CLASSIFICATION AND DEFORMATION OF SINGULARITIES

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INTRODUCTION

Consider the set J^N of real (or complex) polynomials in n variables of degree $\leq$ N. (N a large natural number). The polynomials with critical point $\underline{0}$ are called singularities. They form a linear subspace S of J^N. A polynomial g is called Right-equivalent with f if it can be derived from f by applying a smooth coordinatetransformation of $\mathbb{R}^n$. The Right-equivalenceclasses are immersed submanifolds in J^N.
One can consider the following two problems:
Classificationproblem: Give a list of Right-equivalenceclasses for increasing codimension in S.
Adjacencyproblem: Give a description of the Right-equivalenceclasses, that can occur for arbitrarily small perturbations of a given polynomial. Or more general: Describe the topology of the set of Right-equivalenceclasses.

In part I of this thesis the classificationproblem is treated in the context of germs of real functions. It depends heavily on the work of MATHER [19] and uses also WASSERMANN [27], who gave in his thesis a generalization of Mather's work to the Rightleft-case.
In §1 I recall definitions and theorems concerning Right-equivalence. In §3 a list of equivalenceclasses with codimension $\leq$ 9 is presented. The full proof is given in §4. One of the reasons for treating the problem of equivalence in k-parameter families of germs (in §2) is the existence of 1-parameter families in my list. §5 contains a counterexample to a conjecture of Zeeman, concerning an algebraic condition for a polynomial to be k-determined (for the definition see §1). Moreover I discuss in §5 the classification under Right-left-equivalence.

Professor R. Thom began the theory and classified the singularities in codimension smaller than or equal to four. He used the theory for a study of morphogenesis (to be applied in various sciences), introduced the notion of universal unfolding and posed a large number of important and hard mathematical problems.
In 1968-1970 J. Mather solved a number of these in his fundamental papers on Right-equivalence of functions. A preliminary manuscript was informally distributed. He did not conclude this work in the form of a paper however. Many mathematicians showed interest in the manuscript, which contained interesting new definitions and theorems on universal unfoldings.
In 1970-71 the manuscript was studied in a seminar of Professor N.H. Kuyper at the University of Amsterdam. During the next year (1971-1972) I started my research on the classification of singularities of real smooth functions and improved the classification for codimension ≤ 5 by Mather to the classification in codimension ≤ 8. See [23], in which the results are formulated with some indications of the proof.

After the publication of my list for codimension ≤ 8, independent ARNOLD [1] published end 1972 a paper, in which he gave among others a list of the so-called simple singularities. This was a subset of my list, but complete with respect to the interesting simplicity problem. Very recently there appeared two papers of ARNOLD [2] and [3], in which he gave a very extensive list, namely of all families with 0 and 1 parameter. It refers also to my paper and contains all singularities of codimension ≤ 12. I believe that my presentation is still of some independent interest, since Arnold omits the proofs and only treat Right-equivalence in the complex case, and my presentation includes the real case and Right-left-equivalence.

The adjacencyproblem is treated in part II of this thesis. I study there the complex analytic case, in which the Milnorfibration gives some topological invariants. We refer to p. 62 for the introduction of part II. The results are illustrated in list 3 at the end.

PART I: CLASSIFICATION

§1 EQUIVALENCE AND FINITE DETERMINACY OF GERMS

In this § we recall some definitions and theorems. As general references we give MATHER[19] and WASSERMANN[27].

<u>(1.1)</u> Let X and Y be topological spaces and let $x \in X$. Two C^∞-mappings $f : U \to Y$ and $g : V \to Y$ where U and V are neighborhoods of x in X are called <u>germ-equivalent</u> at x if there is a neighborhood $W \subset U \cap V$ of x in X such that $g|W = h|W$. The equivalenceclasses are called <u>mapgerms</u> at $x \in X$ from X into Y. We denote by $\hat{f} : X \to Y$ the equivalenceclass, containing $f : U \to Y$. Composition of mapgerms is defined by composition of the representatives.
We denote by $\mathcal{E}_n$ the set of germs at $\underline{0} \in \mathbb{R}^n$ of C^∞-functions $f : \mathbb{R}^n \to \mathbb{R}$. $\mathcal{E}_n$ has a natural $\mathbb{R}$-algebra structure induced from $\mathbb{R}$. As a ring $\mathcal{E}_n$ has a unique <u>maximal ideal</u> m_n; m_n is the set of germs $f : \mathbb{R}^n \to \mathbb{R}$ at $\underline{0}$ such that $f(\underline{0}) = 0$.
L_n is the set of germs at $\underline{0} \in \mathbb{R}^n$ of C^∞-diffeomorphisms $\phi : (\mathbb{R}^n,\underline{0}) \to (\mathbb{R}^n,\underline{0})$. Every $\hat{\phi} \in L_n$ has the properties $\phi(\underline{0}) = \underline{0}$ and $d\phi(\underline{0})$ has maximal rank. We can make L_n into a group by taking as the group operation the composition of mapgerms.

<u>(1.2)</u> Let X and Y be C^∞-manifolds and let $x \in X$ and $k \in \mathbb{N} \cup \{0\}$. Two C^∞-mappings $f : U \to Y$ and $g : V \to Y$ where U and V are neighborhoods of x in X are called <u>k-jet-equivalent at x</u> if and only if $f(x) = g(x)$ and all their partial derivatives of order $\leq k$ at x agree (in some, and hence in any system of local coordinates). The equivalence-classes are called <u>k-jets</u>. The equivalenceclass at x, containing

$f : U \to Y$ is denoted by $j_x^k(f)$.
$J^k(n,1)$ is the set of k-jets at $\underline{0} \in \mathbb{R}^n$ of C^∞-mappings $f : \mathbb{R}^n \to \mathbb{R}$
$J^k(n,1)$ has a natural vectorspace structure and is isomorphic with the vectorspace of all polynomials in $x_1,\ldots,x_n$ of degree $\leq k$.
$J^k(n,1)$ contains the subspace $J_o^k(n,1) = \{z = j^k(f) \in J^k(n,1) | f(0)=0\}$.
By f_k we denote the Taylorseries of f at $\underline{0} \in \mathbb{R}^n$ up to the k^{th} degree terms. Two mappings f and g are clearly k-jet-equivalent iff $f_k = g_k$.
$L^k(n)$ is the set of k-jets at $\underline{0} \in \mathbb{R}^n$ of C^∞-diffeomorphisms $\phi : (\mathbb{R}^n,\underline{0}) \to (\mathbb{R}^n,0)$. $L^k(n)$ is a group with the composition (take representatives) as product . This action is well-defined, since $(g.f)_k$ depends only on g_k and f_k.
For simplicity, we shall often indicate germs and jets by giving the name of a representative.

