CLASSIFICATION AND DEFORMATION OF SINGULARITIES

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ACADEMISCH PROEFSCHRIFT

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INTRODUCTION

Consider the set $J^N$ of real (or complex) polynomials in $n$ variables of degree $\leq N$ ($N$ a large natural number). The polynomials with critical point $Q$ are called singularities. They form a linear subspace $S$ of $J^N$. A polynomial $g$ is called Right-equivalent with $f$ if it can be derived from $f$ by applying a smooth coordinate transformation of $\mathbb{R}^n$. The Right-equivalence classes are immersed submanifolds in $J^N$.

One can consider the following two problems:

Classification problem: Give a list of Right-equivalence classes for increasing codimension in $S$.

Adjacency problem: Give a description of the Right-equivalence classes, that can occur for arbitrarily small perturbations of a given polynomial. Or more general: Describe the topology of the set of Right-equivalence classes.

In part I of this thesis the classification problem is treated in the context of germs of real functions. It depends heavily on the work of MATHER [19] and uses also WASSERMANN [27], who gave in his thesis a generalization of Mather's work to the Right-left-case.

In §1 I recall definitions and theorems concerning Right-equivalence. In §3 a list of equivalence classes with codimension $\leq 9$ is presented. The full proof is given in §4. One of the reasons for treating the problem of equivalence in $k$-parameter families of germs (in §2) is the existence of $1$-parameter families in my list. §5 contains a counterexample to a conjecture of Zeeman, concerning an algebraic condition for a polynomial to be $k$-determined (for the definition see §1). Moreover I discuss in §5 the classification under Right-left-equivalence.
Professor R. Thom began the theory and classified the singularities in codimension smaller than or equal to four. He used the theory for a study of morphogenesis (to be applied in various sciences), introduced the notion of universal unfolding and posed a large number of important and hard mathematical problems.

In 1968-1970 J. Mather solved a number of these in his fundamental papers on Right-equivalence of functions. A preliminary manuscript was informally distributed. He did not conclude this work in the form of a paper however. Many mathematicians showed interest in the manuscript, which contained interesting new definitions and theorems on universal unfoldings.

In 1970-71 the manuscript was studied in a seminar of Professor N.H. Kuyper at the University of Amsterdam. During the next year (1971-1972) I started my research on the classification of singularities of real smooth functions and improved the classification for codimension ≤ 5 by Mather to the classification in codimension ≤ 8. See [23], in which the results are formulated with some indications of the proof.

After the publication of my list for codimension ≤ 8, independent ARNOLD [1] published end 1972 a paper, in which he gave among others a list of the so-called simple singularities. This was a subset of my list, but complete with respect to the interesting simplicity problem. Very recently there appeared two papers of ARNOLD [2] and [3], in which he gave a very extensive list, namely of all families with 0 and 1 parameter. It refers also to my paper and contains all singularities of codimension ≤ 12. I believe that my presentation is still of some independent interest, since Arnold omits the proofs and only treat Right-equivalence in the complex case, and my presentation includes the real case and Right-left-equivalence.

The adjacency problem is treated in part II of this thesis. I study there the complex analytic case, in which the Milnorfibration gives some topological invariants. We refer to p. 62 for the introduction of part II. The results are illustrated in list 3 at the end.
PART I: CLASSIFICATION

§ 1 EQUIVALENCE AND FINITE DETERMINACY OF GERMS

In this § we recall some definitions and theorems. As general references we give MATHER[19] and WASSERMANN[27].

(1.1) Let $X$ and $Y$ be topological spaces and let $x \in X$. Two $\mathcal{C}^\infty$-mappings $f : U \to Y$ and $g : V \to Y$ where $U$ and $V$ are neighborhoods of $x$ in $X$ are called germ-equivalent at $x$ if there is a neighborhood $W \subset U \cap V$ of $x$ in $X$ such that $g|_W = h|_W$. The equivalence classes are called mapgerms at $x \in X$ from $X$ into $Y$. We denote by $\hat{f} : X \to Y$ the equivalence class, containing $f : U \to Y$. Composition of mapgerms is defined by composition of the representatives.

We denote by $\mathcal{E}_n$, the set of germs at $0 \in \mathbb{R}^n$ of $\mathcal{C}^\infty$-functions $f : \mathbb{R}^n \to \mathbb{R}$. $\mathcal{E}_n$ has a natural $\mathbb{R}$-algebra structure induced from $\mathbb{R}$. As a ring $\mathcal{E}_n$ has a unique maximal ideal $m_n$; $m_n$ is the set of germs $f : \mathbb{R}^n \to \mathbb{R}$ at $0$ such that $f(0) = 0$.

$L_n$ is the set of germs at $0 \in \mathbb{R}^n$ of $C^\infty$-diffeomorphisms $\phi : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$. Every $\hat{\phi} \in L_n$ has the properties $\hat{\phi}(0) = 0$ and $d\hat{\phi}(0)$ has maximal rank. We can make $L_n$ into a group by taking as the group operation the composition of mapgerms.

(1.2) Let $X$ and $Y$ be $\mathcal{C}^\infty$-manifolds and let $x \in X$ and $k \in \mathbb{N} \cup \{0\}$. Two $\mathcal{C}^\infty$-mappings $f : U \to Y$ and $g : V \to Y$ where $U$ and $V$ are neighborhoods of $x$ in $X$ are called $k$-jet-equivalent at $x$ if and only if $f(x) = g(x)$ and all their partial derivatives of order $\leq k$ at $x$ agree (in some, and hence in any system of local coordinates). The equivalence classes are called $k$-jets. The equivalence class at $x$, containing
\( f : U \to Y \) is denoted by \( j^k_x(f) \).

\( J^k(n,1) \) is the set of \( k \)-jets at 0 \( \in \mathbb{R}^n \) of \( C^\infty \)-mappings \( f : \mathbb{R}^n \to \mathbb{R} \).

\( J^k(n,1) \) has a natural vector space structure and is isomorphic with the vector space of all polynomials in \( x_1, \ldots, x_n \) of degree \( \leq k \).

\( J^k(n,1) \) contains the subspace \( J^k_0(n,1) = \{ z = j^k_0(f) \in J^k(n,1) \mid f(0) = 0 \} \).

By \( f_k \) we denote the Taylor series of \( f \) at 0 \( \in \mathbb{R}^n \) up to the \( k \)th degree terms. Two mappings \( f \) and \( g \) are clearly \( k \)-jet-equivalent iff \( f_k = g_k \).

\( L_k(n) \) is the set of \( k \)-jets at 0 \( \in \mathbb{R}^n \) of \( C^\infty \)-diffeomorphisms \( \phi : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0) \). \( L_k(n) \) is a group with the composition (take representatives) as product. This action is well-defined, since \( (g \cdot f)_k \) depends only on \( g_k \) and \( f_k \).

For simplicity, we shall often indicate germs and jets by giving the name of a representative.

(1.3) There exist canonical projections:

\[ J^{k+1}(n,1) \to J^k(n,1) \to J^{k-1}(n,1) \to \ldots \to J^1(n,1) \to J^0(n,1) \]

and \( \pi_k J^k(n,1) \) defined in an obvious way.

For the maximal ideal \( \mathfrak{m}_n = \text{Ker} \{ \pi^1_0 : \mathbb{C}_n \to J^0(n,1) \} \) we have \( \mathfrak{m}_n^2 = \text{Ker} \{ \pi^1_1 : \mathbb{C}_n \to J^1(n,1) \} \). An element \( \hat{f} \in \mathfrak{m}_n^2 \) is called a singular germ or a singularity. This condition is equivalent to \( f(0) = 0 \) and \( df(0) = 0 \), or to \( f_1 = 0 \).

(1.4) Two germs \( \hat{f}, \hat{g} \in \mathfrak{m}_n \) are called (Right)-equivalent if there exists \( \hat{\phi} \in L_n^R \) such that \( f = g \hat{\phi} \). Notation: \( \hat{f} \sim^R \hat{g} \) (or \( \hat{f} \sim^R L_n \hat{g} \)). Two germs \( \hat{f}, \hat{g} \in \mathfrak{m}_n \) are called Right-left-equivalent if there exist \( \hat{\psi} \in L_n^L \) such that \( \psi f = g \). Notation: \( \hat{f} \sim^L_R \hat{g} \).

Two \( k \)-jets \( j^k(f) \) and \( j^k(g) \in J^k(n,1) \) are called (Right)-equivalent if there exists \( \hat{\phi} \in L(n) \) such that \( f_k = (g \hat{\phi})_k \). Notation: \( j^k(f) \sim^R j^k(g) \) or \( f_k \sim^R g_k \). Two \( k \)-jets \( j^k(f) \) and \( j^k(g) \in J^k_0(n,1) \) are called Right-left-equivalent if there exist \( \hat{\phi} \in L(n) \) and \( \hat{\psi} \in L^L(n) \) such that \( (\hat{\psi} f)_k = (g \hat{\phi})_k \). Notation: \( j^k(f) \sim^R j^k(g) \).

The group \( L_n^R \) acts on \( \mathfrak{m}_n \) by composition on the right; the \( R \)-equivalence-classes are the orbits of this group action. The group \( L_n^L \times L_n^R \) acts on \( \mathfrak{m}_n \) by composition on the right with elements of \( L_n^L \) and on the left with
elements of $L$. The $RL$-equivalence classes are the orbits of this group action. Notations: $\text{Orb}(\hat{f})$ and $\text{Orb}_R(\hat{f})$ for the Right-equivalence classes and $\text{Orb}_{RL}(\hat{f})$ for the Right-left-equivalence classes. The ideals $m_n^k$ are invariant under the two group actions.

In a similar way there are group actions of $L^k(n)$ on $J^{k}(n,1)$ and of $L^k(1) \times L^k(n)$ on $J^{k}(n,1)$. The orbits are denoted by $\text{Orb}^k(f)$ or $\text{Orb}_{RL}^k(f)$ in the $R$-case and by $\text{Orb}_{RL}^k(f)$ in the $RL$-case.

It is very important that the last two actions are algebraic.

\textbf{(1.5) Definitions:}

A germ $\hat{f} \in m_n^k$ is called \textit{Right-$k$-determined} (or $j^k(f)$ is Right-$k$-sufficient) if for any $\hat{g} \in m_n^k$:

$$f_k = g_k = \hat{f} \circ_R \hat{g}.$$ 

A germ $\hat{f} \in m_n^k$ is called \textit{Right-left-$k$-determined} (or $j^k(f)$ is Right-left-$k$-sufficient) if for any $\hat{g} \in m_n^k$:

$$f_k = g_k = \hat{f} \circ_{RL} \hat{g}.$$ 

The property of being $k$-determined is invariant under $RL$-equivalence.

\textbf{Lemma:} Let $f$ be $s$-determined and $f \bowtie g$ then

1. $f \bowtie g$

2. $\hat{g}$ is $s$-determined

\textbf{proof:} $f \bowtie g$, so there is $\phi \in L_n$ such that $f_\phi = (g_\phi)_s$ so $f \bowtie g_\phi$. 

This implies $f \bowtie g_\phi$. 

Since $s$-determinacy is a property of the orbit, also $g$ is $s$-determined.

Other related questions are $C^\infty$-sufficiency and $v$-sufficiency of jets (cf Kuo[15]).

\textbf{Examples:}

1. If $f$ is regular in $\Omega \in \mathbb{R}^n$ there exist coordinates such that $f(x_1,\ldots,x_n) = x_1$.

if $g_1 = f_1$ then also $g$ is regular in $\Omega \in \mathbb{R}^n$ and we can choose coordinates such that $g(x_1,\ldots,x_n) = x_1$. Clearly $g \bowtie f$; so $f$ is
1-determined.

2° If \( 0 \) is non-degenerate critical point of \( f \), then the classical Morse lemma says:

\[
f \wedge f_2 \wedge e_1^2 + \ldots + e_n^2 \quad \text{with} \quad e_i = \pm 1.
\]

If \( g_2 = f_2 \) then also \( 0 \) is a nondegenerate critical point of \( g \) and \( g \wedge g_2 \).

So \( g \wedge g_2 = f_2 \wedge f \); so \( f \) is 2-determined.

(1.6) Nakayama's lemma:

Let \( R \) be a commutative ring with \( 1 \); \( m \) an ideal, \( L \) an \( R \)-module and \( M \) and \( N \) submodules of \( L \). Suppose:

a) \((1 + x)^{-1}\) exists in \( R \) for every \( x \in m \)

b) \( M \) is finitely generated

c) \( M \subseteq N + mM \)

Then: \( M \subseteq N \).

Proof: Let \( e_1, \ldots, e_n \) generate \( M \). By c) there are \( f_i \in N \) and \( \alpha_{ij} \in m \) such that:

\[
e_i = f_i + \sum_{j=1}^{n} \alpha_{ij} e_j.
\]

Hence \((1 - A)^{\hat{e}} = \hat{\Gamma} \) (matrix equation with \( A = (\alpha_{ij}) \) and \( \hat{e} = (e_1, \ldots, e_n)^T \) and \( \hat{\Gamma} = (f_1, \ldots, f_n)^T \)).

Since \( \det(1 - A) = 1 + a \) with \( a \in m \) and \( 1 + a \) is invertible in \( R \), also \((1 - A)^{-1}\) exists and

\[
\hat{e} = (1 - A)^{-1} \hat{\Gamma}.
\]

so \( e_i \in N(i = 1, \ldots, n) \). Hence \( M \subseteq N \).

(1.7) For \( f : R^n \to R \) the ideal, generated by the partial derivatives \( \partial_1 f, \ldots, \partial_n f \) is denoted by \( \Delta(f) \).

Theorem: If \( \hat{f} \in m_n \) obeys \( m_n^{k+1} \subseteq m_n^{2\Delta(f)} + m_n^{k+2} \) then \( f \) is \( k \)-determined.

Proof: Take any \( g \in \mathcal{A}_n \) with \( g_k = f_k \). We define \( F : R^n \times R \to R \) by

\[
F(x,t) = f(x) + t[g(x) - f(x)].
\]

Denote \( F_t(x) = F(x,t) \), hence \( F_0 = f \) and \( F_1 = g \).

We try to find a map \( h : (R^n \times R, \{0\} \times R) \to (R^n, 0) \) such that the map
\( h_t \), defined by \( h_t(x) = h(x,t) \) is a diffeomorphism and moreover
\[
F_t(h_t(x)) = F_0(x),
\]
that is
\[
F(h(x,t),t) = F(x,0). \tag{1}
\]
Differentiating (1) with respect to \( t \) gives:
\[
\prod_{i=1}^{n} \frac{\partial F}{\partial x_i}(h(x,t),t) \cdot \frac{\partial h}{\partial t}(x,t) + \frac{\partial F}{\partial t}(h(x,t),t) = 0
\]
\[
VF(h(x,t),t) \cdot \frac{\partial h}{\partial t}(x,t) + g(h(x,t)) - f(h(x,t)) = 0 \tag{2}
\]
where \( VF = \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right) \) and \( \frac{\partial h}{\partial t} = \left( \frac{\partial h_1}{\partial t}, \ldots, \frac{\partial h_n}{\partial t} \right) \).

Define \( \xi : R^{n+1} \rightarrow R^n \) by \( \xi(h(x,t),t) = \frac{\partial h}{\partial t}(x,t) \). \tag{3}

Substitution in (2) gives
\[
VF(h(x,t),t) \cdot \xi(h(x,t),t) + g(h(x,t)) - f(h(x,t)) = 0.
\]
Since \((x,t)\) is arbitrary and \( h_t \) is a diffeomorphism this is equivalent to: \( VF(x,t) \cdot \xi(x,t) + g(x) - f(x) = 0. \tag{4} \)

We next try to solve the differential equations (3) + (4). We need therefore two lemma's.

\textbf{Lemma 1:} Let \( m_n^{k+1} \subseteq m_n^{2\Delta(f)} + m_n^{k+2} \). Then there exists for all \( t \in R \) a map germ \( \xi : R^{n+1} \rightarrow R^n \) defined on a neighborhood \( U \) of \( (0,t_0) \in R^{n+1} \), which satisfies:

(i) \( \xi(0,t) = 0 \) for all \( (0,t) \in U \)

(ii) \( VF(x,t) \cdot \xi(x,t) + g(x) - f(x) = 0 \) for all \( (x,t) \in U \).

\textbf{Proof:} Let \( \mathcal{E}_{n+1} \) be the ring of germs at \((0,t_0)\) of \( C^\infty \)-functions \( R^{n+1} \rightarrow R \) and \( m_{n+1} \) the maximal ideal of \( \mathcal{E}_{n+1} \). Let
\( \Delta(F) = \mathcal{E}_{n+1}(\partial_1 F, \ldots, \partial_n F) \). We have inclusions \( \mathcal{E}_n \subseteq \mathcal{E}_{n+1} \) and \( m_n \subseteq m_{n+1} \) (subrings). To satisfy (i) and (ii) we need: \( m_n^{k+1} \subseteq \Delta(F)(m_{n+1}^{m_n}) \) or equivalently \( m_n^{k+1} \subseteq \Delta(F)m_n \) (every element of \( \Delta(F)(m_{n+1}^{m_n}) \) has the form \( VF(x,t) \cdot \xi(x,t) \) and \( \xi \in \mathcal{E}_{n+1}m_n \), so \( \xi(0,t_0) = 0 \)).
Now \( \frac{\partial F}{\partial x_1} = \frac{\partial f}{\partial x_1} + t \frac{\partial}{\partial x_1} (g-f) \) hence \( \frac{\partial F}{\partial x_1} = \frac{\partial f}{\partial x_1} - t \frac{\partial}{\partial x_1} (g-f) \).

So \( \Delta(f) \subseteq \Delta^*(F) + \&_{n+1}^k \).

Since \( m_{n+k+1} \subseteq m_n \Delta^*(F) + m_{n+k+2} \) we have:

\[
\&_{n+1}^k m_{n+k+1} \subseteq \&_{n+1}^k m_n \Delta^*(F) + \&_{n+1}^k m_{n+k+2} \subseteq m_{n+k+1} \Delta^*(F) + m_{n+k+2} \Delta^*(F)
\]

\( \&_{n+1}^k m_{n+k+1} \subseteq m_{n+k+1} \Delta^*(F) + m_{n+k+2} \Delta^*(F) \),

So \( \Delta^*(F) \subseteq m_{n+k+1} \Delta^*(F) + m_{n+k+2} \Delta^*(F) \).

We apply Nakayama's lemma with

\[
(R,m,L,M,N) = (\&_{n+1}^k m_{n+1}^k, \&_{n+1}^k m_{n+k+1}^2, \&_{n+1}^k m_{n+k+2}^2, \&_{n+1}^k m_{n+k+3}^2)
\]

and get:

\[
m_{n+k+1} \subseteq \&_{n+1}^k m_{n+k+1} \subseteq m_{n+k+1} \Delta^*(F)
\]

Hence

\[
m_{n+k+1} \subseteq m_{n+k+1} \Delta^*(F) \subseteq m_{n+k+1} \Delta^*(F)
\]

\[
\text{Lemma 2: For each } t_o \in \mathbb{R} \text{ there is } \varepsilon > 0 \text{ such that } F_{t_o} \cap F_{t_o} \text{ for all } t \text{ with } |t-t_o| < \varepsilon.
\]

\[
\text{Proof: It follows from the fundamental existence theorem for solutions of ordinary differential equations that there exists a smooth map germ } h : \mathbb{R}^{n+1} \to \mathbb{R}^n \text{ satisfying the differential equation:}
\]

\[
a) \quad \frac{\partial h}{\partial t}(x,t) = \xi(h(x,t),t)
\]

\[
b) \quad h(x,t_o) = x.
\]

Since \( h_{t_o} \) is the identity, there exist \( \varepsilon > 0 \) such that \( h_t \) a diffeomorphism is for all \( t \) with \( |t-t_o| < \varepsilon. \)

If \( x = 0 \) the differential equation has unique solution \( h(0,t) = 0 \), for

\[
\begin{cases}
\frac{\partial h}{\partial t}(0,t) = \xi(h(0,t),t) = \xi(0,t) = 0 \quad \text{(lemma 1(i))} \\
h(0,t_o) = 0
\end{cases}
\]

Moreover \( \frac{d}{dt}(F_t h_t(x)) = \frac{d}{dt}(F(h(x),t),t) = 0 \text{ according to lemma 1(i).} \)
So \( F_{t_0} = F_{t_0} h_{t_0} = F_t h_t \) for all \( t \) with \( |t - t_0| < \varepsilon \); so \( F_{t_0} \sim F_t \).

The theorem follows now by "continuous induction" over the interval \([0,1]\).

**Remarks:** By the Nakayama-lemma the condition \( m_n^{k+1} \subseteq m_n^2 \Delta(f) + m_n^{k+2} \) is equivalent to \( m_n^{k+1} \subseteq m_n \Delta(f) \). Moreover \( k \)-sufficiency of \( f \) follows also from \( m_n^k \subseteq m_n \Delta(f) + m_n^{k+1} \) or \( m_n^{k-1} \subseteq \Delta(f) + m_n^k \).

(1.8) **Theorem:** If \( f \in m_n \) is \( k \)-determined then \( m_n^{k+1} \subseteq m_n \Delta(f) + m_n^{k+2} \).

**Proof:** We define \( U = \{ g \in \mathbb{A}_n \mid g_k = f_k \} = f + m_n^{k+1} \)

and \( V = \{ g \in \mathbb{A}_n \mid g \sim f \} = \) orbit of \( f = \mathcal{L}_n \).

We consider the natural projection: \( \pi_{k+1} : \mathbb{A}_n \rightarrow j^k+1(n,1) \).

The sets \( U_{k+1} = \pi_{k+1}(U) \) and \( V_{k+1} = \pi_{k+1}(V) \) are submanifolds of \( j^k+1(n,1) \). Let \( \tau(U_{k+1}) \) and \( \tau(V_{k+1}) \) be the tangentspaces to \( U_{k+1} \) resp. \( V_{k+1} \) in \( f_{k+1} \in j^k(n,1) \).

By the assumption \( U \subseteq V \); so also \( U_{k+1} \subseteq V_{k+1} \) and \( \tau(U_{k+1}) \subseteq \tau(V_{k+1}) \).

In order to prove the theorem, it is sufficient to show:

a) \( \tau(U_{k+1}) \equiv m_n^{k+1} \pmod{m_n^{k+2}} \)

b) \( \tau(V_{k+1}) \equiv m_n \Lambda(f) \pmod{m_n^{k+2}} \)

Condition a) follows immediate from the definition of \( U \).

Now we prove condition b): The elements of \( \tau(V_{k+1}) \) can be described as follows: Let for \( t \in [0,\varepsilon) \) \( h_t : (\mathbb{R}^n,0) \rightarrow (\mathbb{R}^n,0) \) be a germ of diffeomorphism with \( h_0 = 1 \). An element of \( \tau(V_{k+1}) \) is equal to

\[
\tau_{k+1}(\frac{d}{dt} fh_t \bigg|_{t=0}).
\]

We have \( \frac{d}{dt} fh_t \bigg|_{t=0} = \nabla f \cdot \frac{dh_t}{dt} \bigg|_{t=0} \in \mathbb{A}_n(\partial_1 f, \ldots, \partial_n f) \).

Let \( \xi = \frac{dh_t}{dt} \bigg|_{t=0} \), then \( \xi(0) = \frac{\partial h_t(0)}{\partial t} \bigg|_{t=0} = 0 \) since \( h_t(0) = 0 \).

So \( \xi \in m_n \) (\( i = 1, \ldots, n \)).
That means \( \frac{d}{dt} (f h_t) \bigg|_{t=0} \in m_n \Delta(f) \), which proves \( \tau(V_{k+1}) \subseteq m_n \Delta(f) \) (modulo \( m_n^{k+2} \)). Moreover every element \( a \) of \( m_n \Delta \) defines an element of \( \tau(V_{k+1}) \). For let \( a(x) = \forall f(x) \cdot \xi \) with \( \xi \in m_n \), and \( h_t(x) = x + t \xi \); then \( h_t \in L_n \) for small \( t \) and we have:

\[
\frac{d}{dt} fh_t \bigg|_{t=0} = \forall f \cdot \frac{dh_t}{dt} \bigg|_{t=0} = \forall f \cdot \xi = a
\]

which proves \( \tau(V_{k+1}) \subseteq m_n \Delta(f) \) (modulo \( m_n^{k+2} \)).

**Remark 1:** According to Nakayama's lemma the condition

\[
m_n^{k+1} \subseteq m_n \Delta(f) + m_n^{k+2}
\]

is equivalent to \( m_n^{k+1} \subseteq m_n \Delta(f) \).

**Remark 2:** In the proof of theorem 2 we showed that for every \( f \in m_n \) the tangent space of the orbit of \( f \) in \( J^k(n,1) \) is equal to \( \pi_k(m_n \Delta(f)) \). Sometimes we will refer to \( m_n \Delta(f) \) also as the tangent space to the orbit of \( f \) in \( E_n \).

(1.9) We shall now discuss the Right-left-case.

This is treated thoroughly by WASSERMANN [27]. He states (pag. 39):

**Theorem:**

If \( f \) is RL-determined then \( m_n^{k+1} \subseteq m_n \Delta(f) + f^*(m_1) + m_n^{k+2} \).

**Remark 1:** \( f^*(m_1) \) is the image of \( m_1 \) under the \( R \)-algebra homomorphism \( f^*: E_{m_1} \to E_n \). Modulo \( m_n^{k+2} \) \( f^*(m_1) \) is spanned as \( R \)-algebra by \( f, x^2, x^3, \ldots, x^q, \ldots \). According to the Malgrange preparation theorem the condition \( m_n^{k+1} \subseteq m_n \Delta(f) + f^*(m_1) + m_n^{k+2} \) is equivalent to \( m_n^{k+1} \subseteq m_n \Delta(f) + f^*(m_1) \).
Remark 2: The tangentspace of the $RL$-orbit of $f$ in $s^k(n,1)$ is equal to $\pi_k[m_n \Delta(f) + f^*(m_1)]$.

(1.10) Definition: codimension: For $\hat{f} \in m_n^2$ we define:

a) $\text{codim} (\hat{f}) = \dim \frac{m_n}{\Delta(\hat{f})}$

b) $\text{codim}_{RL} (\hat{f}) = \dim \frac{m_n}{\Delta(\hat{f}) + f^*(m_1)}$

The definition depends only on the $RL$-equivalence class of $\hat{f}$. 

Lemma: For $\hat{f} \in m_n^2$ we have:

a) $\text{codim} (\hat{f}) = \dim \frac{m_n^2}{\Delta(\hat{f})} - m_n$

b) $\text{codim}_{RL} (\hat{f}) = \dim \frac{m_n^2}{\Delta(\hat{f}) + f^*(m_1)}$

Proof: cf. WASSERMANN [27], proposition 2.19.

Remark: According to remarks (1.8) and (1.9) we can identify $m_n \Delta(f)$ (resp. $m_n \Delta(f) + f^*(m_1)$) with the tangentspace to the $R$-orbit (resp. $RL$-orbit) of $f$ in $m_n^2$. This justifies the use of the term codimension; so the condition of $f$ is equal to the codimension of the $R$-orbit of $f$ in $m_n^2$; and the $RL$-codimension of $f$ is equal to the codimension of the $RL$-orbit of $f$ in $m_n^2$.

Proposition: Equivalent are:

a) $\text{codim} (\hat{f}) < \infty$

b) $\text{codim}_{RL} (\hat{f}) < \infty$

c) $f$ is $k$-determined for some $k \in \mathbb{N}$

d) $f$ is $RL$-$k$-determined for some $k \in \mathbb{N}$

e) For some $k \in \mathbb{N}$: $m_n^k \subseteq m_n \Delta(f) + m_n^{k+1}$

Proof: cf. WASSERMANN [27];
(1.11) Examples: For \( n=2 \) it is possible to compute the codimension and to discover \( k \)-determinacy using a diagram, containing the canonical generators of the vectorspace of formal power series in \( x \) and \( y \):

1) \( f = x^2 + y^4 \); \( \partial_1 f = 2x \) and \( \partial_2 f = 4y^3 \)

\[
\begin{align*}
&x^4 & x^3 & x^2 & x & 1 \\
&x^3 & x^2 & x & 1 \\
&x^2 & x & 1 \\
&x & 1 \\
&1 \\
\end{align*}
\]

a) codim \( (f) = 2 \)
b) As \( m^3 \subseteq \Delta(f) + m^4 \) then \( m^4 \subseteq m\Delta(f) + m^5 \) and so \( f \) is \( 4 \)-determined by (1.7).

