup e \end{array}$$

induce isomorphism of the homology groups (compare the proof of 2.6). Since $\tilde{F}_0$ and $\tilde{F}^* \cup \tilde{e}$ are simply connected, and $F_0 \hookrightarrow F$ induces isomorphisms on fundamental groups (3.3) we can apply Whitehead's theorem. □

In the last step of the proof of our main theorem we consider again the map p as before and lift it to the universal cover

$$\begin{array}{ccc} \tilde{F}_0 \vee S^n \vee \cdots \vee S^n & \longrightarrow & \tilde{F}^* \\ \downarrow \pi & & \downarrow \pi \\ F_0 \vee S^n \vee \cdots \vee S^n & \xrightarrow{p} & F^* \end{array}$$

Proposition 4.3. *The horizontal maps in the above diagram induce isomorphisms on the homology groups.*

Proof. The proof uses the arguments from (2.8), together with (4.2). □

So we conclude:

Theorem 4.4. *Let* $n \geq 3$: $F_0 \vee S^n \vee \cdots \vee S^n$ *and* F *are homotopy equivalent.*

Proof. Lemma (4.3) and Proposition 3.3 allow us to apply Whitehead's theorem. As mentioned before F^* is homotopy equivalent to F. □

5. Special cases and remarks

Example 5.1. Quotient singularities. Let G be a finite group acting on $(\mathbb{C}^{n+1}, 0)$, acting freely outside 0. Let $X = \mathbb{C}^{n+1}/G$ and $\pi : \mathbb{C}^{n+1} \to X$ the projection. Given a function germ $f : (X, 0) \to (\mathbb{C}, 0)$ we consider also the composition

$$g : (\mathbb{C}^{n+1}, 0) \xrightarrow{\pi} (X, 0) \xrightarrow{f} (\mathbb{C}, 0).$$

If f has an isolated singularity then so has g. In this case we can show that $p : F_0 \vee S^n \vee \cdots \vee S^n \longrightarrow F^*$ induces an isomorphism of fundamental groups if $n \geq 2$ by using the information that g is an isolated singularity on a smooth space. We follow in detail the constructions from §§1 and 2 for both f and g. Consider the approximation h (of Morse type) of f and the induced approximation $h \circ \pi$ of g. The Milnor fibres are denoted by capitals, i.e. G is the Milnor fibre of g. The thimbles of f are denoted by e_f, those of g by e_g. Remark that G_0 is the local Milnor fibre of $h \circ \pi$, and this is in general not a complex link.

Since $g : \mathbb{C}^{n+1} \to \mathbb{C}$ has an isolated singularity and therefore its Milnor fibre is a bouquet of spheres of dimension n, it follows that G_0, G, G^* and $G^* \cup e_g$ are connected and simply connected if $n \geq 2$. Therefore the mappings π in the diagram

$$\begin{array}{ccc} G_0 & \longrightarrow & G^* \cup e_g \\ \pi \downarrow & & \downarrow \pi \\ F_0 & \longrightarrow & F^* \cup e_f \end{array}$$

are universal coverings and the horizontal mappings induce isomorphisms on fundamental groups and homology groups. Continuing equivariantly the same type of arguments show that also in

$$\begin{array}{ccc} G_0 \vee S^n \vee \cdots \vee S^n & \xrightarrow{\rho} & G^* \\ \downarrow \pi & & \downarrow \pi \\ F_0 \vee S^n \vee \cdots \vee S^n & \xrightarrow{\rho} & F^* \end{array}$$

the vertical maps are universal coverings and the horizontal mappings induce isomorphisms of fundamental groups.

Next we use also for g the "additivity of the vanishing homology". The same arguments as in Proposition 2.8 make it possible to show that the horizontal mappings in the above diagram also induce isomorphisms of all homology groups.

Next it follows from Whitehead's theorem, that

$$F \overset{h}{\cong} F^* \overset{h}{\cong} F^0 \vee S^n \vee \cdots \vee S^n$$

for quotient singularities if $n \geq 2$.

Remark 5.2. Montaldi and Van Straten [Mo-St] studied $\mathbb{C}^*$-actions on $\mathbb{C}^{n+1}$ and pose similar questions about the quotient Milnor fibre.

Remark 5.3. Tibăr studied in his dissertation [Ti-1] the Lefschetz number of the monodromy transformation of cyclic quotient singularities.

Remark 5.4. A most general bouquet theorem? If this exists, then it should be as follows:

Let X be a stratified space and $f : (X, 0) \to (\mathbb{C}, 0)$ be an isolated singularity, then

$$F \overset{h}{\cong} F_0 \vee F_1 \vee \cdots \vee F_m$$

where F_0 is the complex link of $(X, 0)$ and each F_i is a local Milnor fibre of a "generic" singularity on a stratum, in fact a k-fold repeated suspension of the compex link of a stratum σ (of dimension k) for each critical point of a (generic) perturbation of f. In case (X, x) has "Milnor's property" this is exactly Lê's result in [Lê-4]. Tibăr [Ti-2] announced a proof of the general bouquet theorem with the help of the carrousel method. He also "counts" the numbers of "factors" in the bouquet. The same type of splittings should also exist for nongeneric perturbations.

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