(1.3) There exist canonical projections:

$$\to J^{k+1}(n,1) \xrightarrow{\Pi_k^{k+1}} J^k(n,1) \to J^{k-1}(n,1) \to \cdots \to J^1(n,1) \xrightarrow{\Pi_o^1} J^o(n,1)$$

and $\mathcal{E}_n \xrightarrow{\Pi_k} J^k(n,1)$ defined in an obvious way.
For the maximal ideal $m_n = \mathrm{Ker}[\Pi_o : \mathcal{E}_n \to J^o(n,1)]$ we have $m_n^2 = \mathrm{Ker}[\Pi_1 : \mathcal{E}_n \to J^1(n,1)]$. An element $\hat{f} \in m_n^2$ is called a singular germ or a singularity. This condition is equivalent to $f(\underline{0}) = 0$ and $df(\underline{0}) = \underline{0}$, or to $f_1 \equiv 0$.

(1.4) Two germs $\hat{f},\hat{g} \in m_n$ are called (Right)-equivalent if there exists a $\hat{\phi} \in L_n$ such that $f = g\phi$. Notation: $\hat{f} \sim \hat{g}$ (or $\hat{f} \underset{R}{\sim} \hat{g}$). Two germs $\hat{f},\hat{g} \in m_n$ are called Right-leftequivalent if there exist $\hat{\phi} \in L_n$ and $\hat{\psi} \in L_1$ such that $\psi f = g\phi$. Notation: $\hat{f} \underset{RL}{\sim} \hat{g}$.
Two k-jets $j^k(f)$ and $j^k(g) \in J_o^k(n,1)$ are called (Right)-equivalent if there exists a $j^k(\phi) \in L^k(n)$ such that $f_k = (g\phi)_k$. Notation: $j^k(f) \overset{k}{\sim} j^k(g)$ or $f \overset{k}{\sim} g$ or $f \underset{R}{\overset{k}{\sim}} g$. Two k-jets $j^k(f)$ and $j^k(g) \in J_o^k(n,1)$ are called Right-left-equivalent if there exist $j^k(\phi) \in L^k(n)$ and $j^k(\psi) \in L^k(1)$ such that $(\psi f)_k = (g\phi)_k$. Notation: $f \underset{RL}{\overset{k}{\sim}} g$. The group L_n acts on m_n by composition on the right; the R-equivalence-classes are the orbits of this groupaction. The group $L_1 \times L_n$ acts on m_n by composition on the right with elements of L_n and on the left with

elements of L_1. The RL-equivalenceclasses are the orbits of this groupaction. Notations: $\mathrm{Orb}(\hat{f})$ and $\mathrm{Orb}_R(\hat{f})$ for the Right-equivalence-classes and $\mathrm{Orb}_{RL}(\hat{f})$ for the Right-left-equivalenceclasses. The ideals $m_n{}^k$ are invariant under the two groupactions.
In a similar way there are groupactions of $L^k(n)$ on $J_o^k(n,1)$ and of $L^k(1) \times L^k(n)$ on $J_o^k(n,1)$. The orbits are denoted by $\mathrm{Orb}^k(f)$ or $\mathrm{Orb}_R{}^k(f)$ in the R-case and by $\mathrm{Orb}_{RL}^k(f)$ in the RL-case.
It is very important that the last two actions are algebraic.

<u>(1.5)</u> <u>Definitions</u>:
A germ $\hat{f} \in m_n$ is called <u>Right-k-determined</u> (or $j^k(f)$ is Right-k-sufficient) if for any $\hat{g} \in m_n$:

$$f_k = g_k \Rightarrow \hat{f} \underset{R}{\sim} \hat{g}.$$

A germ $\hat{f} \in m_n$ is called <u>Right-left-k-determined</u> (or $j^k(f)$ is Right-left-k-sufficient) if for any $\hat{g} \in m_n$:

$$f_k = g_k \Rightarrow \hat{f} \underset{RL}{\sim} \hat{g}.$$

The property of being k-determined is invariant under RL-equivalence.

<u>Lemma</u>: *Let* f *be* s-*determined and* $f \overset{s}{\sim} g$ *then*

1^o $f \sim g$

2^o $\hat{g}$ *is* s-*determined*

<u>proof</u>: $f \overset{s}{\sim} g$, so there is $\phi \in L_n$ such that $f_s = (g\phi)_s$ so $f \sim g\phi$ and this implies $f \sim g$.
Since s-determinacy is a property of the orbit, also g is s-determined.

Other related questions are C^o-sufficiency and v-sufficiency of jets (cf KUO[15]).

<u>Examples</u>:
1^o If f is regular in $\underline{0} \in \mathbb{R}^n$ there exist coördinates such that $f(x_1,\dots,x_n) = x_1$.
if $g_1 = f_1$ then also g is regular in $\underline{0} \in \mathbb{R}^n$ and we can choose coördinates such that $g(x_1,\dots,x_n) = x_1$. Clearly $g \sim f$; so f is

1-determined.

2^o if $\underline{0}$ is <u>non-degenerate critical point</u> of f, then the classical Morsse-lemma says:

$$f \sim f_2 \sim e_1 x_1^2 + \ldots + e_n x_n^2 \text{ with } e_i = \pm 1.$$

If $g_2 = f_2$ then also $\underline{0}$ is a nondegenerate critical point of g and $g \sim g_2$.

So $g \sim g_2 = f_2 \sim f$; so f is 2-determined.

<u>(1.6) Nakayama's lemma</u>:

Let R *be a commutative ring with* 1; m *an ideal,* L *an* R*-module and* M *and* N *submodules of* L. *Suppose:*

a) $(1 + x)^{-1}$ *exists in* R *for every* $x \in m$

b) M *is finitely generated*

c) $M \subseteq N + mM$

Then: $M \subseteq N$.

<u>Proof</u>:

Let $e_1,\ldots,e_n$ generate M. By c) there are $f_i \in N$ and $\alpha_{ij} \in m$ such that:

$$e_i = f_i + \sum_{j=1}^{n} \alpha_{ij} e_j.$$

Hence $(1 - A)\vec{e} = \vec{f}$ (matrixequation with $A = (\alpha_{ij})$ and $\vec{e} = (e_1,\ldots,e_n)^T$ and $\vec{f} = (f_1,\ldots,f_n)^T$).

Since $\det(1 - A) = 1 + a$ with $a \in m$ and $1 + a$ is invertible in R, also $(1 - A)^{-1}$ exists and

$$\vec{e} = (1 - A)^{-1}\vec{f}.$$

so $e_i \in N (i = 1,\ldots,n)$. Hence $M \subset N$.