2) \( f = x^4 + y^4 \); \( \partial_1 f = 4x^3 \) and \( \partial_2 f = 4y^3 \)

\[
\begin{align*}
&x^5 & x^4 & x^3 & x^2 & x & 1 \\
&x^4 & x^3 & x^2 & x & 1 \\
&x^3 & x^2 & x & 1 \\
&x^2 & x & 1 \\
&x & 1 \\
&1 \\
\end{align*}
\]

a) codim \( (f) = 8 \)
b) As \( m^5 \subseteq m^2\Delta(f) + m^6 \), \( f \) is \( 4 \)-determined by (1.7).

3) \( f = x^2y \); \( \partial_1 f = 2xy \) and \( \partial_2 f = x^2 \)

\[
\begin{align*}
&x^3 & x^2 & x & 1 \\
&x^2 & x & 1 \\
&x & 1 \\
&1 \\
\end{align*}
\]

a) codim \( (f) = \infty \)
b) \( f \) is not finitely determined.
4) \( f = x^3 + xy^3 \); \( \partial_1 f = 3x^2 + y^3 \) and \( \partial_2 f = 3xy^2 \)

Remark that \( \partial_1 f = 3x^3 + xy \equiv 3x^3 \) Modulo \( m\Delta(f) \)

\( y^2 \partial_2 f = 3x^2y^2 + y^5 \equiv y^5 \) Modulo \( m\Delta(f) \)

Relations that are not in the picture:

\[ 3x^2 + y^3 \equiv 0 \]
\[ 3x^2y + y^4 \equiv 0 \]

a) \( \text{codim}(f) = 8 - 2 = 6 \)

b) As \( m^5 \subseteq m\Delta + m^6 \); so \( f \) is 5-determined by (1.7).

5) \( f = x^3 + y^3 + z^3 \); \( \partial_1 f = 3x^2 \) and \( \partial_2 f = 3y^2 \) and \( \partial_3 f = 3z^2 \)

a) \( \text{codim}(f) = 7 \)

b) As \( m^4 \subseteq m^2 + m^5 \); so \( f \) is 3-determined by (1.7).
§2. Equivalence and non-equivalence in k-parameter families of germs.

(2.1) Introduction: In (1.7) and (1.8) we found:

\[ m_n^{s+1} \subset m_n^2 \Delta(f) + m_n^{s+2} \Rightarrow f \text{ is } s\text{-determined} \Rightarrow m_n^{s+1} \subset m_n^s \Delta(f) + m_n^{s+2} \]

Let \( o(f) \) be the smallest integer \( s \) such that \( f \) is \( s \)-determined. If no such integer exists we write \( o(f) = \infty \). \( o(f) \) is called the degree of determinacy.

In most cases (1.7) and (1.8) do not determine \( o(f) \), but only up to a choice between two consecutive numbers. Further computations are needed to determine \( o(f) \) completely.

Let us consider a polynomial \( f \) of degree \( s \), which satisfies

\[ m_n^{s+1} \subset m_n^2 \Delta(f) + m_n^{s+2} \]

hence

\[ m_n^{s+2} \subset m_n^2 \Delta(f) + m_n^{s+3} \]

So \( f \) is \((s+1)\)-determined and \( o(f) = s+1 \) or \( s \). Let \( \rho \) be a homogeneous polynomial of degree \( s+1 \), with say \( k \) variable coefficients. Then \( f + \rho \) can be considered as a \( k \)-parameter family of germs. In order to prove, that \( f \) is \( s \)-determined, it is sufficient to show that \( f + \rho \sim f \) for all \( \rho \). For this reason we study \( k \)-parameter families of germs. We start with 1-parameter families and try to eliminate the parameter.

(2.2) Proposition: Let \( f_t = f + t\phi \) be defined for \( t \in I \) (a connected interval of \( \mathbb{R} \)). If \( \phi \in m_n \Delta(f + t\phi) \) for all \( t \in I \) then \( f_t \sim f_{t_0} \) for \( t, t_0 \in I \).

Proof: It is sufficient to satisfy the differential equation of (1.7) lemma 1:
\[
\begin{cases}
(i) \quad \xi(0,t) = 0 \\
(ii) \quad \nabla F(x,t) \cdot \xi(x,t) + \phi = 0
\end{cases}
\]
where \(F(x,t) = f_t(x) = f(x) + t\phi(x)\). The conditions (i) and (ii) are equivalent to \(\phi \in \mathfrak{m}_n \Delta(f + t\phi)\). Next we apply (1.7) Lemma 2 and our proposition follows.

**Corollary:** Let \(f_t = f + t\phi\) be defined for \(t \in I\) (connected interval of \(R\)). If for all \(t \in I:\ 1^0 \mathfrak{m}_n^{k+1} \subseteq \mathfrak{m}_n \Delta(f + t\phi) + \mathfrak{m}_n^{k+2} \)

\[2^0 \phi \in \mathfrak{m}_n \Delta(f + t\phi) + \mathfrak{m}_n^{k+1}\]

then: \(f_t \sim f_{t_0}\) for \(t, t_0 \in I\).

**Proof:** Nakayama's lemma gives: \(\mathfrak{m}_n^{k+1} \subseteq \mathfrak{m}_n \Delta(f + t\phi)\) so \(\phi \in \mathfrak{m}_n \Delta(f + t\phi)\).

Apply the proposition (2.2).

(2.3) **Proposition:** Let \(f_t = f + t\phi\) be defined for \(t \in I\) (connected interval of \(R\)). If \(\phi \in \mathfrak{m}_n \Delta(f + t\phi) + (f + t\phi)^* \mathfrak{m}_1\) for all \(t \in I\) then

\[f_t \sim f_{t_0}\] for \(t, t_0 \in I\).

**Proof:** The proof is similar to the proof of theorem (1.7) and proposition (2.2). We try to find maps:

\[h : (R^n \times I, \{t_0\} \times R) \rightarrow (R^n, 0)\]

\[k : (R \times I, \{t_0\} \times R) \rightarrow (R, 0)\]

such that \(h(-,t)\) and \(k(-,t)\) are diffeomorphisms and moreover:

\[(0) \quad k_t^{-1}(F_t(h_t(x))) = F_{t_0}(x)\]

Differentiating (0) with respect to \(t\) gives the following three conditions:

\[(1) \quad \frac{\partial k_t^{-1}}{\partial y}(F(x,t)) \cdot (\nabla F(x,t) \cdot \xi(x,t) + \frac{\partial F(x,t)}{\partial t} + n(F(x,t),t)) = 0\]
\[(2) \quad \frac{\partial h}{\partial t}(x,t) = \xi(h(x,t),t)\]
\[(3) \quad \frac{\partial k}{\partial t}(y,t) = -n(k(y,t),t)\]
together with some initial conditions.

Compare WASSERMANN [27] pag. 22-30.

If it is possible to solve (1) we can solve the equations (2) and (3) locally and in the same way as in theorem (1.7) we find the $RL$-equivalence of $F_t$ and $F_0$ for all $t \in I$.

The condition (1) is implied by

$$(4) \quad \frac{\partial F}{\partial t} \in m_n \Delta(F) + F^*(m_1)$$

In our case $F(x,t) = f(x) + t\psi(x)$.

Since $\frac{\partial F}{\partial t} = \psi$ the condition (4) is a consequence of

$$(5) \quad \psi \in m_n \Delta(f + t\psi) + (f + t\psi)^*m_1 \text{ for all } t \in I.$$
So if \( t \in (0, \infty) \) all \( f_t \) are mutually equivalent, for example
\[
f_t \wedge f_1 = x_1^3 + x_2^4. \]
Also if \( t \in (-\infty, 0) \) all \( f_t \) are mutually equivalent, for example
\[
f_t \wedge f_{-1} = x_1^3 - x_2^4. \]
Remark, that explicit formula's for the diffeomorphisms are obtained from
\[
f_t = x_1^3 + (t^4x_2)^4 \quad \text{if } t > 0 \quad \text{and} \quad f_t = x_1^3 - (|t|^4x_2)^4 \quad \text{if } t < 0.
\]

**Example 2**: Let \( f = x^3 + xy^6 + ay^9 + by^{10} \) with \( b \neq 0 \).

We shall show \( f \underleftarrow{RL} x^3 + xy^6 + ay^9 + y^{10} \)

We have:
\[
\begin{align*}
\partial_1 f &= 3x^2 + y^6 \\
\partial_2 f &= 6xy^5 + 9ay^8 + 10by^9
\end{align*}
\]
Moreover \( y\partial_2 f^* = 6xy^6 + 9ay^9 + 10by^{10} \)
\[
-9f = -9xy^6 - 9ay^9 - 9by^{10} - 9x^3
\]
\[
+3x\partial_1 f = +3xy^6 + 9x^3
\]

So \( y^{10} \in m\Delta(f) + f^*(m_i) \) for all \( a \), and all \( b \neq 0 \). We now apply
proposition (2.3) and obtain \( f \underleftarrow{RL} x^3 + xy^6 + ay^9 + y^{10} \) for \( b \neq 0 \).

After replacing \( x \) by \(-x\); \( y \) by \(-y \) and \( f \) by \(-f \) we can get
\[
f \underleftarrow{RL} x^3 + xy^6 + ay^9 + y^{10}
\]

In this example also it is possible to give explicit formula's for
the diffeomorphisms, since
\[
f \underleftarrow{L} b^9 f = b^9x^3 + b^9xy^6 + ab^9y^9 + b^{10}y^{10} =
\]
\[
= (b^3x)^3 + (b^3x)(by)^6 + a(by)^9 + (by)^{10} \underleftarrow{RL} x^3 + xy^6 + ay^9 + y^{10}
\]

On the other hand, as we show in (2.12) it is impossible to eliminate
the parameter \( a \).

**Example 3**: Let \( g = x_1x_3^2 + x_2^3 + Ax_1^3x_2 + Bx_1^4 + Cx_1^4x_2 + Dx_1^5 \) with \( B \neq 0 \). We show that \( g \) is \( 4 \)-determined, and so
\( g \leftarrow x_1x_3^2 + x_2^3 + Ax_1^3x_2 + Bx_1^4 \).

We shall use this in (4.11).

We have:

\[
\begin{align*}
\mathfrak{a}_1 g &= x_3^2 + 3Ax_1^2x_2 + 4Bx_1^3 + 4Cx_1^3x_2 + 5Dx_1^4 \\
\mathfrak{a}_2 g &= 3x_2^2 + Ax_1^3 + Cx_1^4 \\
\mathfrak{a}_3 g &= 2x_1x_3
\end{align*}
\]

So modulo \( m\Delta(g) + m^6 \) we have:

\[
\begin{align*}
m^3\mathfrak{a}_1 g &= x_3^2m^3 \equiv 0 \\
m^3\mathfrak{a}_2 g &= 3x_2^2m^3 \equiv 0 \\
\text{and} & \quad x_1x_3^m \equiv 0
\end{align*}
\]

Moreover:

a) \( 0 \equiv x_1x_2^3 \mathfrak{a}_1 g = x_1^2x_2^3 + 3Ax_1^3x_2^2 + 4Bx_1^4x_2 \)

so: \( 4Bx_1^4x_2 \equiv 0 \), hence \( x_1^4x_2 \equiv 0 \) (since \( B \neq 0 \))

b) \( 0 \equiv x_1^2\mathfrak{a}_2 g = x_1^2x_3^2 + 3Ax_1^3x_2 + 4Bx_1^5 \)

so: \( 4Bx_1^5 \equiv 0 \), hence \( x_1^5 \equiv 0 \) (since \( B \neq 0 \))

Now it follows that

\[ m^5 \subseteq m\Delta(g) + m^6 \text{ for all values of } C \text{ and } D. \]

So \( g \) is 5-determined for all \( C \) and \( D \).

Because \( Rx_1^4x_2 + Rx_1^5 \subseteq m^5 \subseteq m\Delta(g) \) for all \( C \) and \( D \) theorem (2.4)

gives that

\[ g \leftarrow x_1x_3^2 + x_2^3 + Ax_1^3x_2 + Bx_1^4 \]

and so \( g \) is 4-determined.

(2.6) Sometimes the elimination of a parameter can be shown to be
impossible. First we treat the case of a 1-parameterfamily.

**Definition:** Let \( \{f_t\}_t \subseteq I \) be a family of germs, continuously depending on \( t \), and let \( I \) be an open interval of \( \mathbb{R} \). We call \( t \) a **local invariant** of the family \( \{f_t\}_t \subseteq I \) if \( \forall t_0 \in I \ \exists \epsilon > 0 \text{ such that the} \)

germs \( \{f_t\}_{|t-t_0| < \epsilon} \) are all in different orbits. A similar definition exists for \( RL \)-equivalence.
(2.7) Let $A$ be a subset in $\mathbb{R}^m$. We denote by $A^*$ the closure of $A$ in the Zariski-topology. That is:

$$A^* = \{x \in \mathbb{R}^m | (P(A) = 0) \implies (P(x) = 0) \text{ for all real polynomials } P\}.$$ 

Since $A^*$ is closed in the ordinary topology it contains $A$.

Definition: A closed set $F$ in $\mathbb{R}^m$ is a real algebraic set iff $F^* = F$.

Proposition: $\text{Orb}(z)$ is open in $[\text{Orb}(z)]^*$.


(2.8) Proposition: Let $\mathcal{L}$ be a 1-dimensional affine subspace of $j^k(n,1)$ and let $z \in j^k(n,1)$. Then there are two possibilities for $\mathcal{L} \cap \text{Orb}(z)$:

1° $\mathcal{L} \cap \text{Orb}(z)$ consists of a finite number of points.
2° $\mathcal{L} \cap \text{Orb}(z)$ consists of a collection of open intervals of $\mathcal{L}$.

Proof: Since $[\text{Orb}(z)]^*$ is real algebraic, we have either $\mathcal{L} \cap [\text{Orb}(z)]^*$ is a finite number of points, or $\mathcal{L} \cap [\text{Orb}(z)]^* = \mathcal{L}$.

Since $\mathcal{L} \cap \text{Orb}(z)$ is open in $\mathcal{L} \cap [\text{Orb}(z)]^*$ the proposition follows.

(2.9) Theorem: Let $f_t = f + t\phi$ be a 1-parameter family of germs defined for $t$ in a connected interval $I$ of $\mathbb{R}$.

If 1° $f_t$ is $k$-determined for all $t \in I$,

2° $t \in I : \phi \in m_n\Delta(f + t\phi) + m_n^{k+1}$.

Then $t$ is a local invariant.

Proof: Because $f_t$ is $k$-determined for all $t \in I$, we can work entirely in $j^k(n,1)$. Let $\mathcal{L} = j^k(f_{t_0}) + R\phi$.

According to proposition (2.8) there are only 2 possibilities:

a) $\mathcal{L} \cap \text{Orb}(f_{t_0})$ consists of a finite number of points.

b) $\mathcal{L} \cap \text{Orb}(f_{t_0})$ consists of a collection of open intervals of $\mathcal{L}$.

A necessary condition of b) is that there exists a neighborhood $U$ of $t_0$ in $I$ such that the direction of the line $\mathcal{L}$ is contained in the tangentspace of $\text{Orb}(f_{t_0})$ in $j^k(f_{t_0})$.

So $\phi \in m_n(f + t\phi) + m_n^{k+1}$ for all $t \in U$. 
Since this is not the case we can conclude, that \( \lambda \cap \text{Orb}(f_t) \) consists only of a finite number of points.

Remark: If we have the condition \( \phi \notin m_n \Delta(f + t\phi) + m_n^{k+1} + (f + t\phi)m_n^1 \) in theorem (2.9), we get the conclusion also for RL-equivalence.

\[(2.10) \text{Proposition: If } m_n^{k+1} \subseteq m_n^2 \Delta(f) + m_n^{k+2} \text{ and } \phi \in m_n, \text{ then there exists a } \tau > 0 \text{ such that } m_n^{k+1} \subseteq m_n^2 \Delta(f + t\phi) + m_n^{k+2} \text{ for all } |t| < \tau.\]

Proof:

We consider the canonical projection \( \psi: m_n / m_n^{k+2} \rightarrow m_n / m_n^{k+2} \).

\( \psi(m_n^2 \Delta(f + t\phi)) \) is spanned by vectors \( \{ \hat{a}_1(t), \ldots, \hat{a}_N(t) \} \) continuously depending on \( t \).

If \( \hat{R}_1(t) + \ldots + \hat{R}_N(t) = m_n^{k+1} \) for \( t = 0 \), then the same holds small \( t \) since a determinant (continuously depending on \( t \)) has to be unequal to zero.

So \( \psi(m_n^2 \Delta(f + t\phi)) = m_n^{k+1} \) which is equivalent to \( m_n^{k+1} \subseteq m_n^2 \Delta(f + t\phi) + m_n^{k+2} \).

Corollary: If \( 1^0 m_n^{k+1} \subseteq m_n^2 \Delta(f) + m_n^{k+2} \) and \( 2^0 \phi \notin m_n \Delta(f) + m_n^{k+1} \) then there exist \( \tau' > 0 \) such that \( f \) is not equivalent to \( f + t\phi \) for all \( |t| < \tau' \).

Proof:

1° implies that \( f + t\phi \) is \( k \)-determined for all \( t \) close to 0.

2° implies that \( j^k(f) + R_\phi \cap \text{Orb}(j^k f) \) consists only of a finite number of points.
(2.11) Theorem: If

1. \( m^{k+1} \subseteq m_n^A(f) + m_n^{k+2} \)
2. \( \phi \not\in m_n^A(f) + m_n^{k+1} \)
3. \( \text{codim}(f + t\phi) \) is constant for all \( t \) with \( |t| < \tau \)

then there exists \( \tau' < \tau \) such that \( t \) is a local invariant of \( f + t\phi \) if \( |t| < \tau' \).

Proof:

It is sufficient to prove \( \phi \not\in m_n^A(f + t\phi) + m_n^{k+1} \) for small \( t \). Let

\[ b_1(t), \ldots, b_m(t) \]

be the generators of

\[ m_n^A(f + t\phi) + m_n^{k+1} \]

and let \( \bar{\phi} \) be the representative of \( \phi \) in \( m_n^{k+1} \).

Let \( B(t) \) be the matrix with columnvectors \( b_1(t), \ldots, b_m(t) \) and \( B^r(t) \) be the matrix with columnvectors \( \bar{b}_1(t), \ldots, \bar{b}_m(t), \bar{\phi} \). Since

\[ \text{rank} \ B(t) = k \] for small \( t \) and \( \text{rank} \ B^r(t) = k + 1 \) for \( t = 0 \) (because \( \bar{\phi} \not\in R_0^+(0) + \ldots + R_0^+(0) \)), we have:

\[ \text{rank} \ B^r(t) \geq k + 1 \] for small \( t \); so \( \bar{\phi} \not\in R_0^+(t) + \ldots + R_0^+(t) \) for small \( t \).

(2.12) Example 1: Let \( f_t = x^4 + y^4 + tx^2y^2 \) \((t^2 \neq 4)\).

One can show that \( \text{codim}(f_t) = 8 \) for all \( t^2 \neq 4 \).

Moreover \( m^5 \subseteq m^2A(f_t) + m^6 \) if \( t = 0 \)

and \( x^2y^2 \not\in m^2A(f_t) + m^5 \) if \( t = 0 \).

So according to theorem (2.11) \( t \) is a local invariant of \( f_t \) in a neighborhood of \( t = 0 \).

Remark that this invariant has also a geometrical meaning. Since \( f_t \) is \( 4 \)-determined we can work entirely in \( J^4(2,1) \). Because \( f_t \) is homogeneous of degree 4 the orbit of \( f_t \) under \( L^4(2) \) coincides with the orbit of \( f \) under \( L^1(2) = GL(2) \).
Our local invariant \( t \) depends on the cross ratio of the four (complex) lines with equation \( f_t = 0 \) since every element of \( \phi \in \text{GL}(2) \) induces a projective transformation in the pencil of lines through the origin in the (complex) \( x-y \)-plane, sending \( f_t = 0 \) onto \( f_t \phi = 0 \). Cross ratio is an invariant under complex transformations.

\[
f_t(x, y) = x^4 + y^4 + t x^2 y^2 = 0 \quad f_t(\phi(x, y)) = 0
\]

Example 2: Let \( f = x^3 + xy^6 + ay^9 + y^{10} \)
(compare also 2.5, example 2).
In this case \( y^9 \not\in m \Delta(f) + f^*(m) \quad \forall a \in R. \)
One can deduce this from:

\[
\dim_R \frac{m}{\Delta(f) + f^*(m)} = 14
\]

\[
\dim_R \frac{m}{\Delta(f) + f^*(m) + y^9} = 13.
\]

So \( a \) is a local \( RL \)-invariant.

(2.13) Definition: Let \( D \) be an open connected subset of \( R^k \).
\( \tau = (\tau_1, ..., \tau_k) \) is called a (k-dimensional) local invariant of the family \( \{f_t\}_{t \in D} \) if for every \( \sigma \in D \) there exist \( \varepsilon > 0 \) such that the germs \( \{f_t^{\sigma} \mid \|\sigma - \tau\| < \varepsilon \} \) are all in different orbits. We also say, that \( (\tau_1, ..., \tau_k) \) is a set of local invariants of the family.

Example: The family \( f_t, s = xy(x+y)(x+ty)(x+sy) \) has the set of local invariants \((s, t)\).
They are related to two cross ratio's in the set of 5 lines, defined by \( f_{t,s} = 0 \).

(2.14) Theorem: Let \( f_\tau = f + \tau_1 \phi_1 + \ldots + \tau_k \phi_k \) be a k-parameter family of germs, defined in an open connected subset D of \( \mathbb{R}^k \) and let:

1. \( f_\tau \) be p-determined for all \( \tau \in D \)
2. \( [R_{\phi_1} + \ldots + R_{\phi_k}] \cap [m_n \Delta(f_\tau) + m_n^{p+1}] = \{0\} \)

then \( \tau \) is a (k-dimensional) local invariant.

Proof:
Because \( f_\tau \) is p-determined for all \( \tau \in D \), we can work entirely in \( J^p(n,1) \). Let \( \sigma \in D \) and let \( V = J^p(f_\sigma) + R^p(\phi_1) + \ldots + R^p(\phi_k) \).

We consider \( V \cap \text{Orb} (J^p(f_\sigma)) \). There are two possibilities:

a) \( f_\sigma \) is isolated in \( V \cap \text{Orb} (J^p(f_\sigma)) \).

b) \( f_\sigma \) is not isolated in \( V \cap \text{Orb} (J^p(f_\sigma)) \).

In case b) the curve selection lemma (cf MILNOR[20], pag. 25) implies that there is a real analytic curve:

\[ p : [0,\varepsilon) \rightarrow V \]

with \( p(0) = f_\sigma \) and \( p(t) \in V \cap \text{Orb} (J^p(f_\sigma)) \). In that case the intersection of the tangentspaces onto \( V \) and \( \text{Orb} (J^p(f_\sigma)) \) is at least 1-dimensional, so:

\[ [R_{\phi_1} + \ldots + R_{\phi_k}] \cap m_n \Delta(f_\sigma) \neq \{0\} \]

This gives a contradiction; so we are in case a). Now \( f_\sigma \) is isolated in \( V \cap \text{Orb} (J^p(f_\sigma)) \) and so there is a neighborhood of \( J^p(f_\sigma) \) in \( V \) such that no \( f_\tau \) in this neighborhood is equivalent to \( f_\sigma \). Since \( \sigma \) was arbitrary in \( D \), we are done.

(2.15) Theorem: Let \( f_\tau = f + \tau_1 \phi_1 + \ldots + \tau_k \phi_k \) be a k-parameter family of germs, defined in an open connected subset of \( D \) of \( \mathbb{R}^k \) and let...
1° $f_\tau$ be $p$-determined for all $\tau \in D$

2° $[R_{\phi_1} + \ldots + R_{\phi_k}] \cap [m_n \Delta(f_\tau) + (f_\tau)^*(m_1) + m_{D+1}] = \{0\}$

then $\tau$ is a local $RL$-invariant.

**Proof:** similar to (2.14).

(2.16) **Remark:** Proposition (2.10) and theorem (2.11) remain valid in the case of $k$-parameter families.

(2.17) **Example:** Let $f = x^5 + y^5$

From the picture we conclude:

1° $m^7 \subseteq m^2 \Delta(f) + m^8$ so, $x \quad y$

\[ \begin{array}{cccc}
\text{x}^3 & \text{x}^2 & \text{xy} & \text{y}^3 \\
\text{x} & \text{xy} & \text{y}^2
\end{array} \]

f is 6-determined

2° $\text{codim}(f) = 15$

Consider now $f(u,v,w) = x^5 + y^5 + ux^3y^2 + vy^2x^3 + wx^3y^3$.

Since 15 is the minimal codimension for a germ $f \in m_2$ with $f_4 = 0$, we have $\text{codim} f(u,v,w) \geq 15$ for $(u,v,w)$ small. Moreover

\[ Rx^3y^2 + Rx^2y^3 + Rx^3y^3 \notin m\Delta(x^5 + y^5) + m^7 \]

Theorem (2.11) implies: $(u,v,w)$ is a set of local invariants for $(u,v,w)$ small.

Remark, that $w$ is not a local RL-invariant.
\(3.1\) We consider \(f \in m_n^2\). Since \(f_1 \equiv 0\) the polynomial \(f_2\) is homogeneous of degree 2:

\[ f_2 = \sum_{i,j=1}^{n} a_{ij} x_i x_j \text{ where } a_{ij} = \frac{1}{2} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) (0). \]

The rank of the matrix \(\frac{\partial^2 f}{\partial x_i \partial x_j} (0)\) is invariant under \(RL\)-equivalence.

Definition: The corank of \(f\) is \(n\) minus the rank of the matrix

\[ \frac{\partial^2 f}{\partial x_i \partial x_j} (0). \]

Notation: \(\text{corank } (f) = n - \text{rank} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \).

\(3.2\) \textbf{Splitting lemma:} Let \(f \in m_n^2\), \(\text{codim } (f) < \infty \) and \(\text{corank } (f) = r\).

Then: \(f(x_1, \ldots, x_n) = g_2(x_1, \ldots, x_r) + e_{r+1} x_{r+1}^2 + \cdots + e_n x_n^2\) where

\[ e_{r+1} = \pm 1, \ldots, e_n = \mp 1 \text{ and } g_2 \equiv 0. \]

Proof:

There exists a linear isomorphism such that \(f_2 \equiv e_{r+1} x_{r+1}^2 + \cdots + e_n x_n^2\)

where \(e_{r+1} = \pm 1, \ldots, e_n = \mp 1\) and \(r = \text{corank } (f)\).

So \(f \equiv e_{r+1} x_{r+1}^2 + \cdots + e_n x_n^2\).

We continue now by induction on \(k\).

Let \(f_k g_k(x_1, \ldots, x_r) + e_{r+1} x_{r+1}^2 + \cdots + e_n x_n^2\) with \(k \geq 2\) and

\[ g_k \in m_n^3. \]

Assertion: \(f \equiv g_{k+1}(x_1, \ldots, x_r) + e_{r+1} x_{r+1}^2 + \cdots + e_n x_n^2\) with \(g_{k+1} \in m_n^3.\)
We have \( f \leftarrow^k \mathbf{g}_k(x_1, \ldots, x_r) + \rho(x_1, \ldots, x_n) + e_{r+1}x_{r+1}^2 + \ldots + e_n x_n^2 \)

where \( \rho \) is homogeneous of degree \( k+1 \).

We write \( \rho \) in the following form:

\[
\rho(x_1, \ldots, x_n) = x_n h_n(x_1, \ldots, x_n) + x_{n-1} h_{n-1}(x_1, \ldots, x_{n-1}) + \ldots + x_{r+1} h_{r+1}(x_1, \ldots, x_{r+1}) + \rho(x_1, \ldots, x_r)
\]

where \( h_n, \ldots, h_{r+1} \) are homogeneous of degree \( k \) and \( \rho \) is homogeneous of degree \( k+1 \).

Define \( \phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) by

\[
\begin{aligned}
\phi^1(x_1, \ldots, x_n) &= x_1 \\
\vdots \\
\phi^r(x_1, \ldots, x_n) &= x_r \\
\phi^{r+1}(x_1, \ldots, x_n) &= x_{r+1} + \sigma_{r+1} \\
\vdots \\
\phi^n(x_1, \ldots, x_n) &= x_n + \sigma_n
\end{aligned}
\]

where \( \phi^1, \ldots, \phi^n \) are the components of \( \phi \) and \( \sigma_{r+1}, \ldots, \sigma_n \in m^k \) (to be fixed later).

The Jacobian matrix of \( \phi \) in \( 0 \in \mathbb{R}^n \) is the identity matrix. The inverse-function theorem implies that \( \phi \) is a germ of diffeomorphism.