<u>(1.7)</u> For $f : \mathbb{R}^n \to \mathbb{R}$ the ideal, generated by the partial derivatives $\partial_1 f,\ldots,\partial_n f$ is denoted by $\Delta(f)$.

<u>Theorem</u>: *If* $\hat{f} \in m_n$ *obeys* $m_n^{k+1} \subset m_n^2 \Delta(f) + m_n^{k+2}$ *then* f *is* k*-determined.*

<u>Proof</u>: Take any $g \in \mathcal{E}_n$ with $g_k = f_k$. We define $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by $F(x,t) = f(x) + t[g(x) - f(x)]$. Denote $F_t(x) = F(x,t)$, hence $F_o = f$ and $F_1 = g$.

We try to find a map $h : (\mathbb{R}^n \times \mathbb{R}, \{\underline{0}\} \times \mathbb{R}) \to (\mathbb{R}^n, 0)$ such that the map

h_t, defined by $h_t(x) = h(x,t)$ is a diffeomorphism and moreover

$$F_t(h_t(x)) = F_o(x),$$

that is
$$F(h(x,t),t) = F(x,0). \qquad (1)$$

Differentiating (1) with respect to t gives:

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(h(x,t),t) \cdot \frac{\partial h^i}{\partial t}(x,t) + \frac{\partial F}{\partial t}(h(x,t),t) = 0$$

$$\nabla F(h(x,t),t) \cdot \frac{\partial h}{\partial t}(x,t) + g(h(x,t)) - f(h(x,t)) = 0 \qquad (2)$$

where $\nabla F = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$ and $\frac{\partial h}{\partial t} = (\frac{\partial h^1}{\partial t}, \ldots, \frac{\partial h^n}{\partial t})$.

Define $\vec{\xi} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by $\vec{\xi}(h(x,t),t) = \frac{\partial h}{\partial t}(x,t)$. (3)

Substitution in (2) gives

$$\nabla F(h(x,t),t) \cdot \vec{\xi}(h(x,t),t) + g(h(x,t)) - f(h(x,t)) = 0.$$

Since (x,t) is arbitrary and h_t is a diffeomorphism this is equivalent to: $\nabla F(x,t) \cdot \vec{\xi}(x,t) + g(x) - f(x) = 0$. (4)

We next try to solve the differentialequations (3) + (4). We need therefore two lemma's.

<u>Lemma 1</u>: *Let* $m_n^{k+1} \subseteq m_n^2 \Delta(f) + m_n^{k+2}$. *Then there exists for all* $t \in \mathbb{R}$ *a mapgerm* $\vec{\xi} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *defined on a neighborhood* U *of* $(0,t_o) \in \mathbb{R}^{n+1}$, *which satisfies:*

(i) $\vec{\xi}(0,t) = 0$ *for all* $(0,t) \in U$

(ii) $\nabla F(x,t) \cdot \vec{\xi}(x,t) + g(x) - f(x) = 0$ *for all* $(x,t) \in U$.

<u>Proof</u>: Let $\mathcal{E}_{n+1}$ be the ring of germs at $(0,t_o)$ of C^∞-functions $\mathbb{R}^{n+1} \to \mathbb{R}$ and m_{n+1} the maximal ideal of $\mathcal{E}_{n+1}$. Let $\Delta^*(F) = \mathcal{E}_{n+1}(\partial_1 F, \ldots, \partial_n F)$. We have inclusions $\mathcal{E}_n \subset \mathcal{E}_{n+1}$ and $m_n \subset m_{n+1}$ (subrings). To satisfy (i) and (ii) we need: $m_n^{k+1} \subseteq \Delta^*(F)(\mathcal{E}_{n+1} m_n)$ or equivalently $m_n^{k+1} \subseteq \Delta^*(F) m_n$ (every element of $\Delta^*(F)(\mathcal{E}_{n+1} m_n)$ has the form $\nabla F(x,t) \cdot \vec{\xi}(x,t)$ and $\xi_i \in \mathcal{E}_{n+1} m_n$, so $\xi_i(0,t_o) = 0$).

Now $\frac{\partial F}{\partial x_i} = \frac{\partial f}{\partial x_i} + t\,\frac{\partial}{\partial x_i}(g-f)$ hence $\frac{\partial f}{\partial x_i} = \frac{\partial F}{\partial x_i} - t\,\frac{\partial}{\partial x_i}(g-f)$.

So $\Delta(f) \subseteq \Delta^*(F) + \mathcal{E}_{n+1}m_n^{\ k}$.

Since $m_n^{\ k+1} \subseteq m_n^{\ 2}\Delta(f) + m_n^{\ k+2}$ we have:

$\mathcal{E}_{n+1}m_n^{\ k+1} \subseteq \mathcal{E}_{n+1}m_n^{\ 2}\Delta(f) + \mathcal{E}_{n+1}m_n^{\ k+2} \subseteq m_n^{\ 2}\Delta^*(F) + \mathcal{E}_{n+1}m_n^{\ k+2} \subseteq$
$\subseteq m_n^{\ 2}\Delta^*(F) + m_{n+1}\mathcal{E}_{n+1}m_n^{\ k+1}$.

So $\mathcal{E}_{n+1}m_n^{\ k+1} \subseteq m_n^{\ 2}\Delta^*(F) + m_{n+1}\mathcal{E}_{n+1}m_n^{\ k+1}$.

We apply Nakayama's lemma with

$(R,m,L,M,N) = (\mathcal{E}_{n+1}, m_{n+1}, \mathcal{E}_{n+1}, \mathcal{E}_{n+1}m_n^{\ k+1}, m_n^{\ 2}\Delta^*F)$ and get:

$$m_n^{\ k+1} \subseteq \mathcal{E}_{n+1}m_n^{\ k+1} \subseteq m_n^{\ 2}\Delta^*(F)$$

Hence $$m_n^{\ k+1} \subseteq m_n^{\ 2}\Delta^*(F) \subseteq m_n\Delta^*(F) \text{ as required.}$$

Lemma 2: *For each* $t_o \in \mathbb{R}$ *there is* $\varepsilon > 0$ *such that* $F_t \underset{R}{\sim} F_{t_o}$ *for all* t *with* $|t-t_o| < \varepsilon$.

Proof: It follows from the fundamental existence theorem for solutions of ordinary differential equations that there exists a smooth mapgerm $h : \mathbb{R}^{n+1} \to \mathbb{R}^n$ satisfying the differential equation:

a) $$\frac{\partial h}{\partial t}(x,t) = \vec{\xi}(h(x,t),t)$$

and the initial condition:

b) $$h(x,t_o) = x.$$

Since h_{t_o} is the identity, there exist $\varepsilon > 0$ such that h_t a diffeomorphism is for all t with $|t-t_o| < \varepsilon$.