In stead of (*) we shall use the short-hand notation:

\[
\begin{aligned}
x_1 &= x_1 \\
\vdots \\
x_r &= x_r \\
x_{r+1} &= x_{r+1} + \sigma_{r+1} \\
\vdots \\
x_n &= x_n + \sigma_n
\end{aligned}
\]

By substitution we get (Modulo \( m_n^{k+2} \)):

\[
f \leftarrow^k \mathbf{g}_k(x_1, \ldots, x_r) + \rho(x_1, \ldots, x_n) + e_{r+1}(x_{r+1} + \sigma_{r+1})^2 + \ldots + e_n (x_n + \sigma_n)^2
\]
\[ \begin{aligned}
&= g_k(x_1, \ldots, x_r) + p(x_1, \ldots, x_r) + x_{r+1}h_{r+1}(x_1, \ldots, x_{r+1}) + \ldots + \\
&\quad + x_n h_n(x_1, \ldots, x_n) + e_{r+1}x_{r+1}^2 + 2e_{r+1}x_{r+1}\sigma_{r+1} + \ldots + e_n x_n^2 + 2e_n x_n \sigma_n \\
&= g_k(x_1, \ldots, x_r) + p(x_1, \ldots, x_r) + e_{r+1}x_{r+1}^2 + \ldots + e_n x_n^2 + \\
&\quad + x_{r+1}[h_{r+1}(x_1, \ldots, x_{r+1}) + 2e_{r+1}\sigma_{r+1}] + \ldots + x_n[h_n(x_1, \ldots, x_r) + 2e_n \sigma_n] \\
&= g_{k+1}(x_1, \ldots, x_r) + e_{r+1}x_{r+1}^2 + \ldots + e_n x_n^2.
\end{aligned} \]

If
\[
\begin{aligned}
\sigma_{r+1} &= \frac{-1}{2e_{r+1}} [h_{r+1}(x_1, \ldots, x_{r+1})] \in m_k \\
&\vdots \\
\sigma_n &= \frac{-1}{2e_n} [h_n(x_1, \ldots, x_n)] \in m_k
\end{aligned}
\]

and
\[ g_{k+1}(x_1, \ldots, x_r) = g_k(x_1, \ldots, x_r) + p(x_1, \ldots, x_r). \]

Now the assertion is proved for all \( k \geq 2. \)

Since \( \text{codim}(f) < \infty \) there exists a \( s \) such that \( f \) is \( s \)-determined.

With (1.5) lemma there follows:

\[ f(x_1, \ldots, x_n) \sim g(x_1, \ldots, x_r) + e_{r+1}x_{r+1}^2 + \ldots + e_n x_n^2. \]

(3.3) Remark:

The above proof of the splitting lemma is due to MATHER[19]. Other proofs, not requiring that \( \text{codim}(f) < \infty \) are given by WASSERMANN[27] and GROMOLL-MEYER[13]. In the last case the splitting lemma is given in a Hilbert space context. They mention also an observation of MATHER, that given any two splittings of the form \( f \sim g + Q \) with \( g_2 \equiv 0 \) and \( Q \) a non-degenerate quadratic form, then the corresponding non-degenerate parts and degenerate parts are Right-equivalent.

(3.4) Lemma: Let \( f(x_1, \ldots, x_n) = g(x_1, \ldots, x_r) + e_{r+1}x_{r+1}^2 + \ldots + e_n x_n^2 \)

with \( g_2 \equiv 0; \) then:

1° \( \text{codim } g = \text{codim } f \)

2° \( g \) is \( k \)-determined \( \Rightarrow f \) is \( k \)-determined.
Proof:

1° \( \Delta(f) = (\alpha_1 g, \ldots, \alpha_r g, x_{r+1}, \ldots, x_n) \in m_n \)
\( \Delta(g) = (\alpha_1 g, \ldots, \alpha_r g) \in m_r. \)

2° Let \( \tilde{f}_k = f_k. \) From the proof of the splitting lemma it follows, that:
\[ \tilde{f}(x_1, \ldots, x_n) \sim \tilde{g}(x_1, \ldots, x_r) + e_{r+1}x_{r+1}^2 + \ldots + e_n x_n^2 \]
with \( g_k = \tilde{g}_k. \)

So there is a diffeomorphism \( \phi : (\mathbb{R}^r, 0) \to (\mathbb{R}^r, 0), \) with \( \tilde{g} \phi = g \) and
this implies that \( f \sim \tilde{f}. \) (extending \( \phi \) by the identity).

Corollary: The classification of \( f \in m_n^2 \) follows from the classification of \( g \in m_r^3. \)

\[(3.5) \text{ Lemma: Let } f(x_1, \ldots, x_n) = g(x_1, \ldots, x_r) + e_{r+1}x_{r+1}^2 + \ldots + e_n x_n^2 \]
with \( g_2 = 0. \)
\r = 0 \implies \text{codim } (f) = 0 \]
\r = 1 \implies \text{codim } (f) \geq 1 \]
\r = 2 \implies \text{codim } (f) \geq 3 \]
\r = 3 \implies \text{codim } (f) \geq 7 \]
\r \geq 4 \implies \text{codim } (f) \geq 14 \]
The proof is direct computation, concerning the ideal \( (\alpha_1 g, \ldots, \alpha_r g) \)
for \( g \) a function of lowest degree 3.

\[(3.6) \text{CLASSIFICATION THEOREM:} \]

For \( f \in m_n \) with \( \text{codim } (f) < \infty \) and \( f_1 = 0 \) we have,
either: \( f \sim Q + g \) where \( g \) is a germ of one of the polynomials in the
list on the next page, and \( Q = e_{r+1}x_{r+1}^2 + \ldots + e_n x_n^2 \)
or: \( \text{codim } (f) > 9. \)
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<thead>
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<th>$r$</th>
<th>$g$</th>
<th>type</th>
<th>$R$-codimension</th>
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[Conditions: $J_9$: $4A^3 + 27B^2 \neq 0$; $X_9$: $\alpha \neq 0, -1, 1$; $X_{10}$: $\alpha \neq 0$; $P_8$: $4g_1^3 + 27g_2^2 \neq 0$; $P_9$: $\beta \neq 0$; $P_{10}$: $\beta \neq 0$; $R_{10}$: $A \neq 0$ and $B \neq 0$.]
The proof follows increasing corank $r$ of $f$. In corank $r=0$ and $r=1$ the list is complete. In the sections on $r=2$ and $r=3$ we add remarks about some germs of codimension $>9$. The proof will be given in §4.

(3.7) Remark: Two germs of different type are not equivalent. Within one type, we may have equivalent ones. An example is $A_{k-1}$ with $k$ odd. The parameters in the families $X_9$, $X_{10}$, $P_9$, $P_{10}$, $Q_{10}$ and $R_{10}$ are local invariants. The families $J_{10}$ and $P_8$ have a 1-dimensional local invariant, depending on the two parameters. The equivalence in these families is discussed in (4.5) for $J_{10}$; in (4.10) for $P_8$ and in (4.13) for $R_{10}$. In §4 we also treat the difference between the $R$-classification and the $RL$-classification.

(3.8) Remark: We can consider the classification problem also in other cases: $R$-analytic, $R$-formal power series and also the $C$-analytic case and $C$-formal power series. In all these cases we have the same results as here in the $C^\infty$-$R$-case, because it turns out that classification of germs of finite codimension can be done with polynomial functions. In the $C$-case we can replace all $+$-signs by $-$-signs and sometimes it is possible to give nicer normal forms for the orbits. (See the list I at the end.)
If no confusion is possible, we use the abbreviation \( \Delta \) for \( \Delta(g) \) and \( m \) for \( m^1 \).

\[ \text{corank} = 0 \]

(4.1) Theorem: If \( r = 0 \) then \( f(x_1, \ldots, x_n) \supseteq e_1 x_1^2 + \ldots + e_n x_n^2 \) (A.1).

Proof:
Since \( r = 0 \) we have \( f(x_1, \ldots, x_n) \supseteq e_1 x_1^2 + \ldots + e_n x_n^2 \).
So \( \Delta(f_2) = m \) and \( m^3 \leq m^2 \Delta(f_2) + m^4 \) which implies that \( f_2 \) is 2-determined; so \( f \sim f_2 \).

Remark: We may take \( e_1 \geq e_2 \geq \ldots \geq e_n \). There are \( n+1 \) equivalence-classes, corresponding to +++++, +++++, ++++-, ++++-, etc.

\[ \text{corank} = 1 \]

(4.2) Theorem: If \( r = 1 \) then

either: \( f(x_1, \ldots, x_n) \supseteq x_1^{k+1} + e_2 x_2^2 + \ldots + e_n x_n^2 \) (A.1) \( k \geq 2 \)

or: \( \text{codim}(f) = \infty \).

Proof:
Let \( \text{codim}(f) < \infty \), then \( f(x_1, \ldots, x_n) \supseteq g(x_1) + e_2 x_2^2 + \ldots + e_n x_n^2 \).
Let \( g_k = ax_1^k \) with \( a \neq 0 \), then \( \Delta(g_k) = m^k \); so \( m^{k+1} \leq m^2 \Delta(g_k) + m^{k+2} \) \( k \geq 2 \) and so \( g_k \) is \( k \)-determined, which gives \( g \sim g_k \).

If \( k \) is even and \( a > 0 \): \( g(x_1) \supseteq x_1^k \); and if \( a < 0 \): \( g(x_1) \supseteq -x_1^k \).

Remark: We may take \( e_2 \geq e_3 \geq \ldots \geq e_n \).
If \( k \) is even we have \( 2n \) equivalence-classes.
If \( k \) is odd we have \( n \) equivalence-classes.
In the sequel we shall no longer mention the various quadratic cases.
(4.3) Proposition: If $r = 2$ then
\[ f(x_1, \ldots, x_n) = g(x_1, x_2) + e_3 x_2^2 + \ldots + e_n x_n^2, \]
where $g_3$ is in exactly one of the following four cases:

1. $g_3(x_1, x_2) = x_1^2 x_2 \pm x_2^3$
2. $g_3(x_1, x_2) = x_1^2 x_2$
3. $g_3(x_1, x_2) = x_1^3$
4. $g_3(x_1, x_2) = 0$

Proof:
Because $g_2 = 0$ it follows that $g_3(x_1, x_2)$ is a homogeneous polynomial. We may factor $g_3$ into linear forms over $\mathbb{C}$. The four cases correspond to 3, 2, 1 or 0 linear factors. By a linear map we can arrange, that $g_3$ gets the given form.

(4.4) Theorem: If $r = 2$ and $g_3 = x_1^2 x_2 \pm x_2^3$ or $g_3 = x_1^2 x_2$, we haveither: $f(x_1, \ldots, x_n) = x_1^2 x_2 \pm x_2^{k-1} + e_3 x_3^2 + \ldots + e_n x_n^2$ (D. k) \( k \geq 2 \)
with codim $(f) = k-1$.

or: codim $(f) = \infty$.

Proof: Let codim $(f) < \infty$.

In case 1. we have $g_3(x_1, x_2) = x_1^2 x_2 \pm x_2^3$.

In case 1. we have $g_3(x_1, x_2) = x_1^2 x_2 \pm x_2^3$.

So $g_3$ is 3-determined and $g \between g_3$.

In case 2. $g_3(x_1, x_2) = x_1^2 x_2$ and $\Delta(g_3) = (2x_1 x_2, x_1^2, x_2^2)$, so $g_3$ is finitely determined. So we have to consider higher jets than 3-je
Lemma 1: Let $k > h$ then
\[ x_1^2 x_2 + a_0 x_1^k + a_1 x_1^{k-1} x_2 + \ldots + a_k x_2^k \quad \text{and} \quad x_1^2 x_2 + a_0 x_2^k \]

Proof:
We define an element of $L_2$ by:
\[ \begin{align*}
&x_1: = x_1 + \rho_1 \quad \text{with} \quad \rho_1 \pm \frac{1}{2}[a_0 x_1^{k-2} + \ldots + a_{k-1} x_1^{k-2}] \in m^{k-2} \\
&x_2: = x_2 + \rho_2 \quad \text{with} \quad \rho_2 \pm -a_0 x_1 \in m^{k-2}
\end{align*} \]

So we have:
\[ x_1^2 x_2 + a_0 x_1^k + a_1 x_1^{k-1} x_2 + \ldots + a_k x_2^k \]
\[ (x_1 + \rho_1)^2 (x_2 + \rho_2) + a_0 x_1^k + a_1 x_1^{k-1} x_2 + \ldots + a_{k-1} x_1^{k-1} x_2 + a_k x_2^k \]
\[ x_1^2 x_2 + x_1^2 \rho_2 + 2 x_1 x_2 \rho_2 + a_0 x_1^k + a_1 x_1^{k-1} x_2 + \ldots + a_{k-1} x_1^{k-1} x_2 + a_k x_2^k \]
\[ \pm x_1^2 x_2 + a_0 x_1^k \]

Lemma 2: $g_k = x_1^2 x_2 + x_1^k$ is $k$-determined.

Proof:
\[ \Delta(g_k) = (2 x_1 x_2, x_1^2 + k x_2^{k-1}) \]

We have $m^k \subseteq m \Delta + m^{k+1}$ since
\[ m^{k-2} x_1 x_2 = m^{k-2} 3 g_k \]
\[ x_1^k = x_1^{k-2} \cdot 3 g_k + \frac{k}{2} x_1^k - \frac{1}{2} x_1 x_2 \]
\[ x_2^k = \pm x_2^k \cdot 3 g_k \]

We apply Lemma 1 for $k = 4, 5, \ldots, \ell$ we get
\[ g_{\ell} \quad x_1^2 x_2 + \beta_4 x_2^4 + \ldots + \beta_{\ell-1} x_2^{\ell-1} + \beta_{\ell} x_2^\ell \]

Let $k$ be the smallest integer such that $\beta_k \neq 0$.

In that case is $g_k$ $k$-determined and consequently
\[ g \cup g_k \quad x_1^2 x_2 + x_2^k (D_{k+1}) \]
(4.5) Theorem: If \( r = 2 \) and we are in the case \( g_3 = x_1^3 \) of proposition (4.3) then, either: 
\[ f(x_1, \ldots, x_n) \sim g(x_1, x_2) + e_3 x_3^2 + \ldots + e_n x_n^2, \]
where 
\[
\begin{align*}
g(x_1, x_2) &= x_1^3 + x_2^4, & (E_6) \\
g(x_1, x_2) &= x_1^3 + x_1 x_2^3, & (E_7) \\
g(x_1, x_2) &= x_1^3 + x_2^5, & (E_8) \\
g(x_1, x_2) &= x_1^3 + A x_1 x_2^4 + B x_2^6 \quad (4A^3 + 27B^2 \neq 0 (J_{10}))
\end{align*}
\]
or: \( \text{codim}(f) > 9. \)

In \( J_{10} \) the number \( k = \frac{A^3}{4A^3 + 27B^2} \) is an invariant. Two germs of the family \( J_{10} \) are equivalent if and only if they have equal \( k \) and equal sign of \( B \). If \( B = 0 \) they are equivalent iff the sign of \( A \) is the same.

Proof: Follows from Lemma 1-5.

Remark first, that \( g_3(x_1, x_2) = x_1^3 \) is not finitely determined.

Lemma 1:
\[
x_1^3 + \sigma_4 + \ldots + x_{n-1}^3 + \tau_n \sim x_1^3 + \sigma_4 + \ldots + \sigma_{n-1} + \sigma_n
\]
where \( \sigma_p = a_p x_1 x_2^{p-1} + \beta_p x_2^p \) and \( \tau_n \) homogeneous of degree \( n \).

Proof:
Define an element of \( L_q \) by \( \left\{ \begin{array}{l}
x_1^i = x_1 + \rho_{n-2} \quad \text{with } \rho_{n-2} \in m_{n-2} \\
x_2^i = x_2
\end{array} \right. \)
\[
x_1^3 + \sigma_4 + \ldots + x_{n-1}^3 + \tau_n \sim x_1^3 + 3x_1 \rho_{n-2} + \sigma_4 + \ldots + \sigma_{n-1} + \tau_n = x_1^3 + \sigma_4 + \ldots + \sigma_{n-1} + [3x_1 \rho_{n-2} + \tau_n] = x_1^3 + \sigma_4 + \ldots + \sigma_{n-1} + \sigma_n
\]
if we choose \( \rho_{n-2} \) such that \( 3x_1 \rho_{n-2} + \tau_n = a_n x_1 x_2^{n-1} + \beta_n x_2^n = \sigma_n. \)

[Remark that the coefficients of \( x_1 x_2^{n-1} \) and \( x_2^n \) have not changed].

Corollary: (Normalform):
If \( g^3 x_1^3 \) then \( g \sim x_1^3 + \sigma_4 + \ldots + \sigma_n \) where \( \sigma_p = a_p x_1 x_2^{p-1} + \beta_p x_2^p. \)
Lemma 2: Let \( k \geq 4 \).

- a) \( g_k = x_1^3 + a_k x_2^k \) (\( a_k \neq 0 \)) is \( k \)-determined.
- b) \( g_k = x_1^3 + a_{k-1} x_1 x_2^{k-1} \) (\( a_{k-1} \neq 0 \)) is \((2k-3)\)-determined and not \((2k-5)\)-determined.

Proof:

a) \( \beta_1(g_k) = 3x_1^2 \) and \( \beta_2(g_k) = k a_k x_2^{k-1} \) and this leads directly to:
\[
m^k \subseteq m\Delta + m^{k+1}.
\]

b) \( \beta_1(g_k) = 3x_1^2 + a_{k-1} x_2^{k-1} \)
\[
\beta_2(g_k) = (k-1)a_{k-1} x_1 x_2^{k-2}
\]

We shall show: \( m^{2k-3} \subseteq m\Delta + m^{2k-2} \). We compute now modulo \( m\Delta + m^{2k-2} \):

1. \( 0 = m\beta_2(g_k) = x_1 x_2^{k-2} \)

2. \( 0 \equiv m^{2k-5} \beta_1(g_k) = x_1^{2k-5} + a_{k-1} x_2^{k-5} \)

3. \( 0 \equiv x_2^{k-2} \beta_1(g_k) = 3x_1^2 x_2^{k-2} + a_{k-1} x_2^{k-3} \)

so \( x_2^{2k-3} = 0 \) since \( a_{k-1} \neq 0 \).

We have now all generators of \( m^{2k-3} \), so \( m^{2k-3} \subseteq m\Delta + m^{2k-2} \) and \( g_k \) is \((2k-3)\)-determined.

Since \( x_2^{2k-4} \notin m\Delta + m^{2k-3} \) we have that \( g_k \) is not \((2k-5)\)-determined.

We apply Lemma 1 for \( n=4 \); so let:

\[
g_4 = x_1^3 + a_4 x_1 x_2^3 + \beta_4 x_2^4
\]

If \( \beta_4 \neq 0 \) define an element of \( L_2 \) by:

\[
\begin{cases}
x_1' = x_1 \\
x_2' = x_2 - px_1 \text{ with } p \neq \frac{\beta_4}{4\beta_4} \in \mathbb{R}
\end{cases}
\]

So we have:

\[
\begin{align*}
g_4^h x_1^3 + a_4 x_1 (x_2 - px_1)^3 + \beta_4 (x_2 - px_1)^4 & = \\
& = x_1^3 + \gamma_1 x_1^4 + \gamma_2 x_1^3 x_2 + \gamma_3 x_1^2 x_2^2 + (a_4 - 4p\beta_4) x_1 x_2^3 + 4x_2^4 \\
& \equiv x_1^3 + \gamma_1 x_1^4 + \gamma_2 x_1^3 x_2 + \gamma_3 x_1^2 x_2^2 + \beta_4 x_2^4.
\end{align*}
\]
Next apply again Lemma 1. Since the coefficients of $x_1x_2^3$ and $x_2^4$ don't change we get:

$$g_4 	riangleq x_1^3 + \beta_4 x_2^4 \triangleq x_1^3 + x_2^4$$

$g_4$ is 4-determined (Lemma 2), so

$$g \triangleq x_1^3 + x_2^4 \quad (E_0)$$

If $\beta_4 = 0$ and $\alpha_4 \neq 0$ then $g_4 = x_1^3 + \alpha_4 x_1 x_2 + x_1^3 + x_1 x_2^5$. Then $g_4$ is 5-determined (Lemma 2), so we have to consider

$$g_5 = x_1^3 + x_1 x_2^3 + \gamma_0 x_1^5 + \ldots + \gamma_5 x_2^5.$$

Lemma 3:

a) $g_5 = x_1^3 + x_1 x_2^3 + \gamma_0 x_1^5 + \ldots + \gamma_5 x_2^5$ is 4-determined.

Proof: we shall give an outline of the computation:

step 1: Using \( \begin{cases} x_1' = x_1 + \rho \text{ with } \rho = \frac{1}{3}(\gamma_0 x_1^3 + \ldots + \gamma_3 x_2^3) \in \mathcal{M}^3 \\ x_2' = x_2 \end{cases} \)
we get $g_5 \triangleq x_1^3 + x_1 x_2^3 + \gamma_0 x_1^5 + \gamma_5 x_2^5$.

step 2: Using \( \begin{cases} x_1' = x_1 \\ x_2' = x_2 + \sigma \text{ with } \sigma = \frac{1}{3} \gamma_4 x_2^2 \in \mathcal{M}^2 \end{cases} \)
we get $g_5 \triangleq x_1^3 + x_1 x_2^3 + \gamma_5 x_2^5$.

The coefficient of $x_2^5$ is still the same as in step 1!

step 3: Using \( \begin{cases} x_1' = x_1 \\ x_2' = px_1 + x_2 \text{ with } p = \gamma_5 \in \mathbb{R} \end{cases} \)
we get $g_5 \triangleq x_1^3 + x_1(px_1 + x_2)^3 + \beta_5(px_1 + x_2)^5$.

step 4: Using \( \begin{cases} x_1' = x_1 + \rho \\ x_2' = x_2 \text{ with } \rho = \frac{1}{3}(p_3 x_1^2 + 3p_2 x_1 x_2 + 3px_2^2) \in \mathcal{M}^2 \end{cases} \)
we get $g_5 \triangleq x_1^3 + x_1 x_2^3 + x_1 \cdot [\text{degree 4}] + \rho x_2^3 + \gamma_5 x_2^5$.

The coefficient of $x_2^5$ is equal to $-p + \gamma_5 = 0$. 


step 5: Apply again the system of step 1 and 2. The coefficient of $x_2^5$ does not change, so we get: $g_5 \leftarrow x_1^3 + x_1 x_2^3$.

step 6: Since $g_4$ is 5-determined and $g_5 \leftarrow g_4$ for all $\gamma_0, \ldots, \gamma_5$ we get $g_5$ is 5-determined and so also $g_5 \leftarrow g_4$. Let $h_4 = g_4$ then $h_5 = g_4 + \gamma_0 x_1^5 + \ldots + \gamma_5 x_2^5$ for some values of $\gamma_0, \ldots, \gamma_5$.

Since the righthand side is 5-determined we have $h \leftarrow g_4 + \gamma_0 x_1^5 + \ldots + \gamma_5 x_2^5 \leftarrow g_4$.

So $g_4$ is 4-determined.

**Remark 1:**

It is also possible to prove $x_2^5 \in m\Delta(x_1^3 + x_1 x_2^3 + \beta_5 x_2^5) + m^6$ for all $\beta_5$ and then to use proposition (2.2) to prove $x_1^3 + x_1 x_2^3 + \beta_5 x_2^5 \leftarrow x_1^3 + x_1 x_2^3$.

**Remark 2:**

If $f_k$ is $(k+1)$-determined; it is not always true that $f_k + \beta_0 x_1^k + \beta_1 x_1 x_2 + \ldots + \beta_{k+1} x_2^k$ is also $(k+1)$-determined.

If $m^k \subseteq m\Delta(f_k) + m^{k+1}$ this guarantees only $(k+1)$-determinacy for small values of $\beta_0, \ldots, \beta_{k+1}$ (compare proposition (2.10)).

We return to the case that $\beta_4 = 0$ and $a_4 \neq 0$.

From the Lemma 3 it follows that $g_4 \leftarrow x_1^3 + x_1 x_2^3$ is 4-determined.

So $g \leftarrow x_1^3 + x_1 x_2^3$ (E7)

If $a_4 = 0$ and $\beta_4 = 0$ we have $g_4 = x_1^3$ and we consider $g_5 = x_1^3 + a_5 x_1 x_2 + \beta_5 x_2^5$.

If $\beta_5 \neq 0$ then we can derive in the same way as in the case E6 that $g_5 \leftarrow x_1^3 + \beta_5 x_2^5 \leftarrow x_1^3 + x_2^5$, which is 5-determined; so $g \leftarrow x_1^3 + x_2^5$ (E8)
If $\beta_5 = 0$ then $g_5$ is 7-determined and we have to study higher jets. First we derive a normal form in a more general case.

**Lemma 4:** For $\mu \neq 0$ and $k \geq 5$ we have:

\[
x_1^3 + \mu x_1 x_2^{k-1} + \beta_{k+1} x_2^{k+1} + \ldots + \beta_n x_2^n = \sum_{i=1}^{n-1} \alpha_i x_1 x_2^{n-1} + \beta_{n-1} x_2^{n-1} + \beta_n x_2^n.
\]

**Proof:**

Define an element of $L_2$ by:

\[
\begin{cases}
x_1 = x_1 \\
x_2 = \rho \text{ with } \rho \in \mathbb{R}^{n-k+1}; \rho = \frac{-a_n}{\mu} x_2^{n-k-1}.
\end{cases}
\]

The left-hand side is $n$-equivalent to:

\[
x_1^3 + \mu x_1 x_2^{k-1} + \beta_{k+1} x_2^{k+1} + \ldots + \beta_n x_2^n = \sum_{i=1}^{n-1} \alpha_i x_1 x_2^{n-1} + \beta_{n-1} x_2^{n-1} + \beta_n x_2^n.
\]

**Corollary:** (Normal form): let $\mu \neq 0$ and $n \geq k \geq 5$.

If $g_5 \in x_1^3 + \mu x_1 x_2^{k-1}$ then $g_6 \in x_1^3 + \mu x_1 x_2^{k-1} + \beta_{k+1} x_2^{k+1} + \ldots + \beta_n x_2^n$.

Let us return to $g_5 = x_1^3 + a_5 x_1 x_2^4$.

Since $g_5$ is not 5-determined we study higher jets of $g$:

**Lemma 5:** Let $g_7 = x_1^3 + a_5 x_1 x_2^4 + \beta_6 x_2^6 + \beta_7 x_2^7$ and $(4a_5^3 + 27\beta_6^2) \neq 0$.

1° $g_7$ is 7-determined

2° $g_6$ is in fact 6-determined.

**Proof:**

a) If $a_5 = 0$ and $\beta_6 \neq 0$ we can apply Lemma 2.

b) Let us suppose $a_5 \neq 0$. We shall show that $m^7 \subseteq m\Delta + m^8$.

\[
\begin{align*}
\beta_1 g_7 &= 3x_1^2 + a_5 x_2^4 \\
\beta_2 g_7 &= 4a_5 x_1 x_2^3 + 6\beta_6 x_2^5 + 7\beta_7 x_2^6.
\end{align*}
\]
Modulo $\Delta + \beta$ we have:
\[
\begin{align*}
x_1^{2m} &= 0 \\
x_1^{3m^3} &= 0 \\
0 &= 3x_1^2x_2^3 + \alpha x_2^7 \\
0 &= 4\alpha x_1^2x_2^5 + 6\beta x_2^7
\end{align*}
\]
and
\[
\begin{vmatrix}
\alpha_5 & 6\beta_6 & 0 \\
3 & 0 & 4\alpha_5 \\
0 & 4\alpha_5 & 6\beta_6 \\
\end{vmatrix} = -4[\alpha_5^3 + 27\beta_6^2] \neq 0 \text{ are } \begin{cases} 
x_1^2x_2^3 = 0 \\
x_1^5x_2^5 = 0 \\
x_2^7 = 0 \\
\end{cases}
\]

Now 1° follows:

Because $x_2^7 \in \Delta + \beta$ for all $\beta_7$ we have that $g_7$ is $\beta$-determined.

So if $\beta_5 = 0$ we have $g \sim x_1^3 + Ax_1x_2^4 + Bx_2^6$ \((J_{10})\) if $4A^3 + 27B^2 \neq 0$.

**Remark 3:**

Consider the question: When are two germs \(f_{(A,B)} = x_1^3 + Ax_1x_2^4 + Bx_2^6\) of type \(J_{10}\) equivalent? It turns out that the action of \(L_2\) on the subset $x_1^3 + Rx_1x_2^4 + Rx_2^6$ of \(J^6(2,1)\) coincides with the action of \(GL(2)\). In fact the only possibility is a multiplication in the \(x_2\)-direction:

\[
(*) \quad \begin{cases} 
x_1' = x_1 \\
x_2' = \lambda x_2, \lambda \neq 0.
\end{cases}
\]

A geometrical invariant of $x_1^3 + Ax_1x_2^4 + Bx_2^6$ is constructed as follows:

\[
x_1^3 + Ax_1x_2^4 + Bx_2^6 = (x_1 + k_1x_2^2)(x_1 + k_2x_2^2)(x_1 + k_3x_2^2)
\]

and defines (over $\mathbb{C}$) 3 parabolas in the $x_1-x_2$-plane.
The line \( x_2 = 1 \) intersects the 3 parabolas in 3 points \( A(k_1, 1) \), \( B(k_2, 1) \) and \( C(k_3, 1) \). Let \( D^\infty \) be the point at infinity of \( x_2 = 1 \), then \( (ABCD) = \frac{k_2 - k_1}{k_2 - k_3} \) is an invariant of the \( L_2 \)-action.

Let now \( f(A, B) \), \( f(A, B) \) then (*) implies:
\[
A_1^3 = A_2^3, B_1^3 = B_2^3 \text{ so } A_1^3 1^2 = A_2^3, B_1^3 1^2 = B_2^3.
\]
Hence \( \lambda^2 (4A_1^3 + 27B_1^2) = 4A_2^3 + 27B_2^2 \).

Define \( k(A, B) = \frac{A^3}{4A_1^3 + 27B_2^2} \). Then \( k(A, B) = k(A, B) \).

On the other hand, if \( k(A, B) = k(A, B) \) then \( f(A, B) \) over \( \mathbb{C} \). In the real case we need the additional condition:
\[
(B_1B_2 > 0) \lor (B_1B_2 = 0 \land A_1A_2 > 0)
\]
For each \( k \in \mathbb{R} \) there are two different real germs of type \( J_{10} \). Moreover there are two different topological types; corresponding to 3 real parabolas with \( k \in (-\infty, 1] \) and 1 real parabola with \( k \in [1, \infty) \).

The equivalenceclasses form a system of curves in the \( A-B \)-plane.

A parametrization of the family can be given by a closed curve around the origin, for example: \( C: 4|A|^3 + 27|B|^2 = 1 \).
The two intersection points of $C$ with $4A^3 + 27B^2 = 0$ correspond to more degenerate germs. The other points of the curve are in 1-1-correspondence to the equivalence classes of germs of type $J_{10}$.

(4.6) Proposition: If $r = 2$ and we are in the case $g_3 = 0$ of proposition (4.3) then $f(x_1, \ldots, x_n) = g(x_1, x_2) + e_3 x_3^2 + \ldots + e_n x_n^2$, where $g_4$ is in exactly one of the following cases:

1\° $g_4 = (x_1^2 + x_2^2)(x_1^2 + \alpha x_2^2) \quad \alpha \neq 0, -1, 1$

2\° $g_4 = x_1^2(x_1^2 + x_2^2)$

3\° $g_4 = (x_1^2 + x_2^2)^2$ with codim $(f) \geq 10$

4\° $g_4 = x_1^3 x_2$ with codim $(f) \geq 10$

5\° $g_4 = x_1^4$ with codim $(f) \geq 11$

6\° $g_4 = 0$ with codim $(f) \geq 15$

Proof: Because $g_3 = 0$, $g_4(x_1, x_2)$ is a homogeneous polynomial of degree 4. We may factor $g_4$ into linear forms over $\mathbb{C}$. The six cases correspond to 4, 3, 2, 2, 1 or 0 factors; indicated in the following pictures of the sets $g_4 = 0$.

By linear transformation one can obtain one of the given expressions for $g_4$. In each case one constructs first a normal form of the 5-jet of $f$. Then straight-forward computations show:

3\° $g_4 = (x_1^2 + x_2^2)^2$ \quad \Rightarrow \quad \text{codim} (f) \geq 10

4\° $g_4 = x_1^3 x_2$ \quad \Rightarrow \quad \text{codim} (f) \geq 10

5\° $g_4 = x_1^4$ \quad \Rightarrow \quad \text{codim} (f) \geq 11

6\° $g_4 = 0$ \quad \Rightarrow \quad \text{codim} (f) \geq 15.
(4.7) Theorem: If \( r = 2 \) and we are in case 1\(^0\) of proposition (4.6), then \( f(x_1, \ldots, x_n) \sim (x_1^2 + x_2^2)(x_1^2 + ax_2^2) \) where \( a \neq 0, -1 \) or 1.

The number \( a \) is an invariant under \( R \) and \( RL \)-equivalence.

Proof:

A straightforward computation shows, that \( m^5 \leq m^2 \Delta(f) + m^6 \) for all \( a \neq 0, -1 \) or 1.

The invariance of \( a \) is related to the crossratio of the four (complex) lines with equation: \( (x_1^2 + x_2^2)(x_1^2 + ax_2^2) = 0 \).

Two germs of the family are equivalent if and only if their crossratios are equal (modulo permutation of the lines, which gives a permutation of the six possible answers: \( d, \frac{1}{d}, 1-d, \frac{1}{1-d}, \frac{d}{d-1} \) and \( \frac{d-1}{d} \)).

(Compare also (2.12) example 1).

(4.8) Theorem: If \( r = 2 \) and we are in case \( g_4 = x_1^2(x_1^2 + x_2^2) \) of proposition (4.6) then:

either: \( f(x_1, \ldots, x_n) \sim x_1^4 + x_1^2x_2^2 + ax_2^2 + e_3x_3^2 + \ldots + e_nx_n^2 \)

or: \( \text{codim } (f) = \infty \)

If \( \text{codim } (f) < \infty \) then \( a \) is a local \( R \)-invariant; for \( RL \)-equivalence we can arrange that \( a = \pm 1 \).

Proof: \( x_1^4 + x_1^2x_2^2 \) has infinite codimension. The theorem is a consequence of the following Lemmas:

Lemma 1: (normalform): If \( g \sim x_1^4 + x_1^2x_2^2 \) then

\[ g \sim x_1^4 + x_1^2x_2^2 + \alpha_5x_5^2 + \ldots + \alpha_kx_k^2 \]

Proof: For \( k = 4 \) the statement is true; we proceed by introduction on \( k \). Let \( g \sim \lambda_1x_1^4 + \lambda_1x_1^2x_2^2 + \alpha_5x_5^2 + \ldots + \alpha_kx_k^2 + \tau_{k+1} \)

where

\[ \tau_{k+1} = \lambda_1x_1^{k+1} + \lambda_1x_1^kx_2^2 + \ldots + \lambda_1x_1^{k+1}. \]
Define an element of $L_2$ by:

$$
\begin{align*}
x_1 &:= x_1 + \sigma_1 \text{ with } \sigma_1 \not= -\frac{1}{4}\lambda_0 x_1^{k-2} \pm \frac{1}{2}\lambda_2 x_2^{k-2} \in m^{k-2} \\
x_2 &= x_2
\end{align*}
$$

So we have:

$$
\begin{align*}
g^{k+1} x_1^4 + 4x_1^3 \sigma_1 + x_1^2 \sigma_2^2 + 2x_1 x_2 \sigma_1 + \alpha x_2^5 + \ldots + \alpha_k x_2^k + \tau_{k+1}^+ & = \\
= x_1^4 + x_1^2 x_2 + \alpha x_2^5 + \ldots + \alpha_k x_2^k + x_1^3 \[4\sigma_1 + \lambda \sigma_2 x_2^{k-2} + \\
+ x_1 x_2^2 \[2\sigma_1 + \lambda \kappa_2 x_2^{k-2} + \lambda_1 x_1 x_2 + \ldots + \lambda_{k-1} x_1 x_2^{k-2} + \lambda_k x_2^k
\end{align*}
$$

Next define an element of $L_2$ by:

$$
\begin{align*}
x_1 &:= x_1 \\
x_2 &:= x_2 + \sigma_2 \text{ with } \sigma_2 \not= -\frac{1}{2} \mu_1 x_1^{k-2} + \ldots + \mu_k x_2^{k-2} \in m^{k-2}
\end{align*}
$$

So we have:

$$
\begin{align*}
g^{k+1} x_1^4 + x_1^2 x_2 + \alpha x_2^k + \\
+ x_1^2 x_2 + \alpha x_2^k + x_1 x_2^2 + \ldots + \alpha_k x_2^k + \lambda x_2^{k+1}
\end{align*}
$$

**Lemma 2:** $x_1^4 + x_1 x_2^2 + ax_2^k$ is $k$-determined if $a \not= 0$; $(k \geq 5)$.

**Proof:** A straightforward computation shows: $m^{k+1} \succeq m^2 \Delta(f) + m^{k+2}$ for all $a \not= 0$.

**Lemma 3:** $a$ is a local R-invariant of $x_1^4 + x_1^2 x_2^2 + ax_2^k$; for RL-equivalence we can arrange, that $a = \pm 1$.

**Proof:**

(i) $x_2^k \not\in m\Delta(f) + m^{k+1}$ for all $a \not= 0$; apply (2.9).

(ii) $x_2^k \in m\Delta(f) + f^*(m)$ for all $a \not= 0$; apply (2.3).

Now theorem (4.8) is proved.
Corank $= 3$

**Proposition (4.9):** If $r = 3$ then

$$f(x_1, \ldots, x_n) = g_3(x_1, x_2, x_3) + e_4 x_4^2 + \ldots + e_n x_n^2$$

where $g_3$ has one of the following expressions:

a) $g_3(x_1, x_2, x_3) = x_3 x_2^2 + x_1^3 + g_1 x_1 x_2 + g_2 x_3^2$ with $4g_1^3 + 27g_2^2 \neq 0$

b) $g_3(x_1, x_2, x_3) = x_1 x_3 x_2$ with codim $(f) \geq 10$

c) $g_3(x_1, x_2, x_3) = x_1 x_2^2 + x_3^2$ with codim $(f) \geq 10$

d) $g_3(x_1, x_2, x_3) = x_1 x_2 x_3$ with codim $(f) \geq 11$

e) $g_3(x_1, x_2, x_3) = x_2(x_1 x_2 - x_3^2)$ with codim $(f) \geq 10$

f) $g_3(x_1, x_2, x_3) = x_1 x_2 x_3$ with codim $(f) \geq 10$

g) $g_3(x_1, x_2, x_3) = x_1 x_2^2 + x_3^2$ with codim $(f) \geq 11$

h) $g_3(x_1, x_2, x_3) = x_1 x_2 x_3$ with codim $(f) \geq 15$

Proof:

Since $g_3 = 0$ is the equation of a cubic curve in the projective plane, we can use the projective classification of real cubic curves (cf. BURAU[8] or V.D. WAERDEN[26]).

In case a), b) and c) the curves are irreducible, in the other cases the curves are reducible. Case a) is the elliptic curve (= without multiple points). Case b) is a curve with cusp-point. Case c) is the curve with double point.

By linear transformation we can arrange that $g_4$ gets into one of the given expressions.

Next one constructs in the cases e-f a normalform of the 4-jet of f.

Then straightforward computations show the assertions concerning the codimension in e)-k).
(4.10) Theorem: If $r = 3$ and we are in case a) of proposition (4.9) then $f(x_1, \ldots, x_n) = x_3^2 x_2^3 + x_1^3 + g_1 x_1 x_3^2 + g_2 x_3^3 + e_1 x_4^2 + \ldots + e_n x_n^2$ (P_8), with $4g_1^3 + 27g_2^2 \neq 0$ and codim $(f) = 7$.

The number $j = \frac{4g_1^3}{4g_1^3 + 27g_2^2}$ is an invariant of $f$.

Two elements of this family are equivalent iff their $g_2$'s have the same sign and their $j$'s are equal. If $g_2 = 0$ two elements are equivalent iff their $g_1$'s have the same sign.

Proof: A straight-forward computation shows that $f$ is 3-determined. On the homogeneous polynomials of degree 3 in $x_1, x_2, x_3$ the action of $L_3$ coincides with $\text{GL}(3)$. So $j$ is the classical $j$-invariant of elliptic curves. In the complex case $j \in \mathbb{C}$ classifies the elliptic curves completely. In the real case we have for every $j \in \mathbb{R}$ two different real elliptic curves. Moreover there are 2 different topological types: unipartite with $j \in (-\infty, 1]$ and bipartite with $j \in [1, \infty)$.

![Diagram](image-url)
A parametrization of the family can be given by a circular curve $C$ around the origin, for example

$$C: 4|g_1|^3 + 27|g_2|^2 = 1$$

The two intersection points of $C$ with $4g_1^3 + 27g_2^2 = 0$ correspond to the two types of curves with double point. The other points of the curve are in 1-1-correspondence to the equivalence classes of real elliptic curves.

**4.11 Theorem:** If $r = 3$ and we are in case a) of proposition (4.9) then either: $f(x_1, \ldots, x_n) = x_1^3 + x_2^2x_3 + x_1^2x_3 + \beta_k x_3^k + e_4x_4^2 + \ldots + e_nx_n^2(P_k + \gamma)$

(with $\beta_k \neq 0$ and $k \geq 4$)

or: $\text{codim}(f) = \infty$.

If $\text{codim}(f) < \infty$ then each $\beta_k$ is invariant under $R$-equivalence only; for $RL$-equivalence we can arrange $\beta_k = \pm 1$.

**Proof:** $g_3 = x_1^3 + x_2^2x_3 + x_1^2x_3$ has infinite codimension. The theorem is a consequence of the following Lemmas:

**Lemma 1:** Let $\tau_j$ be a homogeneous polynomial of degree $j$; then

$$x_1^3 + x_2^2x_3 + x_1^2x_3 + \tau_j x_1^3 + x_2^2x_3 + x_1^2x_3 + \lambda x_3^j.$$

**Proof:**

Let an element of $L_3$ be defined by $x_i = x_i + \sigma_i$ with $\sigma_i \in m^{j-2}$ then

$$g_3 = x_1^3 + x_2^2x_3 + x_1^2x_3 + \tau_j g_3 + \sigma_1 \delta_1(g_3) + \sigma_2 \delta_2(g_3) + \sigma_3 \delta_3(g_3) + \tau_j.$$

A direct computation shows that:

$$x_1^m x_2 = m^\lambda g_3$$

so also

$$x_1^m x_2^m \subseteq m^\lambda g_3.$$

This means that we can choose $\sigma_1, \sigma_2$ and $\sigma_3$ in such a way in $m^{j-1}$ that the terms of $\tau_j(x_1, x_2, x_3)$, that are divisible by $x_1$ or $x_2$ vanish.
against $c_1\partial_1(g_3) + c_2\partial_2(g_3) + c_3\partial_3(g_3)$. So $g_3 + \gamma_j \partial_j g_3 + \lambda x_3^j$.

**Corollary:** We have the following normalform for the $k$-jet of $f(k \geq 4)$:

$$g \sim x_1^3 + x_2^2x_3 + x_1^2x_3 + \lambda x_3^k \quad (\lambda \neq 0)$$

**Lemma 2:** $g = x_1^3 + x_2^2x_3 + x_1^2x_3 + \lambda x_3^k$ is $k$-determined $(k \geq 4)$.

**Proof:**

\[
\begin{align*}
\partial_1 g &= 3x_1^2 + 2x_1x_2 \\
\partial_2 g &= 2x_2x_3 \\
\partial_3 g &= x_2^2 + x_1^2 + k\lambda x_3^{k-1}
\end{align*}
\]

We shall show: $m^{k+1} \leq m^2\Delta(g) + m^{k+2}$.

Since $m^3\Delta(g) + m^{k+2} = m^2\Delta(g_3) + m^{k+2}$, we find already all generators of $m^5$, except $x_3^5$. So we have only to show, that $x_3^{k+1} \in m\Delta + m^{k+2}$.

$$x_3 \partial_3 g = x_3^2x_2^2 + x_1^2x_3^2 + k\lambda x_3^{k+1}$$

So $k\lambda x_3^{k+1} \equiv x_1^2x_3^2 \equiv x_3^2x_1x_3^2 \equiv 0 \pmod{m^2\Delta + m^{k+2}}$. Since $\lambda \neq 0$ we have $x_3^{k+1} \in m^2\Delta + m^{k+2}$.

**Lemma 3:** In $x_1^3 + x_2^2x_3 + x_1^2x_3 + \lambda x_3^k \quad (\lambda \neq 0)$ is a local invariant under $R$-equivalence.

**Proof:**

Since $\dim \frac{m}{m\Delta + m^{k+1}} > \dim \frac{m}{m\Delta + m^{k+1} + x_3^k}$ we have $x_3^k \not\in m\Delta + m^{k+1}$ and this implies the lemma.

**Lemma 4:** $x_1^3 + x_2^2x_3 + x_1^2x_3 + \lambda x_3^k \sim RL x_1^3 + x_2^2x_3 + x_1^2x_3 + x_3^k$

**Proof:**

$x_3^k \in m\Delta + f^*(m_1) + m^{k+1}$ for all $\lambda \neq 0$; so $x_3^k$ is contained in the tangentspace to the $RL$-orbit; so (2.3) applies and we are done.

It is possible to give explicit formulas for the diffeomorphisms:
Let $x_i = px_i$ then
\[ x_1^3 + x_2 x_3 + x_1^2 x_3 + \lambda x_3 k \] \[ R \] \[ x_1^3 p^3 x_1 + p^3 x_2 x_3 + p^3 x_1 x_3 + \lambda p^k x_3 k \] \[ RL x_1^3 + x_2 x_3 + x_1^2 x_3 + \lambda p^{k-3} x_3 k = x_1^3 + x_2 x_3 + x_1^2 x_3 + x_3 k \]
if $p = \sqrt[k]{\frac{1}{\lambda}}$.

(4.12) Theorem: If $r = 3$ and we are in case b) of proposition (4.9) then either:
\[ f(x_1, \ldots, x_n) \sim x_1 x_3 + x_2^2 + x_4 + \ldots + e_4 x_4^2 + \ldots + e_n x_n^2 (Q_{10}) \]
with codim $(f) = 9$

or: codim $(f) > 9$

If codim $(f) = 9$ then a is local-invariant under $R$-equivalence only; for RL-equivalence we can arrange that $a = -1, 0$ or 1.

Proof:
The proof is a consequence of the following lemma 1-2.

Lemma 1: If $g \sim x_1 x_3^2 + x_2^3$ we have the following normal form for the n-jet: $g \sim x_1 x_3^2 + x_2^3 + \sigma_4 + \ldots + \sigma_n$ (n \geq 4) where
\[ \sigma_p = \alpha x_3^{p-1} x_2 + \beta x_3^p. \]

Proof:
By introduction on $n$; for $n = 3$ is the statement true.

Let $\tau_n$ be a homogeneous polynomial of degree $n$ and let $g \sim x_1 x_3^2 + x_2^3 + \sigma_4 + \ldots + \sigma_{n-1} + \tau_n$.

Define an element of $L_3$ by
\[ \begin{cases} x_1' = x_1 + \sigma_1 \text{ with } \sigma_1 \in \text{m}^{n-2} \\ x_2' = x_2 + \sigma_2 \text{ with } \sigma_2 \in \text{m}^{n-2} \\ x_3' = x_3 + \sigma_3 \text{ with } \sigma_3 \in \text{m}^{n-2} \end{cases} \]
Then: 

\[ g \left( x_1 x_3^2 + x_2 x_3^3 + x_3^2 \sigma_1 + 2x_1 x_3 \sigma_3 + 3x_2^2 \sigma_2 + \sigma_4 + \ldots + \sigma_{n-1} + \tau_n \right) = \]

\[ = x_1 x_3^2 + x_2 x_3^3 + \sigma_4 + \ldots + \sigma_{n-1} + \] 
\[ + [2x_1 x_3 \sigma_3 + x_2^2 \sigma_2 + 3x_2^2 \sigma_2 + \tau_n] \]

\[ = x_1 x_3^2 + x_2 x_3^3 + \sigma_4 + \ldots + \sigma_{n-1} + \sigma_n \]

by a proper choice of \( \sigma_1, \sigma_2 \) and \( \sigma_3 \).

**Lemma 2:** \( x_1 x_3^2 + x_2 x_3^3 + A x_1^3 x_2 + B x_1^4 \) is \( h \)-determined for all \( B \neq 0 \).

**Proof:** cf (2.5) example 3.

Now we start the classification in case 3b):

**Lemma 1** implies, that

\[ g \left( x_1 x_3^2 + x_2 x_3^3 + a_4 x_1^3 x_2 + \beta_4 x_1^4 \right) \]

If \( \beta_4 \neq 0 \) \( g \) is \( h \)-determined (Lemma 2) and we can arrange that:

\[ g \left( x_1 x_3^2 + x_2 x_3^3 + A x_1^3 x_2 + x_1^4 \right) \]

since \( x_1 x_3^2 \notin m \Delta + m^5 \) for all \( A \); \( A \) is a local \( R \)-invariant. Moreover

A is not a local \( RL \)-invariant, since

\[ x_1 x_2^3 \in m \Delta + f^*(m_1) + m^5 \] for all \( A \neq 0 \).

With \( RL \)-action we can arrange that \( a = 0, +1 \) or \(-1 \).

Next we have to consider the cases \( \beta_4 = 0 \) and \( a_4 \neq 0 \); but this gives already \( \text{codim } (g) > 9 \). A classification in higher codimension is possible, using Lemma 1, but becomes more and more complicated.

**Theorem (4.13):** If \( r = 3 \) and we are in case d) of proposition (4.9) then \( \text{codim } (f) \geq 9 \). If \( \text{codim } (f) = 9 \) then

\[ f(x_1, \ldots, x_n) \leftarrow g(x_1, x_2, x_3) + e_4 x_4^2 + \ldots + e_n x_n^2 \] with

\[ g = x_1 x_3^2 + e_2 x_1 x_2 + e_3 x_1 x_3^2 + A[x_2^4 - 6e_2 e_3 x_2 x_3^2 + x_3^4](A \neq 0) \]

or

\[ g = x_1 x_3^2 + e_2 x_1 x_2 + e_3 x_1 x_3^2 + B[4x_2^3 x_3 - 4e_2 e_3 x_2 x_3^3] \] \( (B \neq 0) \)

\( A \) and \( B \) are local \( R \)-invariants.
Proof: $g_3 = x_1^3 + e_2 x_1 x_2^2 + e_3 x_3^2$ has infinite codimension. The
lemma is a consequence of the following lemmas:

We use the abbreviations:

$$p(x_2, x_3) = x_2^4 - 6e_2 x_2 x_3^2 + x_3^4$$
$$q(x_2, x_3) = 4x_2^3 x_3 - 4e_2 x_2 x_3^3$$

**Lemma 1:** $g \sim g_3 + Ap(x_2, x_3) + Bq(x_2, x_3)$.

**Proof:** Define an element of $L_3$ by $x_i = x_i + \sigma_i$ with $\sigma_i \in m^2$

(i = 1, 2, 3). Let $g_i = g_3 + \tau_i$; where $\tau_i$ is a homogeneous polynomial

of degree 4. Then $g \sim g_3 + \sigma_1^i g + \sigma_2^i g + \sigma_3^i g + \tau_i$.

A straight-forward computation of $m^2 \Delta + m^6$ shows, that we can choose

$\sigma_1, \sigma_2$ and $\sigma_3$ in such a way that $g \sim g_3 + Ap(x_2, x_3) + Bq(x_2, x_3)$.

**Lemma 2:** If $e_2 A^2 + e_3 B^2 \neq 0$ then $g_3 + Ap(x_2, x_3) + Bq(x_2, x_3)$ is

4-determined and codim $(f) = 9$. If $e_2 A^2 + e_3 B^2 = 0$ then codim $(f) > 9$.

**Proof:**

$m^5 \subset m^2 \Delta + m^6$ for all values of $A$ and $B$ with $e_2 A^2 + e_3 B^2 \neq 0$.

**Lemma 3:** If $e_2 = e_3$ then $g \sim g_3 + A' p(x_2, x_3)$ (A' \# 0)

and $g \sim g_3 + B' q(x_2, x_3)$ (B' \# 0)

**Proof:**

The substitution

$$x_2 = x_2 \cos \phi + x_3 \sin \phi$$

$$x_3 = -x_2 \sin \phi + x_3 \cos \phi$$

implies that

$g \sim x_1^3 + e_2 x_1 x_2^2 + e_3 x_3^2 + [A \cos \phi - B \sin \phi] p(x_2, x_3) +$

$+ [A \sin \phi - B \cos \phi] q(x_2, x_3)$.

If $\phi = \arctg \frac{A}{B}$ then the coefficient of $p(x_2, x_3)$ vanishes.

If $\phi = \arctg \frac{B}{A}$ then the coefficient of $q(x_2, x_3)$ vanishes.
We remark that orbits intersect A-B-plane in circles:

![Diagram showing orbits intersecting in circles and hyperbolas.](image)

**Lemma 4:** If \( e_2 \neq e_3 \) then \( g \prec g_3 + A'p(x_2, x_3) \) if \( \frac{A-B}{A+B} < 0 \)

and \( g \prec g_3 + B'q(x_2, x_3) \) if \( \frac{A-B}{A+B} > 0 \)

**Proof:**

The substitution

\[
\begin{aligned}
x_2 &= \frac{1}{2}(\lambda + \frac{1}{\lambda})x_2 + \frac{1}{2}(\lambda - \frac{1}{\lambda})x_3 \\
x_3 &= \frac{1}{2}(\lambda - \frac{1}{\lambda})x_2 + \frac{1}{2}(\lambda + \frac{1}{\lambda})x_3
\end{aligned}
\]

implies that \( g \prec x_1^3 + e_2 x_1 x_2 - e_2 x_1 x_3^2 + \frac{1}{2}[(\lambda^4 + \frac{1}{\lambda^4})A + (\lambda^4 - \frac{1}{\lambda^4})B]p(x_2, x_3) + \frac{1}{2}[(\lambda^4 - \frac{1}{\lambda^4})A + (\lambda^4 + \frac{1}{\lambda^4})B]q(x_2, x_3) \)

a) Let \( (\lambda^4 + \frac{1}{\lambda^4})A + (\lambda^4 - \frac{1}{\lambda^4})B = 0 \) then \( \lambda^8 A + A + \lambda^8 B - B = 0 \)

\( \lambda^8 (A + B) = B - A \) and \( \lambda^8 = \frac{B-A}{A+B} \)

If \( \frac{A-B}{A+B} < 0 \) there are real solutions, so the coefficients of \( p(x_2, x_3) \) can vanish.

b) Let \( (\lambda^4 - \frac{1}{\lambda^4})A + (\lambda^4 + \frac{1}{\lambda^4})B = 0 \)

\( \lambda^8 A - A + \lambda^8 B + B = 0 \)

\( \lambda^8 (A+B) = A - B \) and \( \lambda^8 = \frac{A-B}{A+B} \)

If \( \frac{A-B}{A+B} > 0 \) there are real solutions, so the coefficient of \( q(x_2, x_3) \) can vanish.

We remark, that orbits intersects the A-B-plane in hyperbolas:
Remark:
In the case $e_2 \neq e_3$ we can also use the normalform $x_1^3 + x_1x_2x_3$ for the 3-jet of $g$. This form is easier for the computations.

If $\text{codim}(f) = 9$ then one can show:

$$g \sim x_1^3 + x_1x_2x_3 + Cx_2^4 + Dx_3^4$$

with $C, D \neq 0$

which can be transformed such that $C' = 1$ and $D' \neq 0$ or such that $C' \neq 0$ and $D = 1$. 

\[ x_3 \]
§5 Remarks

A. A conjecture of Zeeman and strong equivalence.

(5.1) In (2.1) I already mentioned, that the theorems (1.7) and (1.8) don't determine the degree of determinacy completely. Zeeman conjectured in a lecture at the IHES (Bures-sur-Yvette) that:

\[ f \text{ is } k\text{-determined} \iff m_n^{k+1} \subseteq \phi_n^{2D(f)} + m_n^{k+2}. \]

In the following example I show that this conjecture is not true.

(5.2) Counterexample: \( f = x_1^3 + x_1^2 x_2^3 \) (\( E_7 \)).

This example is also treated in (1.11).

\( f \) has the following properties:

1° \( m_2^5 \subseteq \phi_2^{2D(f)} \)

2° \( m_2^5 \nsubseteq \phi_2^{2D(f)} \)

3° \( f \) is 4-determined.

In (1.11) I showed 1° and in (4.5) lemma 3 I showed 3°. Since \( \phi_4 f = 3x_1^2 + x_2^3 \) and \( \phi_2 f = 3x_1^2 x_2^2 \) it is impossible that \( x_2^5 \in \phi_2^{2D(f)} \).

So \( m_2^5 \nsubseteq \phi_2^{2D(f)} \).

(5.3) Although the conjecture is not true, the algebraic condition

\[ m_n^{k+1} \subseteq \phi_n^{2D(f)} + m_n^{k+2} \]

of theorem (1.7) can still be translated in terms of determinacy. This is the reason for the following two definitions.
(5.4) Definition: Two germs \( f \) and \( g \) are called \textit{strong-(right)-equivalent} if there is a \( \phi \in L_n \) such that \( g = f \phi \) and the derivative \( d\phi(0) \) is the identity. Notation \( g \overset{R}{=} f \) or \( g \sim f \).