If $x = 0$ the differential equation has unique solution $h(0,t) = 0$, for

$$\begin{cases} \frac{\partial h}{\partial t}(0,t) = \vec{\xi}(h(0,t),t) = \vec{\xi}(0,t) = 0 & \text{(lemma 1(i))} \\ h(0,t_o) = 0 \end{cases}$$

Moreover $\frac{d}{dt}(F_t h_t(x)) = \frac{d}{dt}(F(h(x,t),t) =$

$$= \nabla F(h(x,t),t) \cdot \frac{\partial h}{\partial t}(x,t) + g(h(x,t)) - f(h(x,t)) =$$

$$= \nabla F(h(x,t),t) \cdot \vec{\xi}(h(x,t),t) + g(h(x,t)) - f(h(x,t),t) = 0$$

according to lemma 1(i).

So $F_{t_o} = F_{t_o}h_{t_o} = F_t h_t$ for all t with $|t-t_o| < \varepsilon$; so $F_{t_o} \backsim F_t$.
The theorem follows now by "continuous induction" over the interval $[0,1]$.

Remarks: By the Nakayama-lemma the condition $m_n^{k+1} \subseteq m_n^2\Delta(f) + m_n^{k+2}$ is equivalent to $m_n^{k+1} \subseteq m_n^2\Delta(f)$. Moreover k-sufficiency of f follows also from $m_n^k \subseteq m_n\Delta(f) + m_n^{k+1}$ or $m_n^{k-1} \subseteq \Delta(f) + m_n^k$.

(1.8) Theorem: *If* $f \in m_n$ *is* k-*determined then* $m_n^{k+1} \subseteq m_n\Delta(f) + m_n^{k+2}$.

Proof: We define $U = \{g \in \mathcal{E}_n | g_k = f_k\} = f + m_n^{k+1}$

and $V = \{g \in \mathcal{E}_n | \; g \backsim f\;\} = \text{orbit of } f = fL_n$.

We consider the natural projection: $\pi_{k+1} : \mathcal{E}_n \to J^{k+1}(n,1)$.

The sets $U_{k+1} = \pi_{k+1}(U)$ and $V_{k+1} = \pi_{k+1}(V)$ are submanifolds of $J^{k+1}(n,1)$. Let $\tau(U_{k+1})$ and $\tau(V_{k+1})$ be the tangentspaces to U_{k+1} resp. V_{k+1} in $f_{k+1} \in J^k(n,1)$.

By the assumption $U \subset V$; so also $U_{k+1} \subset V_{k+1}$ and $\tau(U_{k+1}) \subset \tau(V_{k+1})$.

In order to prove the theorem, it is sufficient to show:

a) $\tau(U_{k+1}) \equiv m_n^{k+1} \pmod{m_n^{k+2}}$

b) $\tau(V_{k+1}) \equiv m_n\Delta(f) \pmod{m_n^{k+2}}$

Condition a) follows immediate from the definition of U.

Now we prove condition b): The elements of $\tau(V_{k+1})$ can be described as follows: Let for $t \in [0,\varepsilon)$ $h_t : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ be a germ of diffeomorphism with $h_o = 1$. An element of $\tau(V_{k+1})$ is equal to $\pi_{k+1}(\frac{d}{dt} fh_t\Big|_{t=0})$. We have $\frac{d}{dt}(fh_t)\Big|_{t=0} = \nabla f \cdot \frac{dh_t}{dt}\Big|_{t=0} \in \mathcal{E}_n(\partial_1 f,\ldots,\partial_n f)$.

Let $\vec{\xi} = \frac{\partial h_t}{\partial t}$ then $\vec{\xi}(0) = \frac{\partial h_t(0)}{\partial t}\Big|_{t=0} = 0$ since $h_t(0) = 0$.

So $\xi_i \in m_n$ $(i = 1,\ldots,n)$.

That means $\frac{d}{dt}(fh_t)\big|_{t=0} \in m_n\Delta(f)$, which proves $\tau(V_{k+1}) \subset m_n\Delta(f)$ (Modulo m_n^{k+2}). Moreover every element α of $m_n\Delta$ defines an element of $\tau(V_{k+1})$. For let $\alpha(x) = \nabla f(x) \cdot \vec{\xi}(x)$ with $\xi_i \in m_n$, and $h_t(x) = x + t\vec{\xi}(x)$; then $h_t \in L_n$ for small t and we have:

$$\frac{d}{dt} fh_t\Big|_{t=0} = \nabla f \cdot \frac{dh_t}{dt}\Big|_{t=0} = \nabla f \cdot \vec{\xi} = \alpha$$

which proves $\tau(V_{k+1}) \supset m_n\Delta(f)$ (modulo m^{k+2}).

Remark 1: According to Nakayama's lemma the condition $m_n^{k+1} \subseteq m_n\Delta(f) + m_n^{k+2}$ is equivalent to $m_n^{k+1} \subseteq m_n\Delta(f)$.

Remark 2: In the proof of theorem 2 we showed that for every $f \in m_n$ the tangentspace of the orbit of f in $J^k(n,1)$ is equal to $\pi_k(m_n\Delta(f))$. Sometimes we will refer to $m_n\Delta(f)$ also as the tangent-space to the orbit of f in $\mathcal{E}_n$.

(1.9) We shall now discuss the Right-left-case.
This is treated thoroughly by WASSERMANN [27]. He states (pag. 39):

Theorem:

If f is RL-determined then $m_n^{k+1} \subset m_n\Delta(f) + f^*(m_1) + m_n^{k+2}$.

Remark 1: $f^*(m_1)$ is the image of m_1 under the $\mathbb{R}$-algebra homomorphism $f^*: \mathcal{E}_1 \to \mathcal{E}_n$. Modulo m_n^{k+2} $f^*(m_1)$ is spanned as $\mathbb{R}$-algebra by $f, f^2, f^3, \ldots, f^q, \ldots$. According to the Malgrange preparationtheorem the condition $m_n^{k+1} \subseteq m_n\Delta(f) + f^*(m_1) + m_n^{k+2}$ is equivalent to $m_n^{k+1} \subseteq m_n\Delta(f) + f^*(m_1)$.

<u>Remark 2</u>: The tangentspace of the *RL*-orbit of f in $J^k(n,1)$ is equal to $\pi_k[m_n\Delta(f) + f^*(m_1)]$.