The germs \( \phi \in L_n \) with \( d\phi(0) = 1 \) form a subgroup of \( L_n \), which acts on \( \mathbb{A}_n \) and induces an algebraic action on \( \mathcal{J}^k(n,1) \).

(5.5) Definition: A germ \( f \in \mathbb{A}(n,1) \) is called \textit{strong-\( k \)-determined} if for all \( g \in \mathbb{A}_n \) with \( g_k = f_k \) we have \( g \) is strong-equivalent with \( f \).

(5.6) Theorem: \( m_n^{k+1} \subseteq m_n^{2\Delta(f)} + m_n^{k+2} \Rightarrow f \) \textit{is strong-\( k \)-determined}.

Proof:

1°) The part \( \Rightarrow \) of the proof is similar to that of theorem (1.7) and follows from two lemma's:

Lemma 1: Let \( m_n^{k+1} \subseteq m_n^{2\Delta(f)} + m_n^{k+2} \). Then there exists for all \( t \in \mathbb{R} \) a map germ \( \tilde{\xi}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \) defined in a neighborhood \( U \) of \( (o,t_0) \in \mathbb{R}^{n+1} \) which satisfies:

\begin{enumerate}
  \item \( \tilde{\xi}(o,t) = 0 \) for all \( (o,t) \in U \)
  \item \( (\partial_t \tilde{\xi})(o,t) = 0 \) for all \( (o,t) \in U \)
  \item \( \forall (x,t), \tilde{\xi}(x,t) + g(x) - f(x) = 0 \) for all \( (x,t) \in U \).
\end{enumerate}

Proof: to satisfy (i), (ii) and (iii) we need only to show that

\[ m_n^{k+1} \subseteq \Delta^*(\mathcal{F})m_n^{2} \]

and this follows direct from the proof of (1.7) lemma 1.

Lemma 2: For each \( t_0 \in \mathbb{R} \) there is \( \varepsilon > 0 \) such that \( \mathbb{F} \overset{\varepsilon}{\rightarrow} \mathbb{F} \) for all \( t \) with \( |t-t_0| < \varepsilon \).

Proof: We consider as in (1.7) lemma 2 the differential equation

\[ \frac{\partial h}{\partial t}(x,t) = \tilde{\xi}(h(x,t),t) \]

with the initial condition:

\[ h(x,t_0) = x \]

Since \( \tilde{\xi}_j \in m_n^{2} \) we have \( \frac{\partial^j}{\partial x_i}(\frac{\partial h}{\partial t}) = 0 \) for all \( t \) and \( j = 1,\ldots,n \).
So \( \frac{\partial}{\partial t} \left( \frac{\partial h_t}{\partial x_i} \right) = 0 \), hence \( \frac{\partial h_t}{\partial x_i} \) is constant and \( dh_t \) is constant.

So: \( dh_t = dh_{t_0} = 1 \)

which proves that \( h_t \) is a strong-equivalence.

The rest of the proof is the same as in (1.7) lemma 2.

2°) The part \( = \) of the proof is similar to that of theorem (1.8).

Replace in the proof of theorem (1.8) the set \( V \) by \( W = \{ g \in \mathbb{E}_n ; g \not= f \} \).

In order to prove the theorem is sufficient to show that:

b) \( \tau(W_{k+1}) \equiv m_n^2 \Delta(f) \pmod{m_n^{k+2}} \).

Let \( h_t : (\mathbb{R}^n, \mathcal{O}) \rightarrow (\mathbb{R}^n, \mathcal{O}) \) a germ of diffeomorphism with \( h_0 = 1 \) and \( dh_t(\mathcal{O}) = 1 \). Then we have:

\[
\frac{d}{dt} (fh_t) \bigg|_{t=0} = \nabla f \cdot \frac{dh_t}{dt} \bigg|_{t=0} \in \mathbb{E}_n (\partial_1 f, \ldots, \partial_n f)
\]

Let \( \xi = \frac{\partial h_t}{\partial t} \) then \( \xi(\mathcal{O}) = \frac{\partial h_t(\mathcal{O})}{\partial t} = 0 \) since \( h_t(\mathcal{O}) = \mathcal{O} \) and

\[
\frac{\partial \xi_i}{\partial x_j}(\mathcal{O}) = \frac{\partial^2 h_t}{\partial x_i \partial t}(\mathcal{O}) = \frac{\partial}{\partial t} \frac{\partial h_t}{\partial x_j} = \frac{\partial}{\partial t} \delta_{ij} = 0 \quad \text{so} \quad \xi_i \in m_n^2 \quad (i = 1, \ldots, n).
\]

This means \( \frac{d}{dt} (fh_t) \bigg|_{t=0} \in m_n^2 \Delta(f) \) which proves \( \tau(W_{k+1}) \subseteq m_n^2 \Delta(f) \pmod{m_n^{k+2}} \).

Moreover let \( a \in m_n^2 \Delta(f) ; a(x) = \nabla f(x) \cdot \xi(x) \) with \( \xi_i \in m_n^2 \), and

\( h_t(x) = x + t \xi(x) \); then \( h_t \in L_n \) and \( dh_t(\mathcal{O}) = 1 \) and \( \frac{d}{dt} fh_t \bigg|_{t=0} = a \)

which proves \( \tau(W_{k+1}) \subseteq m_n^2 \Delta(f) \pmod{m_n^{k+2}} \).

(5.7) Example: \( f_\lambda(x_1, x_2) = x_1^3 + x_1^2 x_2^3 + \lambda x_2^5 \) has the property \( m_n^5 \subseteq m_n^5 + m_n^6 \) but it is not true that \( m_n^5 \subseteq m_n^2 \Delta + m_n^6 \).

We found already: \( f_\lambda(x_1, x_2) \) is (ordinary) \( 4 \)-determined. Since \( m_n^6 \subseteq m_n^2 \Delta + m_n^7 \), \( f_\lambda \) is strong-\( 5 \)-determined for every \( \lambda \in \mathbb{R} \) (but not strong-\( 4 \)-determined!).

Since \( x_2^5 \not\in m_n^2 \Delta(f) + m_n^6 \) for all \( \lambda \in \mathbb{R} \) we can conclude that \( \lambda \) is a local invariant under strong-equivalence. [using an extended version of theorem (2.9)].
(5.8) Remark: The classification of germs under strong-equivalence is not so interesting since there are already a lot of invariants in the case of a non-degenerate critical point. Although an arbitrary $f$ with $\text{Rank } (d^2 f)$ is maximal is strong 2-determined, it is not possible to bring the germ to the normal form $e_1 x_1^2 + \ldots + e_n x_n^2$ with $e_i = \pm 1$.

B. Left-multiplication

(5.9) In the list of singularities with codimension $\leq 9$ are some families with one local $R$-invariant, which is not a local $RL$-invariant:

$$X_{10} : x_1^4 + x_1^2 x_2^2 + \alpha x_2^5 \quad \alpha \neq 0$$

$$P_9 : x_1^3 + \frac{2}{\alpha} x_2 x_3 \frac{2}{\alpha} x_1 x_3 + \beta x_3^4 \quad \beta \neq 0$$

$$P_{10} : x_1^3 + \frac{2}{\alpha} x_2 x_3 \frac{2}{\alpha} x_1 x_3 + \beta x_3^5 \quad \beta \neq 0$$

$$R_{10} : x_1^3 + x_1 x_2 x_3 + \alpha x_2 \alpha x_3 \alpha x_1 \alpha x_4 \beta x_3^4 \quad \beta \neq 0$$

$$Q_{10} : x_1 x_3^2 + x_2^3 + x_3 x_2 x_1^4 \quad \alpha \neq 0$$

In most of the cases we used theorem (2.3) for the proof of the $RL$-equivalence of the germs in the family. Sometimes it is also possible to see this in the following way. We treat as example $X_{10}$:

$$g = x_1^4 + x_1^2 x_2^2 + \alpha x_2^5 \quad \alpha \neq 0$$

$$\alpha^4 g = \alpha^4 x_1^4 + \alpha^4 x_1^2 x_2^2 + \alpha^5 x_2^5$$

$$\alpha^4 g = (\alpha x_1)^4 + (\alpha x_2)^2 + (\alpha x_2)^5.$$ 

So

$$g \overset{RL}{\sim} \alpha^4 g \overset{R}{\sim} x_1^4 + x_1^2 x_2^2 + x_2^5$$

The only left-action we used in this computation is scalar multiplication with $\alpha^4$ and this is an element of $\text{GL}(1)$.

It is possible to treat the other cases of the list in the same way.

This raises a more general question: In which cases does the action of $L_n \times \text{GL}(1)$ coincide with the action of $L_n \times L_1$?
(5.10) Theorem: Let $n = 2$ and let $f_t = f + t\psi$.

If $f_t \overset{RL}{\sim} f_t^0$ for all $t$ in a connected interval $I$ of $\mathbb{R}$, then $f_t$ and $f_t^0$ are also equivalent under the action of $L_2 \times GL(1)$ for all $t \in I$.

Proof:

Since $f_t = f + t\psi$ for $t \in I$ are all in the same $RL$-orbit, $\psi$ has to lie in the tangentspace to the $RL$-orbit; so:

$$\phi \in m_2^2 A(f_t) + f_t^*(m_1) \quad \forall t \in I.$$  

This implies $\phi = \sigma_1 \delta_1 f_t + \sigma_2 \delta_2 f_t^j + f_t^j \alpha_j(t)$ with $\sigma_1, \sigma_2 \in m_2$.

BRIANCON ([5] and [6]) proved that:

$$f_t^2 \in g_2^2(x_1 \frac{\partial f_t}{\partial x_1}, x_2 \frac{\partial f_t}{\partial x_2}) \subset m_2^2 A(f_t).$$

So we can find $r_1, r_2 \in m_2$ such that

$$\phi = r_1 \delta_1 f_t + \sigma_2 \delta_2 f_t + \alpha(t) f_t \quad (*)$$

We now return to the situation described in theorem (2.3), where 1-parameter families of diffeomorphisms of sourcespace and targetspace were constructed.

Our equation (*) implies that the diffeomorphism $k$ of the targetspace has to satisfy the differential equation:

$$\begin{cases}
\frac{\partial k(y,t)}{\partial t} = -\alpha(t).k(y,t) \\
k(y,t_o) = x
\end{cases}$$

The solution goes as follows:

$$\frac{\partial k(y,t)}{k(y,t)} = -\alpha(t) dt$$

$$\ln k(y,t) = \delta(t) + C(y)$$

$$k(y,t) = e^{\delta(t) + C(y)} = e^{C(y)} \cdot y(t)$$

The initial condition gives: $y = k(y,t_o) = e^{C(y)} \cdot y(t_o)$

so:

$$k(y,t) = y \cdot \frac{x(t)}{y(t_o)}.$$  

So $k$ is a scalarmultiplication, so we are done.
(5.11) Remark: The proof of theorem (5.10) shows that the condition $f^2_t \in m\Delta$ is enough to get the result.

A similar theorem for $n \geq 3$ doesn't exist, since BRIANCON gives an example with $f^{n-1} \not\in \mathfrak{m} \langle x_1^3 f, \ldots, x_n^3 f \rangle$, namely

$$f = (x_1, x_2, x_3)^3 + [x_1^{3n-1} + x_2^{3n-1} + \cdots + x_n^{3n-1}].$$

In the cases $P_9$, $P_{10}$, $R_{10}$ and $Q_{10}$ we proceed as follows. Since those germs are 4-determined or 5-determined; we have $m^5 \subseteq \mathfrak{m} \Delta$.

Because $f^2 = 0$ we have $f^2 \in \mathfrak{m}^6$ and this implies $f^2 \in m\Delta$. So we can apply the proof of theorem (5.10), which shows that the RL-equivalence of the family can already be done by $L_n \times \text{GL}(1)$-action.
PART II: DEFORMATION OF SINGULARITIES AND ADJACENCY

Introduction:
After the classification problem in part I, I treat in part II the adjacency problem and study approximations of a function germ in its universal deformation.
In §6 and §7 I introduce some known topological invariants of a singularity, namely Milnor number, the intersection form and the monodromy group, and investigate the relation between the invariants of the germ and its approximations.
In §8 I use this in order to explain and prove in a new way some results on adjacency which were partly known already.
In §9 I treat the new notion of μ-adjacency, which describes a relation between families of germs with constant Milnor number.
In §10 I introduce a topology in the orbit space and study it for the set of germs with Milnor number ≤ 10. In their relative topology we get copies of $\mathbb{C}$ or $\mathbb{C} - \{0\}$ for the 1-parameter families in the orbit space. We illustrate the relations adjacency and μ-adjacency in the list III at the end.
This research while in progress was in a later stage to a large extent covered and then influenced by published and unpublished work of Arnold, Lamotke, Saito and Gabrielov. We indicate those references, but we believe that our presentation and survey and some of the proofs still have an independent interest.
Milnor fibration and vanishing cycles.

(6.1) We consider a holomorphic mapping \( f : (U, \emptyset) \to (\mathbb{C}, 0) \), where \( U \) is an open subset of \( \mathbb{C}^{n+1} \) and \( \emptyset \) is the only critical point of \( f \). This situation is studied by Milnor [20] and others.

There exist \( \epsilon > 0 \) and \( \delta > 0 \) such that \( S_\epsilon \) is transversal to \( f^{-1}(t) \) for all \( |t| \leq \delta \). Notation: \( S_\epsilon \cap f^{-1}(t) \) for all \( |t| \leq \delta \).

We write \( B = B_\epsilon^{2n+1} \) and \( D = D_\delta^{2n} \) and define:

\[ E_\delta = f^{-1} (3D) \cap B \]

\( E_\delta \) is a compact oriented manifold with boundary and \( f : E_\delta \to 3D \) is the projection of a fibrewise with typical fibre \( X_\delta = f^{-1}(\delta) \cap B \). (see also (6.2)). It is well-known that \( X_\delta \) has the homotopytype of \( S^n \vee \ldots \vee S^n \); hence \( H_n(X_\delta) \cong \mathbb{Z}^{u(f)} \) for some \( u(f) \in \mathbb{N} \). \( u(f) \) is called Milnor’s number. The intersection form \( \langle - , - \rangle \) on \( H_n(X_\delta) \) is a bilinear form, which is symmetric if \( n \) is even and antisymmetric if \( n \) is odd.

(6.2) Lemma: As before let \( U \subset \mathbb{C}^{n+1} \) and let \( g : (U, \emptyset) \to (\mathbb{C}, 0) \) be a holomorphic mapping such that:

a) \( g \) has no critical points on \( 3B \)

b) \( g \) has no critical values on \( 3D \)

c) \( g_\epsilon \cap g^{-1}(t) \) for all \( t \in D \)

and let \( E \) be the set of critical values of \( g \),

then:
1° The map \( g : g^{-1}(D \setminus \Sigma) \cap \partial D \setminus \Sigma \) is the projection of a (locally trivial) fibrebundle.

2° The map \( g : g^{-1}(D) \cap \partial B \to D \) is the projection of a trivial fibrebundle.

3° Every path \( v : [0,1] \to D - \Sigma \), connecting points \( a \) and \( b \), induces for any connection in the bundle \( g \) a diffeomorphism \( v_* : f^{-1}(a) \cap B \to f^{-1}(b) \cap B \). The isotopy class of this diffeomorphism is unique, that is independent of the connection.

4° This connection can be chosen such that for every closed path the restriction \( v_* : f^{-1}(a) \cap \partial B \to f^{-1}(a) \cap \partial B \) is the identity.

Proof: We use Ehresmann's fibration theorem [11]:

Let \( E \) and \( B \) be smooth manifolds, \( B \) connected and \( p : E \to B \) a smooth surjective mapping, with the property that for all \( x \in B \) the rank of the differential of \( p \) in \( x \) equals the dimension of \( B \) and \( p^{-1}(x) \) is compact and connected. Then \( p : E \to B \) is a smooth fibrebundle and so all fibres \( p^{-1}(x) \) are diffeomorphic.

Our \( g \) has maximal rank on \( g^{-1}(D \setminus \Sigma) \cap \partial B \) and by the transversality-condition also on \( g^{-1}(D) \cap \partial B \); so we can apply this theorem to obtain 1° and 2°.

As moreover every fibrebundle over a contractible space, like a disc is trivial, we have 2°.

Using a suitable connection, we find the required diffeomorphism \( v_* \) and as we have a product structure on the boundary \( f^{-1}(D) \cap \partial B \), we can arrange that \( v_* \) is the identity on the boundary of the fibre \( f^{-1}(a) \cap B \).

Remark: From the above lemma it follows also that \( E_f + \partial D \) is a fibrebundleprojection.

(6.3) Definitions: In the following we shall use deformations and approximations of \( f \). A deformation of \( f \) is a holomorphic mapping \( F : U \times W \to \mathbb{C} \) with \( \partial \in U \subset \mathbb{C}^{n+1} \) and \( \partial \in W \in \mathbb{C}^k \) and the property \( F(x, \partial) = f(x) \).
A deformation $F$ of $f$ is called \textit{infinitesimally versal} if

$$\mathcal{A}_{n+1} = (\partial_0 f, \ldots , \partial_n f) + \mathcal{C}[\phi_1, \ldots , \phi_k].$$

Here $\mathcal{A}_{n+1}$ denotes the ring of germs at $0 \in \mathbb{C}^{n+1}$ of holomorphic functions from $\mathbb{C}^{n+1}$ to $\mathbb{C}$; $(\partial_0 f, \ldots , \partial_n f)$ the ideal in $\mathcal{A}_{n+1}$, spanned by the partial derivatives of $f$ and $\mathcal{C}[\phi_1, \ldots , \phi_k]$ the $\mathbb{C}$-vectorspace, spanned by $\phi_1, \ldots , \phi_k$, where $\phi_i$ is defined by $\phi_i = (\frac{\partial f}{\partial y_i})_{y=0}$ $(i=1, \ldots , k)$ and $y_1, \ldots , y_k$ are coordinates in $\mathbb{C}^k$.

If $f$ has an isolated critical point in $0$ then infinitesimal deformations exist and can be written in the form $F(x, w) = f(x) + \sum_{i=1}^{k} v_i \phi_i$.

A deformation $F : U \times W \to \mathbb{C}$ of $f$ is called \textit{versal} if for any other deformation $G : U \times W' \to \mathbb{C}$ there exist analytic maps:

$$\phi : U \times W' \to U \text{ with } \phi(x, 0) = x$$
$$\psi : W' \to W \text{ with } \phi(0) = 0$$

such that $G(x, u) = F(\phi(x, u), \psi(u))$.

An important theorem of MATHER [19] says that the properties versal (for $W$ small enough) and inf. versal are equivalent.

(6.4) Let $F : U \times W \to \mathbb{C}$ be a deformation of $f$. For $w \in W$ the mapping $F_w : U \to \mathbb{C}$, defined by $F_w(x) = F(x, w)$ is called \textit{approximation} of $f$.

As in (6.1) we can consider the corresponding fibrebundle projection:

$$F_w : E_w = F_w^{-1}(\partial D) \cap B \to \partial D.$$

(6.5) \textbf{Lemma:} There exists $\eta > 0$ so that we have for $\|w\| < \eta$:

a) all critical points of $F_w$ are inside $B$.

b) all critical values of $F_w$ are inside $D$.

c) $\notin \overline{F_w^{-1}(t)}$ for all $t \in D$.

d) the fibrations $E_w \to \partial D$ and $E_f \to \partial D$ are diffeomorphic.

\textbf{Proof:} If we use the continuity of $f$ and $\nabla f$, transversality arguments and an extended version of Ehresmann's fibration theorem, we can in each of the cases a)-d) define an open neighborhood in which the assertion is fulfilled.
(6.6) Lemma: Let $F$ be a versal deformation of $f$. There exist $w \in W$ arbitrarily close to any $w \in W$ such that:

e) all critical points of $F_w$ are non-degenerate.
f) all critical values of $F_w$ are different.

Proof: The points $w \in W$ such that $F_w$ has not $\mu(f)$ (= Milnor's number) distinct critical values from an algebraic variety, the so-called bifurcation variety $\text{Bif}(f)$ (cf. LOOIJENGA [18]).

If $F_w$ has $\mu(f)$ distinct critical values, then all its critical points are non-degenerate (cf. MILNOR [20], appendix B).

Since $f$ is a versal deformation, $w \notin \text{Bif}(f)$ for generic $w \in W$. So $W \setminus \text{Bif}(f)$ is dense in $W$.

(6.7) Let now $w \in W$ be chosen in such a way that the approximation satisfies properties a),...,f) of lemma (6.5) and (6.6). In that case $F_w$ is called a generic approximation of $f$. We next recall the construction of the vanishing cycles as given by BRIESKORN [7] or LAMOTKE [16]. Call $F_w = k$.

Let $a_1,\ldots,a_q$ be the critical points of $k$, and $c_1,\ldots,c_q$ the corresponding critical values. Let $B_1,\ldots,B_q$ be disjoint $(2n+1)$-balls around $a_1,\ldots,a_q$ and inside $B$. Let $D_1,\ldots,D_q$ be disjoint 2-discs around $c_1,\ldots,c_q$ and inside $D$, chosen in such a way that we get local fibrations:

$k : B_i \rightarrow \{a_i\} \rightarrow D_i \quad (i=1,\ldots,q)$

satisfying the usual transversality-conditions:

$s_{B_i} k^{-1}(t) \text{ if } t \in D_i \setminus \{c_i\} \quad (i=1,\ldots,q)$
Take points $d_1, \ldots, d_q$ on $\partial D_1, \ldots, \partial D_q$ and let $d \in \partial D$. We next consider paths $v_i$ in $D \setminus \bigcup_{i=1}^q D_i$ from $d$ to $d_i$. These paths induce the following maps $(i=1, \ldots, q)$:

$$Q_i = k^{-1}(d_i) \cap B_i \xrightarrow{c} k^{-1}(d_i) \cap B \xrightarrow{(v_i)_*} k^{-1}(d) \cap B = X_f,$$

which give in the homology

$$\gamma_i : H_n(Q_i) \xrightarrow{(v_i)_**} H_n(X_f).$$

Since $S^n$ is a deformation retract of $Q_i$ we have $H_n(Q_i) \cong \mathbb{Z}$. Let $s_i \in H_n(Q_i)$ be the cycle represented by $S^n$.

Define $\ell_{v_i} \in H_n(X_f)$ by $\gamma_i(s_i) = \ell_{v_i}$. We set $L_f = \bigcup_{i=1}^q \{ \ell_{v_i} | v \text{ path from } d \text{ to } d_i \text{ in } D \setminus \bigcup_{i=1}^q D_i \}$.

The elements of $L_f$ are called the vanishing cycles of $f$.

(6.8) Let $u_1, \ldots, u_q$ and $u$ be closed paths along $D_1, \ldots, D_q$ and $D$.

An arbitrary path $v$ from $d$ to $d_i$ in $D - \bigcup_{i=1}^q D_i$ induces a map

$$\sigma_v : = (v^{-1}u_1v)^*: H_n(X_f) \to H_n(X_f).$$

The Picard–Lefschetz formula [21] implies

$$\sigma_v(x) = x - (-1)^n \frac{n(n+1)}{2} \langle \ell_v, x \rangle \ell_v.$$

From now on we only consider the case, that $n$ is even, in that case the intersection form is symmetric. The self-intersection number for $a \in L_f$ is given by $\langle a, a \rangle = 2(-1)^n = \begin{cases} 2 & n \equiv 0 \pmod{4} \\ -2 & n \equiv 2 \pmod{4} \end{cases}$ (cf. [21]). So $\sigma_a(x) = x - 2 \frac{\langle a, x \rangle}{\langle a, a \rangle} a$ and we see that $\sigma_a$ is just a reflection in the direction of the vanishing cycle $a$. Note that $\sigma_a^2 = 1$ and that $\sigma_a$ preserves the intersection form. In the sequel we shall restrict the treatment to the case $n \equiv 2 \pmod{4}$; the case $n \equiv 0 \pmod{4}$ is similar.
(6.9) Consider the mapping \( \psi : \pi_1(D \setminus \bigcup_{i=1}^{q} D_i, d) \to \operatorname{Aut}(\mathbb{H}(X_f; \mathbb{Z})) \) that assigns to every closed path \( \gamma \) the induced map \( \psi^* : H_\gamma(X_f) \to H_\gamma(X_f) \).

**Definition:** The image of \( \psi \) is called the **monodromy group** \( W_f \) of \( f \).

Clearly \( W_f \) contains also the reflections in the direction of a vanishing cycle. Also \( \sigma : = \gamma^* : H_\gamma(X_f) \to H_\gamma(X_f) \) is an element of \( W_f \). \( \sigma \) is called the **monodromy operator**.

(6.10) Now we choose \( v_1, \ldots, v_q \) in such a way that:

1° they intersect only in \( d \) and have no selfintersections.
2° \( (v_1^{-1} u_1 v_1) \cdot (v_2^{-1} u_2 v_2) \cdot \ldots \cdot (v_q^{-1} u_q v_q) \overset{h}{=} u \).

In this case the set of vanishing cycles \( \ell_{v_1}, \ldots, \ell_{v_q} \) and the set of the reflections \( \sigma_{v_1}, \ldots, \sigma_{v_q} \) are called **fundamental**; and we use the notations \( \ell_1, \ldots, \ell_q \) and \( \sigma_1, \ldots, \sigma_q \).

With these notations we state:

**Theorem:** (LAMOTKE [16])

- a) \( \{ \ell_1, \ldots, \ell_q \} \) is a basis of \( H_\gamma(X_f) \)
- b) \( W_f(\ell_f) = L_f \)
- c) \( W_f \) is generated by \( \{ \sigma_1, \ldots, \sigma_q \} \)
- d) \( W_f(\ell_1, \ldots, \ell_q) = L_f \)
- e) \( \sigma_q \cdot \sigma_{q-1} \cdot \ldots \cdot \sigma_1 = \sigma \).

(6.11) **Remarks on the basis.**

A basis of vanishing cycles \( \{ \ell_1, \ldots, \ell_q \} \in H_\gamma(X_f) \) induced by paths \( v_1, \ldots, v_q \) from \( d \) to \( d_1, \ldots, d_q \), having the property:

(F1) The paths \( v_1, \ldots, v_q \) intersect only in \( d \) and have no selfintersections
is called a weak distinguished basis (AZZERI [17] calls it a geometric basis). In fact every set of vanishing cycles, having property (P1) is a basis. If moreover the property
\[(P2) (v_1^{-1} u_1 v_1) \cdot (v_2^{-1} u_2 v_2) \cdot \ldots \cdot (v_q^{-1} u_q v_q) \leq u\]
is satisfied, then the basis is called distinguished. The basis \(\{\ell_1, \ldots, \ell_q\}\) in theorem (6.10) can always be chosen in such a way, that the basis is distinguished.
§7 Topological properties of an approximation.

(7.1) We now return to the situation, that $w \in W$ is chosen in such a way, that approximation $F_w$ satisfies the properties of lemma (6.5), but not necessarily those of (6.6).

Let $\{a_1, \ldots, a_p\}$ be the critical points of $g$, not necessarily non-degenerate. For every critical point $a_i$ we can consider the mappings $g_i : (U_i, 0) \to (C, 0)$ with $U_i \subset C^{n+1}$, locally defined by:

$$g_i(z) = g(z-a_i) - g(a_i) \quad (i=1, \ldots, p),$$

each $g_i$ having an isolated critical point in $0$.

For each $g_i$ we can repeat the construction of the vanishing cycles $L$, the monodromy $\sigma_{g_i}$ and the groups $W_{g_i}$. We will compare them with the corresponding notions of $f$.

(7.2) As in (6.7) let $B_1, \ldots, B_p$ be disjoint $(2n+1)$-balls around $a_1, \ldots, a_p$ and inside $B$. Let $D_1, \ldots, D_p$ be small disjoint 2-discs around $g(b_1), \ldots, g(b_p)$ and inside $D$, chosen in such a way, that the transversality condition $\partial B_i \cap g^{-1}(t)$ for all $t \in D_i \setminus \{g(b_i)\}$ is satisfied and such that we have local fibrations

$$g : E_{g_i} = g^{-1}(\partial D_i) \cap B_i \to \partial D_i.$$

We consider again the points $d, d_1, \ldots, d_p$ and the paths $u_1, \ldots, u_p$ and $v_1, \ldots, v_p$. If $g(b_i) = g(b_j)$ we choose $D_i = D_j$; $d_i = d_j$, $u_i = u_j$ and $v_i = v_j$. 
(7.3) We next define a generic approximation $h$ of $f$, near to $W$, which can also be used to obtain generic approximations for the mappings $g_i$ ($i=1, \ldots, p$). We consider also the corresponding fibrations $h : E_i = h^{-1}(3D_i) \cap B_i \to 3D_i$.

(7.4) Lemma: For all $g = F_w$ with $\|w\| < \eta$ (where $\eta$ is defined as in lemma (6.5)) we can find $s \in W$ with $\|s\| < \eta$ arbitrarily close to $w$, such that $h = F_s$ satisfies:

a) all critical points of $h$ are inside $B_1 \cup \ldots \cup B_p$.

b) all critical values of $h$ are inside $D_1 \cup \ldots \cup D_p$.

c) $3B_i \cap h^{-1}(t)$ for all $t \in 3D_i$.

d) the fibrations $E_g \to 3D_i$ and $E_h \to 3D_i$ are diffeomorphic.

e) all critical points of $h$ are non-degenerate.

f) all critical values of $h$ are different.

The proof is a specialization of (6.5) and (6.6) and will be omitted.
enlargement:
(7.5) We now repeat the construction of the vanishing cycles with an approximation $h$, satisfying a)-f) of lemma (7.4). We use a notation with double-indices. Let $\{a_{i1}, \ldots, a_{i r_i}\}$ be the critical points of $h$ inside $B_i$.

The following is defined in the obvious way:

balls $B_{ij}$ with $a_{ij} \in B_{ij} \subset B$

discs $D_{ij}$ with $D_{ij} \subset D_i \subset D$

points $d_{ij}$ with $d_{ij} \in \partial D_{ij}$

paths $v_{ij}$ from $d_i$ to $d_{ij}$ inside $D_i$

paths $u_{ij}$ around $D_{ij}$

Consider the following diagram:

\[
\begin{array}{ccc}
Q_{ij} = h^{-1}(d_{ij}) \cap B_{ij} & \xrightarrow{(v_{ij})_*} & h^{-1}(d_i) \cap B_i \cong X_{g_i} \\
& \downarrow{(v_{i})_*} & \\
& h^{-1}(d) \cap B \cong X_f & 
\end{array}
\]

inducing in the homology:

\[
Z \cong H_n(Q_{ij}) \xrightarrow{(v_{ij})^{**}} H_n(X_{g_i})
\]

Let $s_{ij}$ be a generator of $H_n(Q_{ij})$. Define $\hat{L}_{g_i} \subset H_n(X_{g_i})$ as the set of vanishing cycles with respect to $g_i$ and $L_f \subset H_n(X_f)$ as the set of vanishing cycles with respect to $f$. 

Choosing paths $v_i$ and $v_{ij}$ in such a way that

1° The paths don't intersect each other and are not-selfintersecting

$$\left(v_1^{-1}u_1v_1\right) \cdot (v_2^{-1}u_2v_2) \cdot \ldots \cdot (v_p^{-1}u_pv_p) \equiv u$$

$$\left(v_{i1}^{-1}u_{i1}v_{i1}\right) \cdot (v_{i2}^{-1}u_{i2}v_{i2}) \cdot \ldots \cdot (v_{ir_i}^{-1}u_{ir_i}v_{ir_i}) \equiv u_i$$

the fundamental vanishing cycles $\hat{k}_{ij} \in L_{g_i} \subset H_n(X_{g_i})$ and $\hat{\lambda}_{ij} \in L_f \subset H_n(X_f)$ are defined by: $\hat{k}_{ij} = (v_{ij})^* \cdot s_{ij}$ and $\hat{\lambda}_{ij} = (v_{ij})^* \cdot s_{ij}$.

(7.6) Theorem:

a) $\{\hat{k}_{i1}, \ldots, \hat{k}_{ir_i}\}$ is a basis of $H_n(X_{g_i})$ denoted $\mathbb{Z}^{\mu(g_i)}$ (i=1,...,p)

b) $\{\hat{\lambda}_{11}, \ldots, \hat{\lambda}_{1r_1}, \ldots, \hat{\lambda}_{pr_p}\}$ is a basis of $H_n(X_f) = \mathbb{Z}^{\mu(f)}$

c) The bases in a) and b) are distinguished.

The proof is a consequence of (6.10).

(7.7) Corollary:

The map $(v_i)^*: H_n(X_{g_i}) \rightarrow H_n(X_f)$ is injective and so we can identify $H_n(X_{g_i})$ with a subspace of $H_n(X_f)$ (i=1,...,p) and

$H_n(X_f) = H_n(X_{g_1}) \oplus \ldots \oplus H_n(X_{g_p})$ over $\mathbb{Z}$.

(7.8) The mappings $g_1, \ldots, g_p$ and $f$ define monodromy-groups $\hat{\mathfrak{G}}_1, \ldots, \hat{\mathfrak{G}}_p$ and $\hat{\mathfrak{G}}$ in resp. $\text{Aut}[H_n(X_{g_1})], \ldots, \text{Aut}[H_n(X_{g_p})]$ and $\text{Aut}[H_n(X_f)]$. We shall "extend" the above injections, and also identify $\hat{\mathfrak{G}}_1, \ldots, \hat{\mathfrak{G}}_p$ with subgroups of $\hat{\mathfrak{G}}$.

Set $\Sigma_i = \text{Int} D_{ij}$ and $\Sigma = \text{Int} D_{ij}$. Let $W_{g_i}$ be the image of the composed map:

$$\pi_1(D_{ij} \cdot \Sigma, a_i) \rightarrow \pi_1(D_{ij} \cdot \Sigma, a) \rightarrow \text{Aut}[H_n(X_f)]$$

given by $[w] \mapsto [v_i^{-1}wv_i] \mapsto (v_i^{-1}wv_i)^*$. 
Proposition: \( \hat{W}_i \) and \( W_i \) are isomorphic.

Proof: Set \( X_i = X_{g_i} = h^{-1}(d_i) \cap B_i; X = h^{-1}(d_i) \cap B \) and \( X' = X \setminus X_i \).

We have the following situation:

\[
\begin{array}{ccc}
\pi_1(D_1 - \Sigma_i, d_i) & \xrightarrow{\psi_i} & \text{Aut}[H_n(X_i)] \supset \hat{W}_i \\
\downarrow & & \downarrow \\
\pi_1(D - \Sigma, d) & \xrightarrow{\psi} & \text{Aut}[H_n(X)] \supset \hat{W}_i \\
= & & = \\
\pi_1(D - \Sigma, d) & \xrightarrow{\psi} & \text{Aut}[H_n(X')] \supset W
\end{array}
\]

Define:
\[
\begin{align*}
\hat{W}_i &= \psi_i[\pi_1(D_1 - \Sigma_i, d_i)] = \hat{W}_i \\
\tilde{W}_i &= \psi[\pi_1(D_1 - \Sigma_i, d_i)] \\
W &= \psi[\pi_1(D - \Sigma, d)]
\end{align*}
\]

First we shall show \( \hat{W}_i \cong \tilde{W}_i \).

Let in general \( h : Y \to Y \) be a map with \( h|A = 1 \) then the variation map \( \text{var}_h : H_n(Y,A) \to H_n(Y) \) is defined by \( \text{var}_h[x] = [x - h(x)] \). Considering the composed map

\[
H_n(Y) \xrightarrow{i^*} H_n(Y,A) \xrightarrow{\text{var}_h} H_n(Y)
\]

we see that \( h_* = 1 + i_* \text{var}_h \). Moreover \( \text{var} \) is a natural transformation (cf. [16]).

Let \( [w] \in \pi_1(D_1 - \Sigma_i, d_i) \). We consider the following commutative diagram:

\[
\begin{array}{ccc}
H_q(X, x) & \xrightarrow{\text{var}_w} & H_q(X) \\
\downarrow i^* & & \downarrow q \\
H_q(X, x') & \xrightarrow{\text{var}_w'} & H_q(X) \\
(\text{exc})_* & & (\text{exc})_* \\
H_q(X_i, x_i) & \xrightarrow{\text{var}_w} & H_q(X_i)
\end{array}
\]

The definitions of \( \text{var}_w, \text{var}_w' \) and \( \text{var}_w'' \) are justified, because it is possible to choose \( w \) such that \( w_*|X' = 1 \). From the above diagram follows:

Lemma 1: If \( w \in \pi_1(D_1 - \Sigma_i, d_i) \) then \( \text{var}_w' = 0 \Rightarrow \text{var}_w = 0. \)
Lemma 2: \( \hat{W}_1 \) and \( W_i \) are isomorphic.

Proof: Let \( \text{Aut}[H_n(X); H_n(X_i)] \) be the subset of \( \text{Aut}[H_n(X)] \) consisting of the automorphisms that map \( H_n(X_i) \) into itself. Then \( \hat{W}_i \subset \text{Aut}[H_n(X); H_n(X_i)] \).

The natural map: \( \text{Aut}[H_n(X); H_n(X_i)] \to \text{Aut}[H_n(X_i)] \) defines a surjective morphism: \( \hat{W}_i \to \hat{W}_i \).

We next show the injectivity:

For \( [w] \in \pi_1(D_{i},d_{i},d_{i}) \) we have

\[
\psi_i[w] = 1 + (i_{1})_{*}\text{var}' \, w \quad \quad \psi[w] = 1 + (i_{2})_{*}\text{var}' \, w.
\]

Let \( \psi_i[w] = 1 \) on \( H_n(X_i) \). Then \( (i_{1})_{*}\text{var}' \, w = 0 \) and so \( \text{var}' \, w = 0 \). Lemma 1 implies \( \text{var}' \, w = 0 \) and so \( \psi[w] = 1 \).

Lemma 3: There is an isomorphism \( \phi : W \to W_i \) mapping \( \hat{W}_i \) onto \( W_i \).

Proof: The path \( v \) from \( d \) to \( d_i \) induces a diffeomorphism \( (v_i)_* : X_i \to X \).

By conjugation with \( (v_i)_* \) we get an isomorphism

\( \text{Aut}[H_n(X)] \to \text{Aut}[H_n(X_i)] \).

Clearly the following diagram is commutative:

\[
\begin{CD}
\pi_1(D_{i},d_{i}) \ar[r]^\psi \ar[d] & \text{Aut}[H_n(X)] \ar[d] \\
\pi_1(D_{i},d) \ar[r]^\psi & \text{Aut}[H_n(X_f)]
\end{CD}
\]

and this proves the lemma.

We have now proved our proposition.

(7.9) We denote by \( \sigma_{ij} \) the reflections in the direction of the fundamental vanishing cycles \( \lambda_{ij} \); by \( \sigma = \nu_* : H_n(X_f) \to H_n(X_f) \) the
monodromy operator of \( f \) and by 
\[ \sigma_i = (v_i^{-1} u_i v_i)^* : H_n(X_f) \to H_n(X_f) \]
the transported monodromy operators of \( g_i \) \((i=1,\ldots,p)\).

From the above statements follow:

**Theorem:**
\[ a) \quad H_n(X_f) = H_n(X_{g_1}) \oplus \ldots \oplus H_n(X_{g_p}) \]
\[ b) \quad W_f \text{ is generated by } W_{g_1}, \ldots, W_{g_p} \]
\[ c) \quad \sigma = \sigma_p \cdot \ldots \cdot \sigma_1 \text{ and } \sigma_i = c_{i,r_i} \cdot \ldots \cdot \sigma_1 \]
\[ \quad (i=1,\ldots,p) \]
\[ d) \quad W_f(L_{g_1} \cup \ldots \cup L_{g_p}) = L_f \]

(7.10) **Corollary:** With respect to the chosen basis of fundamental vanishing cycles, the matrix \( M_f \) of the intersection form on \( H_n(X_f) \) is given by

\[
X. \quad M_f = \begin{pmatrix}
M_{g_1} & A_{12} & \ldots & \ldots & A_{1p} \\
A_{21} & M_{g_2} & \ldots & \ldots & A_{2p} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
A_{p1} & A_{p2} & \ldots & M_{g_p}
\end{pmatrix}
\]

where \( M_{g_i} \) are the matrices of the intersection forms on \( H_n(X_{g_i}) \) and
\[ A_{ij}^T = A_{ji} \]. By a proper choice of the basis all the matrices
\( M_{g_1}, \ldots, M_{g_p} \) and \( M_f \) are simultaneously given with respect to a distinguished basis.

Examples will be given in §8.
§8 Adjacency of singularities

(8.1) In this paragraph we apply the theory of intersection forms of §6 and §7 on adjacency of germs. In this way we explain some results, which were in a different way obtained by ARNOLD and SAITO. In the first part of this paragraph we give definitions and report on their results.
In this paragraph we assume $n \equiv 2 \ (\text{mod.} \ 4)$.

(8.2) Definition: A germ $\hat{i} \in m$ is called simple if there exists an open neighborhood $U$ of $\hat{i}$ in $m$ such that $U$ intersects only a finite number of orbits (under the action of biholomorphic map germs).
The orbit of a simple $\hat{i}$ is also called simple.
We remark that $\hat{i}$ being simple implies that codim $(\hat{i})$ is finite and that this definition is equivalent to the definition of ARNOLD [1].

(8.3) Definition: A germ $\hat{i} \in m$ is called simple elliptic (or mildly non-simple) if codim $(\hat{i}) < \infty$ and there exists an open neighborhood $U$ of $\hat{i} \in m$, intersecting only a finite number of orbits with codimension smaller than codim $(\hat{i})$.
The orbit of a simple elliptic $\hat{i}$ is also called simple elliptic.

SAITO [22] proved that the exceptional curve of the resolution of the hypersurface $f = 0$ is an elliptic curve without singularities if and only if $\hat{i}$ is a simple elliptic germ. This explains the chosen name.

The following two classification theorems (8.4) and (8.5) were essentially obtained by ARNOLD [1]. In our list [23] we had already all simple singularities and among other non-simple singularities we had two of the three simple elliptic families. For simple elliptic singularities see also DUISTERMAAT [10].
\textbf{(8.4) Theorem:} \( \hat{f} \) is simple if and only if \( \hat{f} \) is of type \( A_k, D_k, E_k \), where:

\begin{align*}
A_k & : \quad z_0^{k+1} + z_1^2 + z_2^2 + \ldots + z_n^2 \quad (k \geq 1); \quad \text{codim } A_k = \text{dim } A_k - 1 \\
D_k & : \quad z_0^2 z_1 + z_1^{k-1} + z_2^2 + \ldots + z_n^2 \quad (k \geq 4); \quad \text{codim } D_k = \text{dim } D_k - 1 \\
E_6 & : \quad z_0^3 + z_1^4 + z_2^2 + \ldots + z_n^2 \quad \text{codim } E_6 = 5 \\
E_7 & : \quad z_0^3 + z_1 z_2^3 + z_2^2 + \ldots + z_n^2 \quad \text{codim } E_7 = 6 \\
E_8 & : \quad z_0^3 + z_1^4 + z_2^2 + \ldots + z_n^2 \quad \text{codim } E_8 = 7
\end{align*}

\textbf{(8.5) Theorem:} \( \hat{f} \) is simple elliptic if and only if \( \hat{f} \) is of type \( P_8 \), \( X_9 \), or \( J_{10} \) (or in Saito's notation \( \tilde{E}_6 \), \( \tilde{E}_7 \), or \( \tilde{E}_8 \)), where:

\begin{align*}
\tilde{E}_6 & = P_8 : \quad z_0^3 + z_1^3 + z_2^3 + \mu z_0 z_1 z_2 + z_3^2 + \ldots + z_n^2 \\
\tilde{E}_7 & = X_9 : \quad z_0^4 + z_1^4 + \mu z_0 z_1 z_2 + z_2^2 + z_3^2 + \ldots + z_n^2 \\
\tilde{E}_8 & = J_{10} : \quad z_0^5 + z_1^6 + z_2^2 + \mu z_0 z_1 z_2 + z_3^2 + \ldots + z_n^2
\end{align*}

\text{codim } P_8 = 7; \quad \text{codim } X_9 = 8; \quad \text{codim } J_{10} = 9.

\textbf{(8.6) Remark on intersectionmatrices.} PHAM [21] and recently GABRIELOV [12] computed intersectionmatrices for singularities of the form:

\[ z_0^{a_0} + z_1^{a_1} + \ldots + z_n^{a_n}. \]

We refer for the general form for these intersectionmatrices to their papers, and also to HIRZEBRUCH-MAYER [14] p. 88 and give here only a few examples for \( n = 2 \).

An easy way to describe intersectionmatrices is by a diagram. The correspondence between matrix and diagram is as follows:

1. always \( a_{ii} = -2 \)
2. \( a_{ij} = a_{ji} = 0 \)
3. \( a_{ij} = a_{ji} = 1 \)
4. \( a_{ij} = a_{ji} = 2 \)
5. \( a_{ij} = a_{ji} = -1 \)
6. \( a_{ij} = a_{ji} = -2 \)
Examples: (taken from GABRIELOV):

(i) \( z_0^4 + z_1^2 + z_2^2 \) (\( A_3 \)) has diagram

(ii) \( z_0^5 + z_1^3 + z_2^2 \) (\( E_6 \)) has diagram

(iii) \( z_0^4 + z_1^4 + z_2^2 \) (\( X_9 \)) has diagram

(iv) \( z_0^{a_0} + z_1^{a_1} + z_2^2 \) has diagram

(v) \( z_0^3 + z_1^3 + z_2^3 \) has diagram

With the given ordering each basis is distinguished.

HIRZEBRUCH-MAYER \([14]\) showed that

the intersection form of \( z_0^{a_0} + z_1^{a_1} + z_2^2 \) in case \( a_0 \geq a_1 \geq a_2 \) is:

- negative definite \( \iff \frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} > 1 \iff (a_0, a_1, a_2) = (n, 2, 2), (3, 3, 2), (4, 3, 2) \) or \( (5, 3, 2); (n \geq 2) \).

- negative semi-definite \( \iff \frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} = 1 \iff (a_0, a_1, a_2) = (6, 3, 2), (4, 4, 2) \) or \( (3, 3, 3) \).

In the case of simple singularities one can also apply the following:

There is a 1-1-correspondence between simple germs \( \tilde{f} \) and algebraic varieties \( X \), given by \( f = 0 \), having in \( 0 \) a rational doublepoint. In that case we can use the minimal resolution \( \pi: \tilde{X} \rightarrow X \) of this singular variety. TJURINA \([24]\) and Brieskorn showed that if \( f \) is of type \( A_k, D_k \) or \( E_k \) then \( \pi^{-1}(0) \) is diffeomorphic with the typical fibre \( X_f \) of the Milnor fibration. The corresponding intersection forms have all been computed; their matrices can be given with respect to a distinguished basis by diagrams as before, and these diagrams happen to be the usual Dynkindiagrams for \( A_k, D_k, E_k \) and their intersection forms are all negative definite.
Recently ARNOLD [3] announced that GABRIELOV had also computed
intersection matrices in other cases. For the singularity
\[ z_0^p + z_1^q + z_2^r + \lambda z_0 z_1 z_2 \]
with \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \), the intersection form with respect to a weak
distinguished basis is given by:

\[ \begin{array}{ccc}
 & p & \\
q & & \\
r & & \\
\end{array} \]

and \( u = p + q + q - 1 \).

(8.7) Change of basis.
The intersection matrix can change if we use another basis. The
question arises if there is a nice form for these matrices over \( \mathbb{Z} \).
One can ask this question with respect to:
a) a basis of cycles in \( H_n(X_\tau) \)
b) a weak distinguished basis in \( H_n(X_\tau) \)
c) a distinguished basis in \( H_n(X_\tau) \)
Moreover one has to say, what one likes to call a "nice" matrix. In
the case of simple singularities one can arrange that with respect
to a distinguished basis, the diagram is just the corresponding
Dynkin diagram. This diagram has the form of a tree; and the matrix
has the properties \( a_{ii} = 2 \) and \( a_{ij} \leq 0 \) if \( i \neq j \).
So a definition of "nice" could be: \( a_{ii} = 2 \land a_{ij} \leq 0 \) if \( i \neq j \).
But already in the case of the simple elliptic singularities it is
impossible to obtain this nice situation, even if we allow a basis
of type a).

Namely let \( -q_{ij} \) be the matrix of the intersection form of a simple
elliptic singularity. Then:
1° the quadratic form \( \sum q_{ij} x_i x_j \) on \( \mathbb{R}^n \) is positive with kernel-
dimension 2.
2° there is no partition of \( \{1, \ldots, n\} \) into two non-empty sets I and J
such that \((i,j) \in I \times J\) implies \(q_{ij} = 0\) (cf. LAZZERI [17]; proposition 2).

If now \(q_{ij} \leq 0\) for all \(i \neq j\), then BOURBAKI [4] (p. 78) implies that the kernel dimension of the quadratic form is 0 or 1. This gives a contradiction.

(8.8) Definition: A germ \(\hat{g}\) is called adjacent to \(\hat{f}\) if in any neighborhood of \(\hat{f}\) there are germs of the orbit of \(\hat{g}\).

Notation: \(\hat{g} \leq \hat{f}\).

If \(\hat{g} \leq \hat{f}\) the orbit of \(\hat{g}\) is also called adjacent to the orbit of \(\hat{f}\).

Examples:

1° \(f_t(x) = x^6 + tx^7\) is for \(t = 0\) of type \(A_7\) and if \(t \neq 0\) of type \(A_6\), so \(A_6 \leq A_7\).

2° \(f_t(x,y) = x^2y + y^6 + tx^2\) is for \(t = 0\) of type \(D_9\) and if \(t \neq 0\) of type \(A_7\) so \(A_7 \leq D_9\).

3° \(f_t(x,y) = \phi(x,y) + tx^2 + ty^2\) is for \(t \neq 0\) of type \(A_4\). This shows \(A_4 \leq \phi\) for all .

(8.9) Proposition: If \(\hat{g} \leq \hat{f}\) then there exists an injection \(H_n(X_g) \to H_n(X_f)\) preserving the intersection form \(\langle \cdot, \cdot \rangle\) and mapping a distinguished basis of \(H_n(X_g)\) into a distinguished basis of vanishing cycles of \(H_n(X_f)\), such that the intersection matrix of \(\hat{g}\) can be identified with a diagonal submatrix of the intersection matrix of \(\hat{f}\).

Proof:

From the definition of adjacency follows that with respect to a versal deformation \(F : U \times W \to \mathbb{C}\) of \(f\) there exist \(w \in W\) arbitrarily close to \(0 \in W\) such that \(\hat{g}\) is equivalent to the approximation \(\hat{F}_w\).

Then we can apply theorem (7.6). So we can consider \(H_n(X_g)\) as a subset of \(H_n(X_f)\) and there is a basis of vanishing cycles \(\{l_1^g, \ldots, l_q^g\}\) of \(H_n(X_f)\) such that \(\{l_{p+1}^g, \ldots, l_q^g\}\) is a basis of vanishing cycles of \(H_n(X_g)\).
The following two theorems characterize simple and simple elliptic singularities by properties of the intersection form. They were stated in a letter of Arnold to the international mathematical conference on manifolds and related topics in Tokyo (1973).

(8.10) Theorem: \( \hat{f} \) is simple if and only if the intersection form on \( H_n(X_f) \) is negative definite.

Proof:

Remark (8.6) shows that a simple singularity has a negative definite intersection form.

If \( g \) is not simple, then some germ in at least one of the following three families is adjacent to \( \hat{g} \) (cf. Arnold [3]):

- \( \mathcal{E}_6 = P_8: z_o^3 + z_1^3 + z_2^3 + \mu z_o z_2 z_3 + z_2^2 + \ldots + z_n^2 \)
- \( \mathcal{E}_7 = X_9: z_o^4 + z_1^4 + z_2^2 + \mu z_o^2 z_1^2 + z_3^2 + \ldots + z_n^2 \)
- \( \mathcal{E}_8 = J_{10}: z_o^6 + z_1^3 + z_2^2 + \mu z_o z_1^2 + z_3^2 + \ldots + z_n^2 \)

In those families of germs with constant Milnor number, the intersection form is also constant and can be computed from \( z_o^3 + z_1^3 + z_2^3, z_o^4 + z_1^4 + z_2^2 \) and \( z_o^6 + z_1^3 + z_2^2 \).

These are negative semi-definite with a 2-dimensional kernel. The intersection matrix of \( \hat{g} \) contains a negative semi-definite matrix as diagonal submatrix and cannot be negative definite.

(8.11) Theorem: \( \hat{f} \) is simple elliptic if and only if the intersection form on \( H_n(X_f) \) is negative semi-definite.

Proof:

If \( \hat{f} \) is simple elliptic it follows from (8.5) and (8.6) that the intersection form is negative semi-definite.

Let \( \hat{g} \) be not simple elliptic. We already know from (8.10) that a simple germ has a negative definite intersection form. So let us assume, that \( \hat{g} \) is not simple or simple elliptic.

In the same way is in (8.10) one shows now that some germ in at least one of the following three families is adjacent to \( \hat{g} \):
Arnold announced in [3] (see also Demazure [9]) that Gabriev had computed the intersection forms and that in all these cases there is a vector with positive value. So \( \hat{\mathfrak{g}} \) cannot have a negative semi-definite intersection form.

(8.12) Theorem:

For simple singularities we have:

\[
\hat{\mathfrak{g}} \leq \hat{\mathfrak{f}} \Rightarrow \text{Dynkindiagram (\( \hat{\mathfrak{g}} \)) \subset \text{Dynkindiagram (\( \hat{\mathfrak{f}} \))}
\]

where the Dynkindiagram of a germ of type \( A_k, D_k, \) or \( E_k \) equals the usual Dynkindiagram of \( A_k, D_k, E_k \) in the theory of semi-simple Lie-algebra's:

- \( A_k \): \[\bullet \cdots \bullet \bullet \bullet \quad (k \text{ points})\]
- \( D_k \): \[\bullet \cdots \bullet \bullet \quad (k \text{ points})\]
- \( E_6 \): \[\bullet \bullet \bullet \quad \]
- \( E_7 \): \[\bullet \bullet \bullet \bullet \quad \]
- \( E_8 \): \[\bullet \bullet \bullet \bullet \bullet \bullet \quad \]

Proof: Arnold [1] proved the theorem by direct computations, using the definitions of adjacency and by comparison of the results with the possible subdiagrams of the corresponding Dynkindiagrams.

We next show, that it is possible to prove that the adjacency implies the inclusion of Dynkindiagrams, using the theory developed in §7.

In this alternative proof the relation with the theory of the intersection forms becomes clearer.

Let \( \hat{\mathfrak{g}} \leq \hat{\mathfrak{f}} \). In (8.9) we found that we can consider \( H_n(X_{\mathfrak{g}}) \) as a subset of \( H_n(X_{\mathfrak{f}}) \) and that there exists a basis of vanishing cycles of \( H_n(X_{\mathfrak{f}}) \) such that \( \{ \ell_{p+1}, \ldots, \ell_q \} \) is a basis of vanishing cycles of \( H_n(X_{\mathfrak{g}}) \).

If \( \hat{\mathfrak{g}} \) is simple, then it is always possible to choose a distinguished basis \( \{ \ell_{p+1}, \ldots, \ell_q \} \) in such a way, that the intersection matrix of \( \hat{\mathfrak{g}} \)
is in the normal form, given by the Dynkin diagram. The intersection-
matrix of \( f \) contains this matrix as submatrix, but it is not
necessarily in the normal form.

Since \( f \) is simple, the intersection form is negative definite and the
set \( L_f \) and the bilinear form \( <-,-> \) satisfy the definition of
root system. We shall apply now a customary argument in the classifi-
cation theory of root systems (cf. BOURBAKI [4]).

The ordered basis \( \{ \ell_1, \ldots, \ell_p, \ell_{p+1}, \ldots, \ell_q \} \) defines an ordering of
the roots of \( L_g \) and \( L_i \). Because the intersection matrix on \( H_n(P_g) \)
has Cartan form with respect to \( \{ \ell_{p+1}, \ldots, \ell_q \} \) these roots are funda-
mental (with respect to \( L_i \)). Using the ordering we can now select
fundamental roots \( \{ m_1, \ldots, m_p \} \) (with respect to \( L_i \)), such that:

\[
m_1 < \ldots < m_p < \ell_{p+1} < \ldots < \ell_q.
\]

Moreover we write \( m_i = \ell_i \) if \( p+1 \leq i \leq q \).

We shall prove that \( \{ m_1, \ldots, m_p, m_{p+1}, \ldots, m_q \} \) is fundamental with
respect to \( L_f \).

**Lemma:** \( <m_i, m_j> \geq 0 \) for \( i \neq j \).

**Proof:**

(i) if \( i \leq p \land j \leq p \): then \( <m_i, m_j> \geq 0 \) because \( m_i \) and \( m_j \) are
fundamental.

(ii) if \( i > p \land j > p \): then \( <m_i, m_j> = <\ell_i, \ell_j> \geq 0 \).