<u>(1.10) Definition</u> : <u>codimension</u>: For $\hat{f} \in m_n^2$ we define:

a) $\operatorname{codim}(\hat{f}) = \dim_{\mathbb{R}} \frac{m_n}{\Delta(f)}$

b) $\operatorname{codim}_{RL}(\hat{f}) = \dim_{\mathbb{R}} \frac{m_n}{\Delta(f) + f^*(m_1)}$

The definition depends only on the *RL*-equivalenceclass of $\hat{f}$.

<u>Lemma</u>: *For* $\hat{f} \in m_n^2$ *we have:*

a) $\operatorname{codim}(\hat{f}) = \dim_{\mathbb{R}} \frac{m_n^2}{m_n\Delta(f)}$

b) $\operatorname{codim}_{RL}(\hat{f}) = \dim_{\mathbb{R}} \frac{m_n^2}{m_n\Delta(f) + f^*(m_1)}$

<u>Proof</u>: WASSERMANN [27], proposition 2.19.

<u>Remark</u>: According to remarks (1.8) and (1.9) we can identify $m_n\Delta(f)$ (resp. $m_n\Delta(f) + f^*(m_1)$) with the tangentspace to the *R*-orbit (resp. *RL*-orbit) of f in m_n^2. This justifies the use of the term codimension; so the condition of f is equal to the codimension of the *R*-orbit of f in m_n^2; and the *RL*-codimension of f is equal to the codimension of the *RL*-orbit of f in m_n^2.

<u>Proposition</u>: *Equivalent are:*

a) $\operatorname{codim}(\hat{f}) < \infty$

b) $\operatorname{codim}_{RL}(\hat{f}) < \infty$

c) f *is* k-*determined for some* $k \in \mathbb{N}$

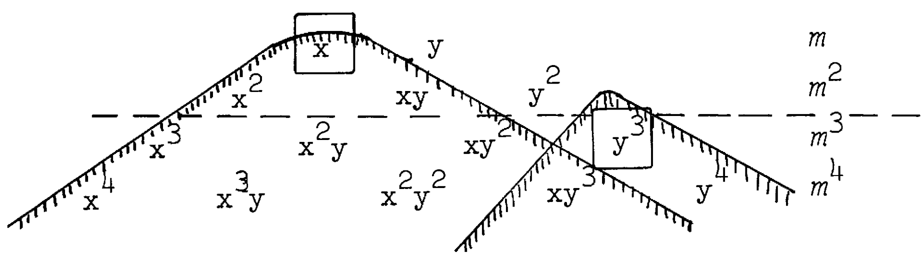
d) f *is* *RL*-k-*determined for some* $k \in \mathbb{N}$

e) *For some* $k \in \mathbb{N}$: $m_n^k \subseteq m_n\Delta(f) + m_n^{k+1}$

<u>Proof</u>: cf WASSERMANN [27];

(1.11) Examples: For n=2 it is possible to compute the codimension and to discover k-determinacy using a diagram, containing the canonical generators of the vectorspace of formalpower series in x and y:

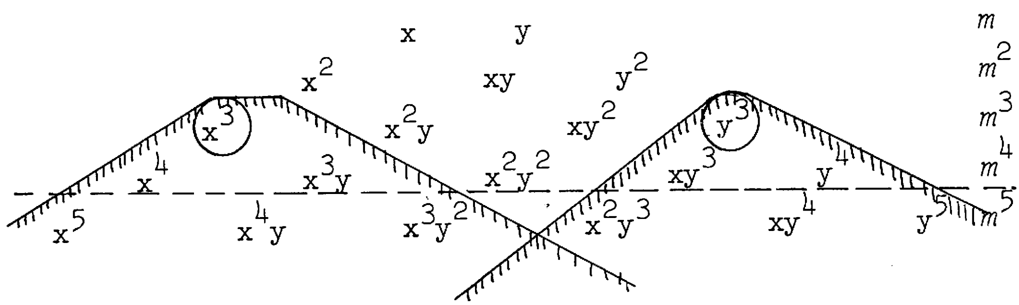
1) $f = x^2 + y^4$; $\partial_1 f = 2x$ and $\partial_2 f = 4y^3$



a) codim (f) = 2

b) As $m^3 \subseteq \Delta(f) + m^4$ then $m^4 \subseteq m\Delta(f) + m^5$ and so f is 4-determined by (1.7).

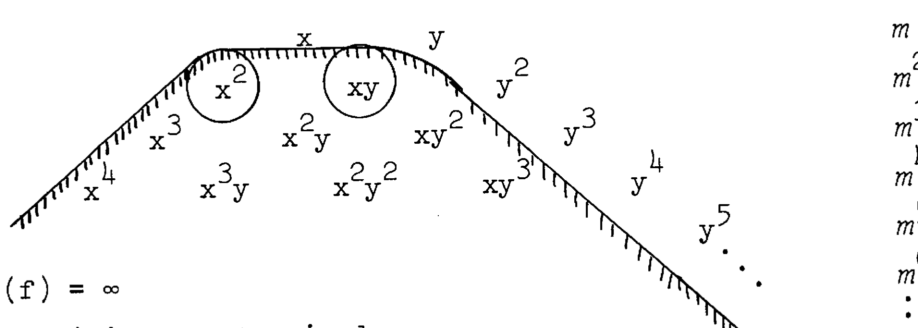
2) $f = x^4 + y^4$; $\partial_1 f = 4x^3$ and $\partial_2 f = 4y^3$



a) codim (f) = 8

b) As $m^5 \subseteq m^2\Delta(f) + m^6$, f is 4-determined by (1.7).

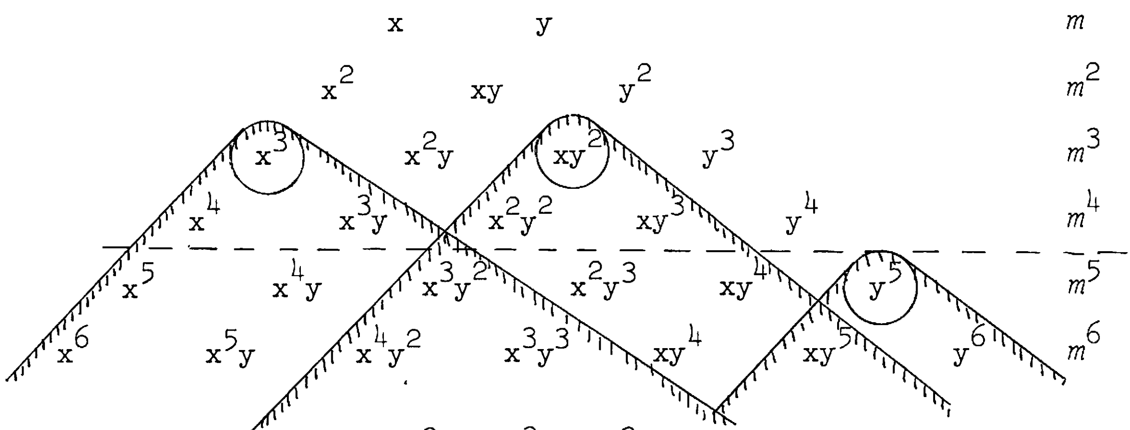
3) $f = x^2y$; $\partial_1 f = 2xy$ and $\partial_2 f = x^2$



a) codim (f) = ∞

b) f is not finitely determined.