(iii) if \( i \leq p \land j > p \) (or resp. \( i > p \land j \leq p \) : We have that
\( m_i - m_j \) is not a root, for otherwise \( m_i = (m_i - m_j) + m_j \) with
\( m_i - m_j > 0 \), so \( m_i \) is not fundamental.

The \( m_j \)-chain through \( m_i : m_i + sm_j, \ldots, m_i + tm_j \) starts with \( m_i \);
so \( s = 0 \). The formula:

\[
\frac{<m_i, m_j>}{-<m_i, m_i>} = \frac{s-t}{2} \text{ gives } <m_i, m_j> \geq 0 \text{ since } <m_i, m_i> = -2.
\]

So with respect to the basis \( \{ m_1, \ldots, m_p, m_{p+1}, \ldots, m_q \} \), resp.
\( \{ m_1, \ldots, m_p \} \) the intersection matrices of \( f \) and \( g \) are simultaneously
in Cartan form. So the Dynkin diagram of \( f \) is a subdiagram of the
Dynkin diagram of \( g \).
(8.13) Theorem:
a) If $\hat{g} \leq \hat{f}$ and $\hat{f}$ is a simple elliptic singularity, then
   either: $\hat{g}$ is equivalent to $\hat{f}$
or: $\hat{g}$ is simple.
b) Moreover if $\hat{g}$ is simple we have:
   $\hat{g} \leq \hat{f} \iff$ Dynkindiagram ($\hat{g}$) $\subseteq$ Dynkindiagram ($\hat{E}_\lambda$)
   where: $\lambda = 6$ if $\hat{f}$ has type $P_8$
   $\lambda = 7$ if $\hat{f}$ has type $X_9$
   $\lambda = 8$ if $\hat{f}$ has type $J_{10}$
   and the Dynkindiagram of $\hat{E}_\lambda$ are the so-called extended Dynkindiagrams for $E_\lambda$ in the theory of semi-simple Liegroups (see BOURBAKI [4], p.199):

Proof: Our proof and calculation that the inclusion implies the
adjacency was more complicated than that given in the paper of
SAITO [22], which appeared recently. Therefore we show only that
adjacency implies inclusion. SAITO proved that part by direct
computations, using the definition of adjacency. Our proof shall
use the intersectionform and the monodromy group.

The intersectionforms of the simple elliptic singularities can be
given by the following diagrams (with respect to a weak distinguished
basis; compare GABRIELOV):
The kernel dimension is 2. If \( i \) and \( j \) are such that \( i = j \) (so \( \langle l_i, l_j \rangle = -2 \)), then \( \langle l_i - l_j, l_i - l_j \rangle = 0 \), so \( e_i - e_j \) is a kernelvector. After dividing out by the subspace, spanned by \( e_i - e_j \), the intersectionform is given by the following diagrams:

These diagrams correspond with negative quadratic forms with a 1-dimensional kernel.

Let \( g \leq f \) then \( W_g \subseteq W_f \) and since \( H_n(X_g) \cap R[e_i - e_j] = \{0\} \) this implies \( W_g \subseteq W(E_k) \), where \( W(E_k) \) is the Weyl group of \( E_k \).

With the Weyl groups \( W(E_k) \) there correspond a (infinite) set \( \mathcal{H} \) of hyperplanes in a vectorspace \( V \), which divides \( V \) into chambers. The reflections in the hyperplanes generate \( W(E_k) \). The reflections in the walls of one Weylchamber already generate \( W(E_k) \).

A vertex \( P \) of a Weylchamber is called a special vertex if for every hyperplane \( H \in \mathcal{H} \) there is a parallel hyperplane in \( \mathcal{H} \) through \( P \). The reflections in the hyperplanes through a special vertex \( P \) generate the group \( W(E_k) \). Any finite subgroup of \( W(E_k) \) is also a subgroup of \( W(E_k) \).

In general the subgroup of \( W(\tilde{X}) \) fixing a vertex \( Q \) has a Coxetergraph, that can be derived from the Coxetergraph \( \tilde{X} \) by removing one of the nodes:

- \( \tilde{E}_6 \) gives \( E_6 \) and not \( A_6 \) and \( D_6 \).
- \( \tilde{E}_7 \) gives \( E_7 \) and \( A_7 \) and not \( D_7 \).
- \( \tilde{E}_8 \) gives \( E_7, D_7 \) and \( A_7 \).
So we know already that:
1° $W(E_7)$ has subgroup $W(A_7)$
2° $W(E_8)$ has subgroups $W(A_8)$ and $W(D_7)$.

**Assertion 1**: $W(A_6)$ and $W(D_6)$ are not subgroups of $W(E_6)$.

**Proof**:
Order of $W(E_6) = 2^7 \cdot 3^4 \cdot 5$
Order of $W(D_6) = 2^5 \cdot 6!$
Order of $W(A_6) = 7!$

The assertion follows now from the Lagrange-theorem on the order of a subgroup.

**Assertion 2**: $W(D_7)$ is not a subgroup of $W(E_7)$.

**Proof**:
Order $W(D_7) = 2^6 \cdot 7!$ divides on order $W(E_7) = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$ so we cannot apply the arguments of assertion 1. A.M. Cohen (Utrecht) pointed out to me, that the (following) straightforward computation shows, that it is impossible to find within $R^7$ an extension of the root system $D_7$, containing only vectors of length $\sqrt{2}$ and with innerproducts $-1$, $0$ or $1$ with the vectors of $D_7$.

We proceed as follows:

$D_7$ has a realization in $R^7$ by the following combinations of basis-vectors: $\pm e_i \pm e_j$ ($1 \leq i < j \leq 7$). Also $E_7$ can be realized in $R^7$.

Extend the system with $x = \sum_{i \neq 1} a_i e_i$ (with $\sum_{i \neq 1} a_i^2 = 2$) and such that $\langle x, \pm e_i \pm e_j \rangle \in \{-1,0,1\}$.

Then we must have: $\pm a_i \pm a_j \in \{-1,0,1\}$ $1 \leq i < j \leq 7$.

This implies $a_i \in \{-1,0,1\}$.

When $a_1^2 = 1$ then $\exists j \neq i$ with also $a_j^2 = 1$ and $a_k = 0$ if $k \neq i,j$.

This gives just the elements of $D_7$.

If $x \not\in D_7$ then $a_i \in \{-\frac{1}{2},0,\frac{1}{2}\}$ and consequently:

$\|x\|^2 \leq \frac{7}{14} < 2$ and this is not possible since $\|x\|^2 = 2$. 

Lemma: Let $\hat{g} \leq f$. If $f$ is simple elliptic and $g$ is simple then $u(g) \leq u(f) - 2$.

Proof:
Let $K$ be the kernel of $\langle \cdot, \cdot \rangle$ on $H_n(X_f;Q) = q^{u(f)}$.
$$\dim H_n(X_g;Q) + \dim K = \dim [K + H_n(X_g;Q)] + \dim [K \cap H_n(X_g;Q)]$$
so: $u(g) + 2 \leq u(f) + 0$
and: $u(g) \leq u(f) - 2$

We now consider various cases:

a) If $f$ is of type $P_6$, then $u(g) \leq 6$ and so $g$ is of type $D_k (k \leq 6)$, $A_k (k \leq 6)$ or $E_6$. Assertion 1 gives that $g$ is not of type $A_6$, $D_6$; so the only possibilities are the connected subgraphs of $\hat{E}_6$.

b) If $f$ is of type $X_9$, then $u(g) \leq 7$ and so $g$ is of type $D_k (k \leq 7)$, $A_k (k \leq 7)$, $E_6$ or $E_7$. Assertion 2 gives that $g$ is not of type $D_7$; so the only possibilities are the connected subgraphs of $\hat{E}_7$.

c) If $f$ is of type $J_{10}$, then $u(g) \leq 8$ and so $g$ is of type $D_k (k \leq 8)$, $A_k (k \leq 8)$, $E_k (k \leq 8)$ and they correspond just with the connected subgraphs of $\hat{E}_8$.

Now we are done.

(8.14) Corollary: Adjacency diagram for simple and simple elliptic germs.

$$X \leftarrow Y \text{ means } X \leq Y$$

$$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow A_4 \leftarrow A_5 \leftarrow A_6 \leftarrow A_7 \leftarrow A_8 \leftarrow \ldots$$

$$D_4 \leftarrow D_5 \leftarrow D_6 \leftarrow D_7 \leftarrow D_8 \leftarrow \ldots$$

$$E_6 \leftarrow E_7 \leftarrow E_8 \leftarrow J_{10} = \hat{E}_8$$

$$X_9 = \hat{E}_7$$

$$F_8 = \hat{E}_6$$
(8.15) Definition: The germs \( \hat{g}_1, \ldots, \hat{g}_p \) are called simultaneously adjacent to \( \hat{f} \) if there exists a (germ of) deformation of \( \hat{f} \) such that for every neighborhood \( U \) of \( 0 \in \mathbb{C}^k \) there is \( \lambda \in U \) such that \( f_\lambda \) has exactly \( p \) critical points \( a_1, \ldots, a_p \) and the germs at \( 0 \in \mathbb{C}^m \) of \( g(x - a_i) - g(a_i) \) are equivalent to the germs \( \hat{g}_i \). A similar definition holds for orbits.

Corollary: If \( \hat{g}_1, \ldots, \hat{g}_p \) are simultaneously adjacent to \( \hat{f} \) then the conclusions of theorem (7.9) and remark (7.10) are valid.

(8.16) Problem: Let \( \hat{f} \) be a simple germ and let \( \hat{g}_1, \ldots, \hat{g}_p \) be simultaneously adjacent to \( \hat{f} \). Can one construct the Dynkin diagram of \( \hat{f} \) from the disjoint union of the Dynkin diagrams of \( \hat{g}_1, \ldots, \hat{g}_p \) by adding branches between different components? The answer is no. We give the following counterexample:

Let \( f_t = x_1^3 + x_2^4 + tx_1^2 \).
Then: \( \frac{\partial f_t}{\partial x_1} = 3x_1^2 + 2tx_1 = 0 \rightarrow x_1 = 0 \lor x_1 = \frac{2t}{3} \), \\
\( \frac{\partial f_t}{\partial x_2} = 4x_2^3 = 0 \rightarrow x_2 = 0 \).
So we have critical points: \((0,0)\) and \((\frac{2t}{3},0)\).

In \((0,0)\) we have for \( t \neq 0 \) a germ of type \( A_3 \). In \((\frac{2t}{3},0)\) we have for \( t \neq 0 \) a singularity with Milnor number equal 3. So the singularity must be of type \( A_3 \).

If \( t = 0 \) \( f_t = x_1^3 + x_2^4 \) is of type \( E_6 \).

So two germs of type \( A_3 \) are simultaneously adjacent to a germ of type \( E_6 \):

\[
A_3 \quad \quad \quad \quad \quad A_3 \quad \quad \quad \quad \quad ?? \quad \quad \quad \quad \quad E_6
\]

Remark: In the following matrix for \( E_6 \) it is possible to see two submatrices, equivalent to \( A_3 \):

\[
A_3 \quad \quad \quad \quad \quad A_3 \quad \quad \quad \quad \quad E_6
\]
§9 $\mu$-homotopic germs

Throughout this paragraph we study germs with isolated singularity at $0$.

(9.1) **Definition:** Two germs $\hat{g}_a$ and $\hat{g}_b$ are called $\mu$-homotopic, if there exists a continuous 1-parameter family $g_t$, $t \in [a,b] \subset \mathbb{R}$ connecting $\hat{g}_a$ and $\hat{g}_b$ and such that $\mu(g_t)$ is constant for all $t \in [a,b]$.

**Examples:**

a) $g_t = z_1^3 + z_2^3 + tz_1^2 z_2^2$ contains $\mu$-homotopic germs for all $t \in \mathbb{C}$; this is not surprising as all $g_t$ are in one and the same orbit.

b) $g_t = z_1^4 + z_2^4 + tz_1^2 z_2^2$ contains $\mu$-homotopic germs for all $t^2 \neq 4$.

(9.2) **Proposition:** $\mu$-homotopy is an equivalence relation.
The proof is a straightforward verification of the definition of equivalence relation.

The equivalence classes are called $\mu$-homotopy classes or $\mu$-classes.

(9.3) **Proposition:** If $\hat{g}_a$ and $\hat{g}_b$ are $\mu$-homotopic, then there exists an isomorphism $H_n(X_{\hat{g}_a}) \rightarrow H_n(X_{\hat{g}_b})$ preserving the intersection form $\langle -,- \rangle$.

**Proof:**

Define $G(t,x) = g_t(x)$.

The set $\{(t,x) \mid \frac{\partial G}{\partial x_0}(t,x) = \ldots = \frac{\partial G}{\partial x_n}(t,x) = 0\} \subset \mathbb{C}^2$ contains $[a,b] \times 0$ as isolated component. This follows from the fact, that for any $g_c$ with $c \in [a,b]$ every small deformation $g_t$ of $g_c$ has only one critical point inside a small ball $B$, with radius depending...
on c. It is possible to find $\varepsilon > 0$ such that $g_t$ has an isolated critical point at $0 \in \mathbb{C}^{n+1}$ and no other critical points inside a ball of radius $\varepsilon$ for all $t \in [a,b]$. Next we can apply the proof of theorem 1 of TJURINA [24] and our proposition follows.

\[(9.4)\] Definition:
\[
Z(p) = \{f \in \mathcal{G}_n \mid u(f) \geq p\}
\]
\[
\mathcal{E}(p) = \{f \in \mathcal{G}_n \mid u(f) = p\}
\]
\[
\mathcal{Z}^k(p) = \{f \in \mathcal{J}^k(n,1) \mid u(f) \geq p\}
\]
\[
\mathcal{Z}^k(p) = \{f \in \mathcal{J}^k(n,1) \mid u(f) = p\}
\]

Remark: The sets $Z(p)$, $E(p)$, $Z^k(p)$ and $E^k(p)$ are invariant under the right-action of biholomorphic mappings. Moreover they are unions of $u$-classes.

\[(9.5)\] Proposition:

a) $Z^k(p)$ is an algebraic subset of $J^k(n,1)$

b) $E^k(p)$ is a difference of two algebraic subsets in $J^k(n,1)$

C) $Z^k(p)$ and $E^k(p)$ have only a finite number of topological components.

Proof:

a) Remember: $u(f) = \dim_{\Delta(f)} A$, where $\Delta(f) = (\Delta_1 f, \ldots, \Delta_n f)$.

Assertion: $\dim_{\Delta(f)} A \geq p \iff \dim_{\Delta(f) + m^p} A \geq p$

$\iff$ is trivial

$\Rightarrow$ (following MATHER [19]):

Let $\dim_{\Delta(f) + m^p} A < p$; consider the following increasing sequence of ideals:

$\Delta \subset \Delta + m \subset \Delta + m^k \subset \Delta + m^p$

since $\dim_{\Delta(f) + m^p} A \geq 0$ and $\dim_{\Delta(f) + m^p} A < p$, there exists a $k < p$

such that $\dim_{\Delta(f) + m^k} A = \dim_{\Delta(f) + m^{k+1}} A$. 

So $\Delta(f) + m^k = \Delta(f) + m^{k+1}$ and $m^k \subseteq \Delta(f) + m^{k+1}$.

From the Nakayama lemma it follows that: $m^k \subseteq \Delta(f)$.

So $\dim \frac{\mathfrak{A}}{\Delta(f)} = \dim \frac{\mathfrak{A}}{\Delta(f) + m^k} \leq \dim \frac{\mathfrak{A}}{\Delta(f) + m^p} < p$.

Now the assertion is proved.

The condition $\dim \frac{\mathfrak{A}}{\Delta(f) + m^p} \geq p$ is clearly algebraic, since it is a rank-condition on a subspace of the finite dimensional vectorspace $\mathfrak{A}$ and gives rise to determinants in the coordinates of $J^k(n,1)$.

b) follows from the fact that $\Xi^k(p) = S^k(p) \setminus S^{k+1}(p)$.

c) A theorem of Whitney says, that for any pair of algebraic sets, the difference has at most a finite number of topological components (cf. MILNOR [20]).

Corollary: Every topological component of $\Xi^k(p)$ coincides with a $\mu$-class.

(9.6) List of $\mu$-classes with $\mu \leq 10$.

$\Sigma(1): \ A_1$

$\Sigma(2): \ A_2$

$\Sigma(3): \ A_3$

$\Sigma(4): \ A_4, \ D_4$

$\Sigma(5): \ A_5, \ D_5$

$\Sigma(6): \ A_6, \ D_6, \ E_6$

$\Sigma(7): \ A_7, \ D_7, \ E_7$

$\Sigma(8): \ A_8, \ D_8, \ E_8, \ P_8$

$\Sigma(9): \ A_9, \ D_9, \ X_9, \ P_9$

$\Sigma(10): \ A_{10}, \ D_{10}, \ J_{10}, \ X_{10}, \ P_{10}, \ Q_{10}, \ R_{10}$

The symbols correspond to those in §3. The complex normal forms are given in list I at the end.
(9.7) Proposition: The classes of the list are in different topological components of $\Sigma(p)$.

Proof:
The intersection forms are different, so by proposition (9.3) there is no $\mu$-homotopy, joining any two different classes in the list.

(9.8) Definition: $g$ is called $\mu$-adjacent to $f$ if every neighborhood of $f$ contains an element, that is $\mu$-homotopic to $g$.

Since $g$ and $f$ have isolated singular point at $0$, we can work entirely in $H^k(n,1)$ for $k$ large enough. The following lemma shows, that the definition of $\mu$-adjacency depends only on the $\mu$-class of $g$ and $f$.

(9.9) Lemma: Let $A^k(p)$ be a topological component of $\Sigma^k(p)$ and $B^k(p)$ be a topological component of $\Sigma^k(q)$ ($q \leq p$).

Then either: $A^k(p) \cap B^k(q) = \emptyset$

or: $A^k(p) \subseteq B^k(q)$.

Proof:
Let $C^k(q)$ be the top. component of $S^k(q)$ such that $A^k(q) \subset C^k(q)$.

The sets $S^k(q) = S^k(q) \setminus S^k(q+1)$ and $S^k(q)$ have the same number of topological components; so

either $1^o \ B^k(q) \subset C^k(q)$

or $2^o \ B^k(q) \cap C^k(q) = \emptyset$

$1^o$ gives $A^k(p) \subset C^k(q) = B^k(q)$

$2^o$ gives $B^k(q) \cap C^k(q) \neq \emptyset$ and so $B^k(q) \cap A^k(q) = \emptyset$.

(9.10) Theorem: If $g$ is $\mu$-adjacent to $f$ then there exists an injection $H_n(X_g) \rightarrow H_n(X_f)$ preserving $\cdot, \cdot$ and a distinguished basis of vanishing cycles such that the intersection matrix of $g$ is a diagonal submatrix of the intersection matrix of $f$.

Proof:
similar to (8.9).
Theorem: If \( g \) is a simple singularity and \( f_t \) a 1-parameter family with \( u \) constant.
If \( g \leq f_{t_0} \) then also \( g \leq f_t \).

Proof:
The \( u \)-homotopyclass of \( g \) is \( u \)-adjacent to the \( u \)-class of \( f_{t_0} \) (and so also of \( f_t \)).
So in every neighborhood of \( f_t \), there are germs \( u \)-homotopic with \( g \).
Since \( g \), is simple \( u \)-homotopy implies equivalence.

Corollary: If a simple singularity \( g \) is \( u \)-adjacent to \( f \), then \( g \) is (ordinary) adjacent to \( f \).

Remark: (difference between adjacency and \( u \)-adjacency).
Arnold gave a complete graph of the adjacency relation between simple singularities. Saito also considered \( F_8, X_9, \) and \( J_{10} \) (the simple elliptic singularities \( E_6, E_7 \) and \( E_8 \)). In these cases adjacency and \( u \)-adjacency between the different classes coincide. This is in general not the case. As an example we have the following result.

Let \( g \) be of type \( X_9 \):
\[
g = z_0^4 + 2z_0^2 z_1^2 + az_1^4 \quad (a \neq 0, 1)
\]
and \( f \) of type \( X_{10} \):
\[
f = z_0^4 + 2z_0^2 z_1^2 + bz_1^5 \quad (b \neq 0)
\]
then \( g \) is not adjacent to \( f \) for no (fixed) values of the parameters \( a \) and \( b \). Also here the crossratio gives the obstruction. Indeed \( g \) is \( u \)-adjacent to \( f \). The following picture illustrates this situation:

Consider the 2-parameter family:
\[
z_0^4 + 2z_0^2 z_1^2 + az_1^4 + bz_1^5
\]

\( \square \) orbits of type \( X_9 \)
\( \bullet \) singularity of infinite codimension

\( \square \) orbits of type \( X_{10} \)
invariant (of \( X_9 \))
invariant of \( X_{10} \)
cross ratio
Remark: It is possible to extend the graph of the adjacency relation of simple and simple elliptic singularities (cf. (8.14)) with the other singularities of the list. Further computations are then needed. We treat this in §10. As an example: consider the path of germs:

$$f_t = t^2 z_o^2 z_1 + z_o^4 + 2z_o^2 z_1^2 + 2tz_o z_1^3 + z_1^5$$

If $t \neq 0 f_t$ is of type $D_7$ and if $t = 0 f_t$ is of type $X_{10}$. So $D_7$ is adjacent to $X_{10}$ (for all $b \neq 0$).

Saito proved that $D_7$ is not adjacent to $X_9$. So in the family considered in (9.12) the polynomials of type $X_{10}$ differ from those of type $X_9$ by the property, that $X_{10}$ is in the closure of $D_7$ and $X_9$ is not.
§10 On the topology of the orbit space.

(10.1) Let $G_k$ be the set of germs of holomorphic mappings $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with codimension $\leq k$, having in $0$ a critical point. Remark, that all germs in $G_k$ are $(k+2)$-determined. We define in $G_k$ a topology in the following way: An open set is $\{ g \in G_k \mid g_{k+2} \text{ lies in an open set of } \mathbb{C}^{N(k+2)} \}$, where $N(k)$ is the number of coefficients of polynomials of degree $k$ in $n$ variables.

The natural injection $G_k \to G_{k+1}$ is a continuous map with respect to this topology. We define $G = \bigcup_{k=1}^{\infty} G_k$ and derive the topology of $G$ from the topology of the spaces $G_k$.

(10.2) Let $W$ be the set of orbits in $G$ under the right action of biholomorphic mappings: We give $W$ the quotient topology; the projection $\pi : G \to W$ is then a continuous mapping. So a set $U$ in $W$ is open if and only if $\pi^{-1}(U)$ is open in $G$. We use the symbol $Y$ of an orbit to denote also its projection in $W$. So in fact $\pi(Y)$ is denoted by $Y_w$.

Each simple singularity defines one point in $W$. The projections of some non-simple singularities will be discussed in (10.6).

(10.3) Theorem: The topology of $W$ is not Hausdorff, even not $T_1$.

Proof:
Since the orbit of type $A_1$ (non-degenerate quadratic form) is dense in $G_k$ for every $k \geq 1$, every open neighborhood of $w \in W$ contains the point $A_1$.
So there is no open neighborhood of $w$ avoiding $A_1$. 
(10.4) We now consider $\mu$-classes and denote by $\Theta(w)$ the codimension of the $\mu$-class, containing $w \in W$. Since $\mu$-classes are topological components of differences of algebraic sets and so a finite union of manifolds, this number is well-defined.

Let $U_k$ be the union of the $\mu$-classes with $\Theta(w) \leq k$, so

$$U_k = \{w \in W \mid \Theta(w) \leq k\}$$

List of $\mu$-classes in $U_8$:

- $U_0 : A_1$
- $U_1 \setminus U_0 : A_2$
- $U_2 \setminus U_1 : A_3$
- $U_3 \setminus U_2 : A_4, D_4$
- $U_4 \setminus U_3 : A_5, D_5$
- $U_5 \setminus U_4 : A_6, D_6, E_6$
- $U_6 \setminus U_5 : A_7, D_7, E_7, P_8$
- $U_7 \setminus U_6 : A_8, D_8, E_8, X_9, P_9$
- $U_8 \setminus U_7 : A_9, D_9, J_{10}, X_{10}, P_{10}, Q_{10}, R_{10}$

In the case of simple singularities there exist normal forms without local invariants, so $A_k (k \geq 1), D_k (k \geq 4), E_6, E_7$ and $E_8$ are points in $W$. The orbitspaces for the families $J_{10}, X_9, X_{10}, P_8, P_9, P_{10}, Q_{10}$ and $R_{10}$ will be described next.

(10.5) We gave normal forms for these families in the real case, already in (3.6). These forms can also be used in the complex case, but sometimes other normal forms are more practical. They are mentioned in the proof of (10.6) and also in the list at the end.

We shall investigate in these eight cases those values of the parameters for which the germs are equivalent. Next we take the quotient-space to this equivalence. We get a topological space (even a complex space), which can be identified with the corresponding subset in $W$. 
In each of these 8 cases, $f$ is finitely determined, say by its $k$-jet. So we can work entirely in $J^k(n,1)$ and have only to consider $k$-jets of mappings.

Let $f_t$ be a $k$-parameter family of germs. The condition $f_t(\phi(z)) = f_0(z)$ gives restrictions on the coefficients of $j^k(\phi)$. It can be verified in each case separately that $\phi$ has to be an element of $GL(n)$. This is left to the reader. Even in most of the cases the only possible action is multiplication by a scalar of each coordinate:

$$
\begin{align*}
  z_1 &= \alpha z_1 \\
  z_2 &= \beta z_2 \\
  z_3 &= \gamma z_3
\end{align*}
$$

We call this a **diagonal isomorphism**.

(10.6) Theorem:

a) The orbit spaces of $P_8$, $X_9$, $J_10$ and $Q_{10}$ are complex isomorphic with $C$.
b) The orbit spaces of $P_9$, $P_{10}$, $R_{10}$ and $X_{10}$ are complex isomorphic with $C\setminus\{0\}$.

**Proof:**

case $P_8$: $f(A,B) = z_1^3 + z_2^2 z_3 + Az_1z_3^2 + Bz_3^3$ with $4A^3 + 27B^2 \neq 0$.

If $f(A,B)(\phi(z)) = f(A',B')$ then $\phi$ must be a diagonal isomorphism and we get:

$$f(A',B') = \alpha^3 z_1^3 + \beta^2 y z_2^2 z_3 + A\alpha^2 z_1 z_3^2 + B\gamma^3 z_3^3$$

So $f(A,B)$ and $f(A',B')$ are equivalent $\iff$

$\exists \alpha, \beta, \gamma \in C\setminus\{0\}$ with $\alpha^3 = 1 \wedge \beta^2 \gamma = 1 \wedge A\alpha^2 B = A' \wedge \gamma^3 B = B'$ $\iff$

$\exists \alpha, \beta \in C\setminus\{0\}$ with $\alpha^3 = 1 \wedge A\beta^4 = A' \wedge \beta^6 B = B'$ $\iff$

$\exists \beta \in C\setminus\{0\}$ with $\beta^4 A = A' \wedge \beta^6 B = B'$.

Hence: $f(A,B) \sim f(A',B') \iff j(A,B) = j(A',B')$ where $j(A,B) = \frac{A^3}{4A^3 + 27B^2}$; the so-called $j$-invariant.

The orbits are characterized by $j \in C$; so the orbit space of $P_8$ is $C$. 
case \( J_{10} \): \( f(A, B) = z_1^3 + A z_2^4 + B z_2^6 \) with \( 4A^3 + 27B^2 \neq 0 \).

If \( f(A, B)(\phi(z)) = f(A', B')(z) \) then \( \phi \) can only be a diagonal isomorphism.

So \( f(A, B) \sim f(A', B') \iff \exists \alpha, \beta \in C - \{0\} \) with \( \alpha^3 = 1 \land \alpha^4 B = A' \land B^6 = B' \).

This case is similar to \( P_8 \) and the orbitspace is \( C \).

The orbits can be characterized by \( k(A, B) = \frac{A^3}{4A^3 + 27B^2} \in C \).

case \( X_9 \): \( f_d = z_1 z_2 (z_1 - z_2)(z_1 - d z_2) \) with \( d \neq 0, 1 \).

\( d \) is the cross ratio of the four complex lines \( f_d = 0 \).

\( f_d = f_d' \iff \) crossratio of \( f_d = 0 \) and \( f_d' = 0 \) are equal \( \iff \) \( d' \in \{ \frac{d}{1-d}, 1-d, \frac{1}{1-d}, \frac{d}{d-1}, \frac{d-1}{d} \} \).