4) $f = x^3 + xy^3$; $\partial_1 f = 3x^2 + y^3$ and $\partial_2 f = 3xy^2$



Remark that $x\partial_1 f = 3x^3 + xy^3 \equiv 3x^3$ Modulo $m\Delta(f)$

$y^2\partial_2 f = 3x^2y^2 + y^5 \equiv y^5$ Modulo $m\Delta(f)$

Rélations that are not in the picture:

$$3x^2 + y^3 \equiv 0$$

$$3x^2y + y^4 \equiv 0$$

a) codim (f) = 8 - 2 = 6

b) As $m^5 \subseteq m\Delta + m^6$; so f is 5-determined by (1.7).

5) $f = x^3 + y^3 + z^3$; $\partial_1 f = 3x^2$ and $\partial_2 f = 3y^2$ and $\partial_3 f = 3z^2$

a) codim (f) = 7

b) As $m^4 \subseteq m^2 + m^5$; so f is 3-determined by (1.7).

§2. Equivalence and non-equivalence in k-parameter families of germs.

(2.1) Introduction: In (1.7) and (1.8) we found:

$$m_n^{s+1} \subset m_n^2\Delta(f) + m_n^{s+2} \Rightarrow f \text{ is s-determined} \Rightarrow m_n^{s+1} \subset m_n\Delta(f) + m_n^{s+2}$$

Let $\sigma(f)$ be the smallest integer s such, that f is s-determined. If no such integer exists we write $\sigma(f) = \infty$. $\sigma(f)$ is called the degree of determinacy.

In most cases (1.7) and (1.8) do not determine $\sigma(f)$, but only up to a choice between two consecutive numbers. Further computations are needed to determine $\sigma(f)$ completely.

Let us consider a polynomial f of degree s, which satisfies

$$m_n^{s+1} \subset m_n\Delta(f) + m_n^{s+2}$$

hence

$$m_n^{s+2} \subset m_n^2\Delta(f) + m_n^{s+3}$$

So f is (s+1)-determined and $\sigma(f) = s+1$ or s. Let ρ be a homogeneous polynomial of degree s+1, with say k variable coefficients. Then $f + \rho$ can be considered as a k-parameter family of germs. In order to prove, that f is s-determined, it is sufficient to show that $f + \rho \sim f$ for all ρ. For this reason we study k-parameter families of germs. We start with 1-parameter families and try to eliminate the parameter.

(2.2) Proposition: *Let* $f_t = f + t\phi$ *be defined for* $t \in I$ *(a connected interval of* $\mathbb{R}$*). If* $\phi \in m_n\Delta(f + t\phi)$ *for all* $t \in I$ *then* $f_t \sim f_{t_0}$ *for* $t, t_0 \in I$.

Proof: It is sufficient to satisfy the differentialequation of (1.7) lemma 1:

$$\begin{cases} \text{(i)} & \vec{\xi}(0,t) = 0 \\ \text{(ii)} & \nabla F(x,t) \cdot \vec{\xi}(x,t) + \phi = 0 \end{cases}$$

where $F(x,t) = f_t(x) = f(x) + t\phi(x)$. The conditions (i) and (ii) are equivalent to $\phi \in m_n\Delta(f + t\phi)$. Next we apply (1.7) lemma 2 and our proposition follows.

Corollary: *Let* $f_t = f + t\phi$ *be defined for* $t \in I$ *(connected interval of* $\mathbb{R}$*). If for all* $t \in I$: 1^o $m_n^{k+1} \subseteq m_n\Delta(f + t\phi) + m_n^{k+2}$

2^o $\phi \in m_n\Delta(f + t\phi) + m_n^{k+1}$

then: $f_t \backsim f_{t_o}$ *for* $t, t_o \in I$.

Proof: Nakayama's lemma gives: $m_n^{k+1} \subseteq m_n\Delta(f + t\phi)$ so $\phi \in m_n\Delta(f+t\phi)$. Apply the proposition (2.2).

(2.3) Proposition: *Let* $f_t = f + t\phi$ *be defined for* $t \in I$ *(connected interval of* $\mathbb{R}$*). If* $\phi \in m_n\Delta(f + t\phi) + (f + t\phi)^*m_1$ *for all* $t \in I$ *then* $f_t \underset{RL}{\backsim} f_{t_o}$ *for* $t, t_o \in I$.

Proof: The proof is similar to the proof of theorem (1.7) and proposition (2.2). We try to find maps:

$$h : (\mathbb{R}^n \times I, \{t_o\} \times \mathbb{R}) \to (\mathbb{R}^n, \underline{0})$$

$$k : (\mathbb{R} \times I, \{t_o\} \times \mathbb{R}) \to (\mathbb{R}, 0)$$

such that $h(-,t)$ and $k(-,t)$ are diffeomorphisms and moreover:

(0) $k_t^{-1}(F_t(h_t(x))) = F_{t_o}(x)$

Differentiating (0) with respect to t gives the following three conditions:

(1) $\dfrac{\partial k_t^{-1}}{\partial y}(F(x,t)) \cdot [\nabla F(x,t) \cdot \vec{\xi}(x,t) + \dfrac{\partial F}{\partial t}(x,t) + \eta(F(x,t),t)] = 0$

(2) $\dfrac{\partial h}{\partial t}(x,t) = \vec{\xi}(h(x,t),t)$

(3) $\dfrac{\partial k}{\partial t}(y,t) = -\eta(k(y,t),t)$

together with some initial conditions.

Compare WASSERMANN[27] pag. 22-30.

If it is possible to solve (1) we can solve the equations (2) and (3) locally and in the same way as in theorem (1.7) we find the RL-equivalence of F_t and F_o for all $t \in I$.

The condition (1) is implied by

(4) $\frac{\partial F}{\partial t} \in m_n \Delta^*(F) + F^*(m_1)$

In our case $F(x,t) = f(x) + t\phi(x)$.

Since $\frac{\partial F}{\partial t} = \phi$ the condition (4) is a consequence of

(5) $\phi \in m_n \Delta(f + t\phi) + (f + t\phi)^* m_1$ for all $t \in I$.

<u>(2.4)</u> In the case of k-parameter families we have:

<u>Theorem</u>: *Let* $f_\tau = f + \tau_1\phi_1 + \ldots + \tau_k\phi_k$ *with* $\tau = (\tau_1,\ldots,\tau_k) \in \mathbb{R}^k$. *Let* $\sigma \in \mathbb{R}^k$ *and* D *a subset of* $\mathbb{R}^k$, *such that for every* $\tau \in D$ *also the linesegment* $\sigma\tau$ *is contained in* D, *then:*

a) *if* $\mathbb{R}\phi_1 + \ldots + \mathbb{R}\phi_k \subset m_n\Delta(f_\tau)$ $\forall \tau \in D$ *then* $f_\tau \backsim f_\sigma$ $\forall \tau \in D$

b) *if* $\mathbb{R}\phi_1 + \ldots + \mathbb{R}\phi_k \subset m_n\Delta(f_\tau) + (f_\tau)^* m_1$ $\forall \tau \in D$ *then* $f_\tau \underset{RL}{\backsim} f_\sigma$ $\forall \tau \in D$.

<u>Proof</u>: Let $\tau \in \mathbb{R}^k$ be given and let $\sigma = (\sigma_1,\ldots,\sigma_k)$. Define $\phi = (\tau_1 - \sigma_1)\phi_1 + \ldots + (\tau_k - \sigma_k)\phi_k$. Let $g_t = f_\sigma + t\phi$ and let $I = [0,1] \subset \mathbb{R}$.

In case a): g_t and I satisfy the conditions of proposition (2.2); so $g_1 \backsim g_o$ and $f_\tau \backsim f_\sigma$.

In case b): g_t and I satisfy the conditions of proposition (2.3); so $g_1 \underset{RL}{\backsim} g_o$ and $f_\tau \underset{RL}{\backsim} f_\sigma$.

<u>(2.5)</u> <u>Example 1</u>: Let $f_t = x_1^3 + tx_2^4 \quad t \neq 0$.

We have $\Delta(f_t) = (3x_1^2, 4tx_2^3)$.

$x_2^4 \in m\Delta(f_t)$ $\forall t \neq 0$ since $x_2^4 = \frac{1}{4t}\partial_2 f_t$.

We can apply proposition (2.2):

So if $t \in (0,\infty)$ all f_t are mutually equivalent, for example $f_t \backsim f_1 = x_1^3 + x_2^4$. Also if $t \in (-\infty,0)$ all f_t are mutually equivalent, for example $f_t \backsim f_{-1} = x_1^3 - x_2^4$. Remark, that explicit formula's for the diffeomorphisms are obtained from $f_t = x_1^3 + (t^{\frac{1}{4}}x_2)^4$ if $t > 0$ and $f_t = x_1^3 - (|t|^{\frac{1}{4}}x_2)^4$ if $t < 0$.

Example 2: Let $f = x^3 + xy^6 + ay^9 + by^{10}$ with $b \neq 0$.

We shall show $f \underset{RL}{\backsim} x^3 + xy^6 + ay^9 + y^{10}$

We have: $\begin{cases} \partial_1 f = 3x^2 + y^6 \\ \partial_2 f = 6xy^5 + 9ay^8 + 10by^9 \end{cases}$

$$\left.\begin{array}{rcl} \text{Moreover } y\partial_2 f &=& 6xy^6 + 9ay^9 + 10by^{10} \\ -9f &=& -9xy^6 - 9ay^9 - 9by^{10} - 9x^3 \\ +3x\partial_1 f &=& +3xy^6 \qquad\qquad\qquad\qquad 9x^3 \end{array}\right\}+$$

$$y\partial_2 f - 9f + 3x\partial_1 f = by^{10}$$

So $y^{10} \in m\Delta(f) + f^*(m_1)$ for all a, and all $b \neq 0$. We now apply proposition (2.3) and obtain $f \underset{RL}{\backsim} x^3 + xy^6 + ay^9 \pm y^{10}$ for $b \neq 0$. After replacing x by $-x$; y by $-y$ and f by $-f$ we can get

$$f \underset{RL}{\backsim} x^3 + xy^6 + ay^9 + y^{10}$$

In this example also it is possible to give explicit formula's for the diffeomorphisms, since

$$f \underset{L}{\backsim} b^9 f = b^9x^3 + b^9xy^6 + ab^9y^9 + b^{10}y^{10} =$$
$$= (b^3x)^3 + (b^3x)(by)^6 + a(by)^9 + (by)^{10} \overset{R}{\backsim} x^3 + xy^6 + ay^9 + y^{10}$$

On the other hand, as we show in (2.12) it is impossible to eliminate the parameter a.

Example 3: Let $g = x_1x_3^2 + x_2^3 + Ax_1^3x_2 + Bx_1^4 + Cx_1^4x_2 + Dx_1^5$ with $B \neq 0$. We show that g is 4-determined, and so

$$g \backsim x_1x_3^2 + x_2^3 + Ax_1^3x_2 + Bx_1^4.$$

We shall use this in (4.11).

We have: $\begin{cases} \partial_1 g = x_3^2 + 3Ax_1^2x_2 + 4Bx_1^3 + 4Cx_1^3x_2 + 5Dx_1^4 \\ \partial_2 g = 3x_2^2 + Ax_1^3 + Cx_1^4 \\ \partial_3 g = 2x_1x_3 \end{cases}$

So modulo $m\Delta(g) + m^6$ we have: $m^3\partial_1 g \equiv x_3^2m^3 \equiv 0$

$$m^3\partial_2 g \equiv 3x_2^2m^3 \equiv 0$$

and $\quad x_1x_3m \equiv 0$

Moreover:

a) $0 \equiv x_1x_2\partial_1 g \equiv x_1x_2x_3^2 + 3Ax_1^3x_2^2 + 4Bx_1^4x_2$

so: $4Bx_1^4x_2 \equiv 0$, hence $x_1^4x_2 \equiv 0$ (since $B \neq 0$)

b) $0 \equiv x_1^2\partial_1 g \equiv x_1^2x_3^2 + 3Ax_1^4x_2 + 4Bx_1^5$

so: $4Bx_1^5 \equiv 0$, hence $x_1^5 \equiv 0$ (since $B \neq 0$)

Now it follows that

$$m^5 \subseteq m\Delta(g) + m^6 \text{ for all values of C and D.}$$

So g is 5-determined for all C and D.