Define \( c : C - \{0, 1\} \to C \) by:

\[
c(d) = d^2 + (\frac{1}{d})^2 + (1-d)^2 + (\frac{1}{1-d})^2 + (\frac{d}{d-1})^2 + (\frac{d-1}{d})^2,
\]

\[
c(d) = \frac{2d^6 - 6d^5 + 9d^4 - 8d^3 + 9d^2 - 6d + 2}{(d-1)^2 d^2}
\]

The map \( c : C - \{0, 1\} \to C \) is surjective and for every \( q \in C \) there are at most six solutions of \( c(d) = q \). The definition of \( c \) implies, that with any solution \( d \) also \( \frac{1}{d}, 1-d, \frac{1}{1-d}, \frac{d}{d-1}, \frac{d-1}{d} \) are solutions.

This shows that the orbitspace of \( X_9 \) is \( C \).

case \( P_9 \): \( f_A = z_1 z_2 z_3 + z_1^3 + z_2^3 + A z_3^4 \) with \( A \neq 0 \).

If \( f_A(\phi(z)) = f_{A'}(z) \) then either \( \phi \) is a diagonal isomorphism or \( \phi \) is defined by: \( \phi(z_1) = \beta z_2; \phi(z_2) = az_1; \phi(z_3) = \gamma z_3 \).

In both cases:

\( f_A \sim f_{A'} \iff \exists \alpha, \beta, \gamma \in C - \{0\} \) with \( a^3 = 1, a^4 = \alpha^3 A = A' \iff \exists \gamma \in C - \{0\} \) with \( \gamma^3 = 1 \land \gamma A = A' \).

We get the orbitspace of \( P_9 \) if we divide \( C - \{0\} \) by the \( Z_3 \)-action of multiplication by 3rd root of unity. This gives \( C - \{0\} \).
case $P_{10}: f_A = z_1z_2z_3 + z_1^3 + z_2^3 + Az_3^5$ with $A \neq 0$.

This case is similar to $P_9$, we get again $Z_3$-action on $\mathbb{C}-\{0\}$.

\[ \text{case } Q_{10}: f_A = z_1^3 + z_2^2z_3 + Az_3z_3^3 + z_3^4. \]

This case is similar to $P_9$, we get now $Z_{12}$-action on $\mathbb{C}$.

\[ \text{case } R_{10}: f_A = z_1z_2z_3 + z_1^3 + z_2^4 + Az_3^4 \text{ with } A \neq 0. \]

This case is similar to $P_9$, we get $Z_3$-action on $\mathbb{C}-\{0\}$.

\[ \text{case } X_{10}: f_A = z_1^4 + z_2^2z_3^2 + Az_3^5 \text{ with } A \neq 0. \]

This case is similar to $P_9$, we get $Z_4$-action on $\mathbb{C}-\{0\}$.

(10.7) We define $K(w) = \{ w' \in W | w' \in U \text{ for every open set } U \text{ in } W, \text{ containing } w \}$.

**Lemma:** $w_1$ adjacent to $w_2$ if and only if $K(w_1) \subseteq K(w_2)$.

This is clear from the definitions of adjacency and of $K$.

**Examples:**

\[
\begin{align*}
K(A_s) &= A_s \cup A_{s-1} \cup \ldots \cup A_1 \\
K(D_s) &= D_s \cup D_{s-1} \cup \ldots \cup D_4 \cup A_s \cup A_{s-1} \cup A_{s-2} \cup \ldots \cup A_1 \\
K(E_6) &= E_6 \cup E_5 \cup E_4 \cup A_5 \cup A_4 \cup \ldots \cup A_1 \\
K(E_7) &= E_7 \cup E_6 \cup E_5 \cup E_4 \cup A_6 \cup A_5 \cup A_4 \cup \ldots \cup A_1 \\
K(E_8) &= E_8 \cup E_7 \cup E_6 \cup E_5 \cup E_4 \cup A_7 \cup A_6 \cup A_5 \cup A_4 \cup \ldots \cup A_1 \\
K(F_8(j_0)) &= F_8(j_0) \cup K(E_6) \text{ (not open)}
\end{align*}
\]

(10.8) We are interested in the orbits that occur, when we perturb a given orbit $w$ a "little". This means that we have to study small open neighborhoods of $w$ in $W$. Those open sets certainly contain $K(w)$. In the case of simple singularity $K(w)$ is the smallest open set containing $w$.

In the sequel we try to describe some of the open neighborhoods of $w$ if $w \in U_8$. We remark that $U_8$ consists of a finite number of points and a finite number of copies of $\mathbb{C}$ and $\mathbb{C}-\{0\}$, each in itself having induced the usual Hausdorff topology. So if $w$ is not-simple in $U_8$ every
neighborhood of \( w \) contains at least an open neighborhood of \( w \) in \( C \) or \( C-(0) \). \( C \) can be embedded in \( S^2 \) by adding one point (call it \( \omega \)). Then open neighborhoods of \( \omega \) in \( C \) are defined in the usual way.

**10.9 Adjacency in corank 3.**

We consider now \( V_3 = P_9 \cup P_9 \cup P_{10} \cup P_{10} \cup P_{10} \) in the relative topology (see figure).

**Case \( P_9 \):** A point \( j \in P_9 = C \) corresponds with a germ

\[
z_1^3 + z_2^2 z_3 + \varepsilon_1 z_1 z_3^2 + \varepsilon_2 z_3^3 \text{ such that } \frac{4\varepsilon_1^2}{4\varepsilon_1^3 + 27\varepsilon_2^2} = j.
\]

An open set of \( j \) in \( V_3 \) is an open neighborhood of \( j \in P_9 = C \) in the usual topology of \( C \).

**Case \( P_9 \):** Points \( w \) of \( P_9 = C-(0) \) can be given by:

\[
z_1^3 + z_2^2 z_3 + z_1^2 z_3 + A z_3^4 \text{ with } A \neq 0, \text{ or also by:}
\]

\[
z_1^3 + z_2^2 z_3 + \varepsilon_1 z_1 z_3^2 + \varepsilon_2 z_3^3 + A' z_3^4 \text{ with } A' \neq 0, \text{ where } \varepsilon_1 \text{ and } \varepsilon_2 \text{ satisfy } 4\varepsilon_1^3 + 27\varepsilon_2^2 = 0 \text{ and } (\varepsilon_1, \varepsilon_2) \neq (0,0).
\]

So an open neighborhood of \( w \in P_9 = C-(0) \) in \( V_3 \) consists of:

1° an open neighborhood of \( w \) in \( P_9 = C-(0) \)
2° an open neighborhood of \( \omega \) in \( C = P_9 \).

\( P_9 \) is \( u \)-adjacent to \( P_9 \), but not adjacent.

**Case \( P_{10} \):** In a similar way as in case \( P_9 \) one concludes that an open neighborhood of \( w \in P_{10} = C-(0) \) in \( V_3 \) consists of:

1° an open neighborhood of \( w \) in \( C-(0) = P_{10} \)
2° an open neighborhood of \( 0 \) in \( C-(0) = P_9 \)
3° an open neighborhood of \( \omega \) in \( C = P_8 \).

\( P_8 \) is \( u \)-adjacent to \( P_{10} \), but not adjacent.

\( P_9 \) is \( u \)-adjacent to \( P_{10} \), but not adjacent.
case $Q_{10}$: A point $w \in Q_{10} = \mathbb{C}$ can be given by $z_1^3 + z_2^2 z_3 + A z_3^3 + z_4^4$.

We consider its universal deformation and omit terms of degree $\leq 2$:

$$z_1^3 + z_2^2 z_3 + \lambda_1 z_2 z_3^2 + \lambda_2 z_3^3 + (A + \lambda_3) z_4 z_3^3 + z_3^4.$$  

Let $4 \lambda_1^3 + 27 \lambda_2^2 \neq 0$. The equation $\frac{1}{4} \lambda_1^3 + 27 \lambda_2^2 = j$ has for every $j \in \mathbb{C}$ solutions $(\lambda_1, \lambda_2)$ arbitrarily close to $(0,0)$. So we get all the members of the family $P_8$ in the deformation.

If $4 \lambda_1^3 + 27 \lambda_2^2 = 0$ and $(\lambda_1, \lambda_2) \neq (0,0)$ we get germs of type $P_9$. We next change coordinates and transform the germ in the normalform $z_1^3 + z_2^2 z_3 + z_2 z_3^2 + w z_3^4$ of $P_9$. Then the coefficient of $z_3^4$ goes to $0$ if $(\lambda_1, \lambda_2, \lambda_3) \to (0,0,0)$.

So an open neighborhood of $w \in Q_{10} = \mathbb{C}$ in $V_3$ consists of:

1. an open neighborhood of $w$ in $\mathbb{C} = Q_{10}$
2. an open neighborhood of $w$ in $\mathbb{C} - \{0\} = P_9$
3. the whole set $P_8$

This discussion shows:

$P_8$ is adjacent to every germ in the family $Q_{10}$.

$P_9$ is $u$-adjacent to $P_{10}$, but not adjacent.

case $R_{10}$: A point $w \in R_{10}$ can be given by $z_1^3 + z_2 z_3^2 + z_4^4 + A z_3^4$.

We consider its universal deformation and omit terms of degree $\leq 2$:

$$z_1^3 + z_1 z_2 z_3 + \lambda_1 z_2^3 + \lambda_2 z_3^3 + z_4^4 + (A + \lambda_3) z_3^4.$$  

The 3-jet is of type $P_8$ if $\lambda_1 \lambda_2 \neq 0$. The $j$-invariant tends to $0$ if $(\lambda_1, \lambda_2) \to (0,0)$ since $R_{10}$ has doublepoints. If $\lambda_2 = 0 \land \lambda_1 \neq 0$ we have a germ of type $P_9$. A coordinatechange to the normalform $z_1^3 + z_1 z_2 z_3 + z_2^3 + w z_3^4$ shows that $u \to 0$ if $(\lambda_1, \lambda_2) \to (0,0)$.

So an open neighborhood of $w \in R_{10} = \mathbb{C} - \{0\}$ in $V_3$ consists of
1° an open neighborhood of \( w \) in \( C - \{0\} = R_{10} \\
2° an open neighborhood of 0 in \( C - \{0\} = P_9 \\
3° an open neighborhood of \( \infty \) in \( C = P_8 \)

\( P_8 \) is \( u \)-adjacent to \( R_{10} \), but not adjacent.
\( P_9 \) is \( u \)-adjacent to \( R_{10} \), but not adjacent.

(10.10) Adjacency relations of \( X_9 \).

The adjacency of \( X_9 \) to \( X_{10} \) is already discussed in (9.12). Every open
neighborhood of \( w \in X_9 = C \) contains an open neighborhood of \( w \) in
\( C = X_9 \). Every open neighborhood of \( w \in X_{10} = C - \{0\} \) contains:
1° an open neighborhood of \( w \) in \( C - \{0\} = X_{10} \\
2° an open neighborhood of \( \infty \) in \( C = X_9 \).

Next we study the adjacency of \( X_9 \) with \( P_{10}, Q_{10} \) and \( R_{10} \).

Let \( f_t(z_1, z_2, z_3) \) be of type \( X_9 \) if \( t \neq 0 \) and of type \( P_{10}, Q_{10} \) or \( R_{10} \)
if \( t = 0 \). After change of coordinates we can arrange, that the 3-jet
of \( f_t \) has the form:

\[
g_t = t z_3^2 + z_3 \phi_t(z_1, z_2, z_3) + \sigma_t(z_1, z_2).
\]

For \( t \neq 0 \) holds:

\[
g_t = t(z_3^2 + \frac{\phi_t(z_1, z_2)}{2t} z_3^2) + \sigma_t(z_1, z_2) - \frac{[\phi_t(z_1, z_2)]^2}{4t}.
\]

Since \( f_t \) is of type \( X_9 \) we have \( \sigma_t(z_1, z_2) = 0 \) for \( t \neq 0 \). The continuity
of \( f_t \) implies \( \sigma_t(z_1, z_2) \equiv 0 \). Since \( g_0 = 0 \) is a reducible curve in
\( P^2(C) \), \( f \) is not of type \( P_{10} \) or \( Q_{10} \).

Remark: The same reasoning shows that there are no adjacency relations
between:

\[
X_k: z_1 \frac{4}{k} + z_1^2 z_2^2 + A z_2^{k-5} + z_3^2 \quad (k \geq 9)
\]

and \( P_\ell: z_1 z_2 z_3 + z_1^3 + z_2^3 + A z_3^{\ell-5} \) \( (\ell \geq 9) \)

This is the first example of such a situation.
The following curve shows that $X_9$ is $u$-adjacent to $R_{10}$ and that every open neighborhood of $w \in X_{10} = \mathbb{C}\setminus \{0\}$ contains a neighborhood of $\infty$ in $\mathbb{C} = X_9$:

$$f_t = t z_3^2 + z_1 z_2 z_3 + z_3^3 + A z_1^4 + z_2^4 - \frac{2 z_1^2 z_2^2}{ht} + z_2^4.$$  

This shows that $X_9$ is $u$-adjacent to $R_{10}$. Whether $X_9$ is adjacent to $X_{10}$ is unknown to me.

(10.11) Theorem:

a) $E_7$ is adjacent to $P_9$

b) $E_8$ is adjacent to $P_{10}$

c) $E_8$ is adjacent to $Q_{10}$

d) $E_7$ is adjacent to $R_{10}$ and $E_8$ is not adjacent to $R_{10}$

e) $E_8$ is adjacent to $X_{10}$.

Proof:

a) $f_t = t^2 z_3^2 + z_3^3 (z_1 z_2 + 2 t z_2^2 + z_3^2) + z_1^3 + z_2^4$  
$f_0 = z_1 z_2 z_3 + z_1^3 + z_3^3 + z_2^4$ has type $P_9$

For $t \neq 0$ we can transform $f_t$ in the normalform of $E_7$.

b) $f_t = t z_3^2 + z_3^3 (z_1 z_2 + 2 t z_2^2 + z_3^2) + z_1^3 + t z_1 z_2^3 + t^2 z_2^4 + z_2^5$  
$f_0 = z_1 z_2 z_3 + z_1^3 + z_3^3 + z_2^5$ has type $P_{10}$

For $t \neq 0$ we can transform $f_t$ in the normalform of $E_8$.

c) $f_t = t^2 z_3^2 + z_3^3 (2 t z_2^2 + t z_1 z_2^3 + z_2 z_3) + z_1^3 + z_1 z_2^3 + z_2^4 + t z_2^5$.  
$f_0 = z_2^2 z_3 + z_1^3 + z_1 z_2^3 + z_2^4$ has type $Q_{10}$

For $t \neq 0$ we can transform $f_t$ in the normalform of $E_8$.

d) We shall show in (10.12) that $D_{10}$ is not adjacent to $R_{10}$. Since $D_7$ is adjacent to $E_8$, this shows that $E_8$ is not adjacent to $R_{10}$.

Since $P_9$ is $u$-adjacent to $R_{10}$ and $E_7$ is adjacent to $P_9$ also $E_7$ is adjacent to $R_{10}$.
e) $f_t = t z_3^3 + z_1^4 + z_1^2 z_2^2 + Az_2^5$

$f_0 = z_1^4 + z_1^2 z_2^2 + Az_2^5$ is of type $X_{10}$

If $t \neq 0$ we can transform $f_t$ in the normal form of $E_8$.

(10.12) Theorem:

a) $D_7$ is not adjacent to $P_9$
b) $D_7$ is not adjacent to $R_{10}$
c) $D_8$ is not adjacent to $Q_{10}$
d) $D_8$ is not adjacent to $P_{10}$
e) $D_8$ is not adjacent to $X_{10}$.

Proof:
a) and b):

Let $\phi_t(z_1, z_2, z_3)$ be of type $D_7$ if $t \neq 0$. After change of coordinates we can assume that the $4$-jet of $\phi_t$ has the form:

$$t^2 z_3^2 + z_3 f(z_1, z_2, z_3) + z_1^2 z_2^2 + z_3 g(z_1, z_2, z_3) + o(z_1, z_2)$$

where:

$$f(z_1, z_2, z_3) = \gamma_{11} z_1^2 + \gamma_{22} z_2^2 + \gamma_{33} z_3^2 + \gamma_{12} z_1 z_2 + \gamma_{13} z_1 z_3 + \gamma_{23} z_2 z_3$$

$$g(z_1, z_2, z_3) = p_0 z_1^4 + p_1 z_1^2 z_2^2 + p_2 z_1^2 z_3^2 + p_3 z_1 z_2^3 + p_4 z_2^4$$

Assume that $\phi_0$ is of type $P_9$ or $R_{10}$ then the universal deformation shows that it is sufficient to study only the $4$-jet of $\phi_t$.

For $t \neq 0$ we apply the substitution $z_3 = z_3 - \frac{1}{2t^2} f(z_1, z_2, z_3)$.

Then the coefficient of $z_2^4$ becomes $-\frac{\gamma_{22}}{4t^2} + p_4$.

The coefficient of $z_1 z_2^3$ becomes $A = -\frac{\gamma_{12} \gamma_{22}}{2t^2} + p_3$.

The coefficient of $z_2^5$ becomes $B = \frac{\gamma_{23} \gamma_{22}}{4t^4} - \frac{\gamma_{22} \gamma_{33}}{2t^2}$.

If $-\frac{\gamma_{22}}{4t^2} + p_4 \neq 0$ we can transform $\phi_t$ in the normal form of $D_7$.

Let now $\gamma_{22} = 2t \sqrt{p_4}$, so the coefficient of $z_2^4$ vanishes. Then $\phi$ can be transformed in $tz_3^2 + z_1^2 z_2 + Az_2^3 + Bz_2^5$, and next in $tz_3^2 + z_1^2 z_2 + (B - \frac{A^2}{2})z_2^5$. 

If \( B - \frac{A^2}{4} \neq 0 \) we have an orbit of type \( D_6 \).

If \( B - \frac{A^2}{4} = 0 \) then:

\[
\frac{\gamma_{23}p_4}{t^2} - \sqrt{p_4} a_{222} = \frac{1}{4} \left( - \frac{\gamma_{12}p_4}{t^2} + p_3 \right)^2
\]

\[
\gamma_{23}p_4 - \sqrt{p_4} a_{222} = \frac{1}{4} \left( - \frac{\gamma_{12}p_4}{t^2} + p_3 t \right)^2.
\]

When we take the limit for \( t \to 0 \) we get:

\[
\gamma_{23}p_4 = \frac{1}{4} \gamma_{12}^2 p_4.
\]

1° If \( \gamma_{23} = \frac{1}{4} \gamma_{12}^2 \), then the point \((0:1:0)\) is a multiplepoint of the cubic curve:

\[
\gamma_{11}z_1^2z_3 + \gamma_{33}z_3^3 + \gamma_{12}z_1z_2z_3 + \gamma_{23}z_2^2z_3^2 + \gamma_{13}z_1z_3^2 + z_1^2z_2z_3.
\]

The tangents in this point satisfy:

\[
z_1^2 + \gamma_{12}z_1z_3 + \gamma_{23}z_3^2.
\]

So if \( \gamma_{12}^2 - 4\gamma_{23} = 0 \) the tangents coincide. Hence the point \((0:1:0)\) is no doublepoint and so \( \phi_0 \) is not of type \( P_9 \) or \( R_{10} \).

2° If \( \lim_{t \to 0} p_4 = 0 \) we proceed as follows. We can assume \( 4\gamma_{23} - \gamma_{12}^2 \neq 0 \).

A coordinate change in the \( z_1-z_3 \)-space takes the 3-jet in the form:

\[
z_1z_2z_3 + az_1^3 + bz_3^3.
\]

Since \( p_4 \) is still the coefficient of \( z_3^4 \) the singularity is not of type \( P_9 \) or \( R_{10} \).

This shows part a) and b).

The proof of part c) and d) is similar, although longer and more complicated. It will be omitted.

e) Let \( \phi_t(z_1,z_2) \) be of type \( D_8 \) if \( t \neq 0 \); after change of coordinates we can assume that 5-jet of \( \phi_t \) has the form:

\[
tz_1^2z_2 + G_4(z_1,z_2) + G_5(z_1,z_2)
\]

where:

\[
G_4(z_1,z_2) = p_0z_1^4 + p_1z_1^3z_2 + p_2z_1^2z_2^2 + p_3z_1z_2^3 + p_4z_2^4
\]

\[
G_5(z_1,z_2) = q_0z_1^5 + \ldots + q_4z_1z_2^4 + q_5z_2^5.
\]
If $\phi$ is of type $X_{10}$, then it is sufficient to study the 4-jet.

If $p_4 \neq 0$, then $\phi_t$ is of type $D_5$ if $t \neq 0$. Suppose $p_4 = 0$.

After change of coordinates in the following way:

\[
\begin{align*}
    z_1 &= z_1 - \frac{1}{2t} p_3 z_2^2 \\
    z_2 &= z_2 - \frac{1}{t} \{ p_0 z_1^2 + p_1 z_1 z_2 + p_2 z_2^2 \}
\end{align*}
\]

the 4-jet of $\phi_t(t \neq 0)$ is given by: $t z_1^2 z_2^2$.

The coefficients of $z_2^5$ is now: $q_5 = \frac{1}{4t} p_3^2$.

The coefficient of $z_2^6$ is now: $\frac{3}{2t^2} p_2^2 p_3^2 - \frac{1}{2t} q_4 p_3$.

If $\phi_t$ is of type $D_5$ then these two coefficients must vanish.

(modifications of the 5-jet give no contributions on terms of degree 6).

So:

1. $p_3^2 = 4 t q_5$. Hence $\lim_{t \to 0} p_3 = 0$.

Since $p_4 = 0$ this implies that $\lim_{t \to 0} q_5 \neq 0$, for otherwise $\phi$ is not of type $X_{10}$.

2. $3 p_2 p_3^2 = t q_4 p_3$. Now is $p_3 \neq 0$ since otherwise $q_5 = 0$.

So: $3 p_2 = t q_4 p_3$ and $\lim_{t \to 0} p_2 \neq 0$ since otherwise $\phi$ is not of type $X_{10}$.

So $p_3 = \frac{t q_4}{3 p_2}$; hence $q_5 = \frac{t q_4^2}{36 p_2}$.

There follows that $\lim_{t \to 0} q_5 = 0$ and this contradicts the fact that $\phi$ is of type $X_{10}$.

(10.13) Remark

I didn't succeed in computing the adjacency of $A_7$ to $P_9$ and $A_8$ to $P_{10}$, $Q_{10}$ and $R_{10}$. For the adjacency of $A_8$ to $X_{10}$ see (10.15).

All the other $(\mu)$-adjacency relations are given in the list at the end. This list gives also information about the partial ordering of open sets in the part $U_8$ of the orbitspace $W$. 
The graph of simple and simple elliptic singularities is extended to $U_8$. If we add dimension arguments, semi-continuity of corank, etc, the proof is given in the sections (10.9) to (10.12).

(10.14) Remark
A comparison of list II of the diagrams of intersection matrices and list III of the $\mu$-adjacency raises the question if the following remains true:
$g$ is $\mu$-adjacent to $f$ if and only if the diagram of $g$ is contained in the diagram of $f$.
In list III there is no counterexample to this conjecture.

(10.15) Remark (added in proof):
A'CAMPO informed me that he has developed a new geometric way of computing the intersection matrix for singularities with corank 0 and 1. With this method he can also show that $A_8$ is adjacent to $X_{10}$. 
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SAMENVATTING

We bestuderen de Rechts-equivalentie van kiemen van reële en complexe functies.

In deel I geven we de volledige classificatie voor codimensie kleiner dan of gelijk aan negen. Speciale aandacht wordt besteed aan de equivalentie in k-parameter families, het verschil tussen Rechts-equivalentie en Rechts-links-equivalentie en aan de algebraïsche conditie voor k-bepaaldheid.

In deel II beschouwen we in het complex-analytische geval benaderingen van een functiekiem. We bestuderen de relatie tussen de intersectievormen en de monodromiegroepen van een kiem en zijn benaderingen. Als toepassing behandelen we stellingen over de nabijheidsrelatie van simpele en simpele elliptische singulariteiten van Arnold en Saito. We besluiten met een gedeeltelijke beschrijving van de topologie van de ruimte van de Rechts-equivalentieklassen.

SUMMARY

We study the Right-equivalence of germs of real and complex functions.

In part I the complete classification for codimension smaller than or equal to nine is given. Special attention is given to equivalence in k-parameter families, the difference between Right-equivalence and Right-left-equivalence, and to the algebraic condition for a germ to be k-determined.

In part II we consider in the complex-analytic case approximations of a function germ. We study the relation between intersection forms and monodromy groups of a germ and its approximations. As an application we cover theorems on the adjacency relation of simple and simple elliptic singularities by Arnold and Saito. We conclude with a partial description of the topology of the orbit space.
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1. Laten $f$ en $g$ elementen van $\mathcal{A}_n$ zijn met de eigenschap, dat de algebra's $\frac{m_n}{\Delta(f)}$ en $\frac{m_n}{\Delta(g)}$ isomorf zijn. De vraag van Takens of dan $f$ en $g$ rechts-equivalent zijn, moet in het algemeen ontkennend beantwoord worden.

Takens, F.: Singularities of functions and vectorfields.
Nieuw Archief voor Wiskunde (3), XX, (1972), 107-130.

2. Stel $V$ een algebraïsch oppervlak in $\mathbb{C}^3$ met een geïsoleerde singulariteit in de oorsprong. Laat $f$ de intersectiematrix zijn van de locale naburige vezel en zij $f'$ de intersectiematrix van een goede resolutie van $V$.
De bewering van Durfee, dat $f$ en $f'$ stabiel equivalent zijn, is onjuist.


3. Zij $G$ een eindige ondergroep van $GL(n)$. Noteer door $\mathcal{A}(G)_n$, resp. $L(G)_n$ de elementen van $\mathcal{A}_n$, resp. $L_n$, die invariant zijn onder alle elementen van $G$.
$f \in \mathcal{A}(G)_n$ heet $k$-$G$-bepalend als voor elke $g \in \mathcal{A}(G)_n$ geldt:
Als $f_k = g_k$ dan is er een $\phi$ in $L(G)_n$ met $f\phi = g$.
Er geldt dan:
1) Als $m_n^{k+1} \cap \mathcal{A}(G)_n \subseteq m_n(\Delta(f) \cap \mathcal{A}(G)_n)$ dan is $f$ $k$-$G$-bepalend.
2) Als $f$ $k$-$G$-bepalend is, dan is $m_n^{k+1} \cap \mathcal{A}(G)_n \subseteq m_n(\Delta(f) \cap \mathcal{A}(G)_n)$.
4. Het tegenvoorbeeld (5.2) uit dit proefschrift toont tevens voor elke \( p > 0 \) de onjuistheid aan van de bewering:

\[ f \text{ is } k\text{-bepalend dan en slechts dan als } m^{k+p} \subset m^{1+p} \Delta(f) + m^{k+p+1}. \]

5. De Boardmansymbolen van \( f \) en van zijn universele ontvouwing \( F \)

zijn gelijk.

Mather, J.: On Thom-Boardman singularities.

6. De opgave 4a van het herexamen Wiskunde I van het V.W.O. in 1972 (Gymnasium en Atheneum) luidde als volgt:

"Een functie \( f \) is voor \(-6 \leq x \leq 3\) gedefinieerd door

\[ f(x) = 2x + 3 \sqrt[3]{(x-2)^2}. \]

Onderzoek of de functie differentieerbaar is voor \( x = 2.\)"

De commissie bedoeld in art. 27 lid 5 van het Besluit eindexamens V.W.O.-H.A.V.O.-M.A.V.O. maakt een essentiële gedachtenfout als zij in de bindende normen voor de beoordeling van het schriftelijk werk aangeeft, dat er 2 punten moeten worden afgetrokken indien

\[ \lim_{x \to 2} f'(x) \text{ en } \lim_{x \to 2} f''(x) \]

niet apart onderzocht zijn.

7. Het door Hadeler gegeven bewijs van de stelling, dat elke continue functie op \([a,b]\) daar ook integreerbaar is, is onvolledig.

Hadeler, K.P.: "Mathematik für Biologen".
Heidelberger Taschenbücher Band 129 (1974).


9. Een verdere ontsluiting van het gebergte door wegen, kabelbanen en hotels in de hoogalpine regionen dient voorkomen te worden.
10. Gezien de hoogte van de prijzen van wiskundeboeken in Nederland kan men deze beter uit het buitenland betrekken. Met name de Universiteitsbibliotheek zou van deze mogelijkheid gebruik moeten kunnen maken.

11. Het is merkwaardig, dat in het verplichte wiskunde-programma voor scheikunde-studenten aan de Universiteit van Amsterdam geen lineaire algebra voorkomt.

12. De periode van 3 jaar, waarin een eervol ontslagen hoogleraar als promotor kan optreden, dient verlengd te kunnen worden.

Stellingen behorende bij het proefschrift "Classification and deformation of singularities" van D. Siersma, Amsterdam, juli 1974.
List of diagrams of intersection matrices.

\[ \text{negative definite} \]

\[ \text{After dividing by 1-dim. kernel} \]