Because $\mathbb{R}x_1^4x_2 + \mathbb{R}x_1^5 \subseteq m^5 \subseteq m\Delta(g)$ for all C and D theorem (2.4) gives that $\quad g \backsim x_1x_3^2 + x_2^3 + Ax_1^3x_2 + Bx_1^4$

and so g is 4-determined.

<u>(2.6)</u> Sometimes the elimination of a parameter can be shown to be impossible. First we treat the case of a 1-parameterfamily.

<u>Definition</u>: Let $\{f_t\}_{t \in I}$ be a family of germs, continuously depending on t, and let I be an open interval of $\mathbb{R}$. We call t a <u>local invariant</u> of the family $\{f_t\}_{t \in I}$ if $\forall t_o \in I \;\exists \varepsilon > 0$ such that the germs $\{f_t \mid |t-t_o| < \varepsilon\}$ are all in different orbits. A similar definition exists for *RL*-equivalence.

<u>(2.7)</u> Let A be a subset in $\mathbb{R}^m$. We denote by A^* the closure of A in the <u>Zariski-topology</u>. That is:
$A^* = \{x \in \mathbb{R}^n | (P(A) = 0)$ implies $(P(x) = 0)$ for all real polynomials $P\}$.
Since A^* is closed in the ordinary topology it contains $\bar{A}$.

<u>Definition</u>: A closed set F in $\mathbb{R}^m$ is a <u>real algebraic set</u> iff $F^* = F$.

<u>Proposition</u>: $\mathrm{Orb}(z)$ *is open in* $[\mathrm{Orb}(z)]^*$.
<u>Proof</u>: cf THOM-LEVINE[25] p. 18-19 propositions 1 and 2.

<u>(2.8)</u> <u>Proposition</u>: *Let* ℓ *be a* 1*-dimensional affine subspace of* $J^k(n,1)$ *and let* $z \in J^k(n,1)$. *Then there are two possibilities for* $\ell \cap \mathrm{Orb}(z)$:
1° $\ell \cap \mathrm{Orb}(z)$ *consists of a finite number of points.*
2° $\ell \cap \mathrm{Orb}(z)$ *consists of a collection of open intervals of* ℓ.
<u>Proof</u>: Since $[\mathrm{Orb}(z)]^*$ is real algebraic, we have either $\ell \cap [\mathrm{Orb}(z)]^*$ is a finite number of points, or $\ell \cap [\mathrm{Orb}(z)]^* = \ell$.
Since $\ell \cap \mathrm{Orb}(z)$ is open in $\ell \cap [\mathrm{Orb}(z)]^*$ the proposition follows.

<u>(2.9)</u> <u>Theorem</u>: *Let* $f_t = f + t\phi$ *be a* 1*-parameter family of germs defined for* t *in a connected interval* I *of* $\mathbb{R}$.
If 1° f_t *is* k*-determined for all* $t \in I$,
2° $t \in I : \phi \notin m_n\Delta(f + t\phi) + m_n^{k+1}$.
Then t *is a local invariant.*
<u>Proof</u>: Because f_t is k-determined for all $t \in I$, we can work entirely in $J^k(n,1)$. Let $\ell = j^k(f_{t_o}) + \mathbb{R}\phi$.
According to proposition (2.8) there are only 2 possibilities:

a) $\ell \cap \mathrm{Orb}\,(f_{t_o})$ consists of a finite number of points.

b) $\ell \cap \mathrm{Orb}\,(f_{t_o})$ consists of a collection of open intervals of ℓ.

A necessary condition of b) is that there exists a neighborhood U of t_o in I such that the direction of the line ℓ is contained in the tangentspace of $\mathrm{Orb}(f_{t_o})$ in $j^k(f_{t_o})$.
So $\phi \in m_n(f + t\phi) + m_n^{k+1}$ for all $t \in U$.

Since this is not the case we can conclude, that $\ell \cap \mathrm{Orb}\,(f_{t_o})$ consists only of a finite number of points.

Remark: If we have the condition $\phi \notin m_n\Delta(f + t\phi) + m_n^{k+1} + (f + t\phi)^*m_1$ in theorem (2.9), we get the conclusion also for RL-equivalence.

(2.10) Proposition: *If* $m_n^{k+1} \subseteq m_n^2\Delta(f) + m_n^{k+2}$ *and* $\phi \in m_n$, *then there exists a* $\tau > 0$ *such that* $m_n^{k+1} \subseteq m_n^2\Delta(f + t\phi) + m_n^{k+2}$ *for all* $|t| < \tau$.

Proof:

We consider the canonical projection $\Psi\colon m_n/m_n^{k+2} \to m_n^{k+1}/m_n^{k+2}$.
$\Psi(m_n^2\Delta(f + t\phi))$ is spanned by vectors $\{\vec{a}_1(t), \ldots, \vec{a}_N(t)\}$ continuously depending on t.

If $\mathbb{R}\vec{a}_1(t) + \ldots + \mathbb{R}\vec{a}_N(t) = m_n^{k+1}/m_n^{k+2}$ for $t = 0$, then the same holds small t since a determinant (continuously depending on t) has to be unequal to zero.

So $\Psi(m_n^2\Delta(f + t\phi)) = m_n^{k+1}/m_n^{k+2}$ which is equivalent to $m_n^{k+1} \subseteq m_n^2\Delta(f + t\phi) + m_n^{k+2}$.

Corollary: *If* 1° $m_n^{k+1} \subseteq m_n^2\Delta(f) + m_n^{k+2}$.

and 2° $\phi \notin m_n\Delta(f) + m_n^{k+1}$

then there exist $\tau' > 0$ *such that* f *is not equivalent to* $f + t\phi$ *for all* $|t| < \tau'$.

Proof:

1° implies that $f + t\phi$ is k-determined for all t close to 0.

2° implies that $j^k(f) + \mathbb{R}\phi \cap \mathrm{Orb}\,(j^kf)$ consists only of a finite number of points.

<u>(2.11)</u> <u>Theorem</u>: *If* 1° $m_n^{k+1} \subseteq m_n\Delta(f) + m_n^{k+2}$

2° $\phi \notin m_n\Delta(f) + m_n^{k+1}$

3° codim $(f + t\phi)$ *is constant for all* t *with* $|t|<\tau$

then there exists $\tau' < \tau$ *such that* t *is a local invariant of* $f + t\phi$ *if* $|t| < \tau'$.

<u>Proof</u>:

It is sufficient to prove $\phi \notin m_n(f + t\phi) + m_n^{k+1}$ for small t. Let $\vec{b}_1(t), \ldots, \vec{b}_m(t)$ be the generators of

$$m_n\Delta(f + t\phi) + m_n^{k+1} \Big/ m_n^{k+1} \quad \text{in} \quad m_n \Big/ m_n^{k+1}$$

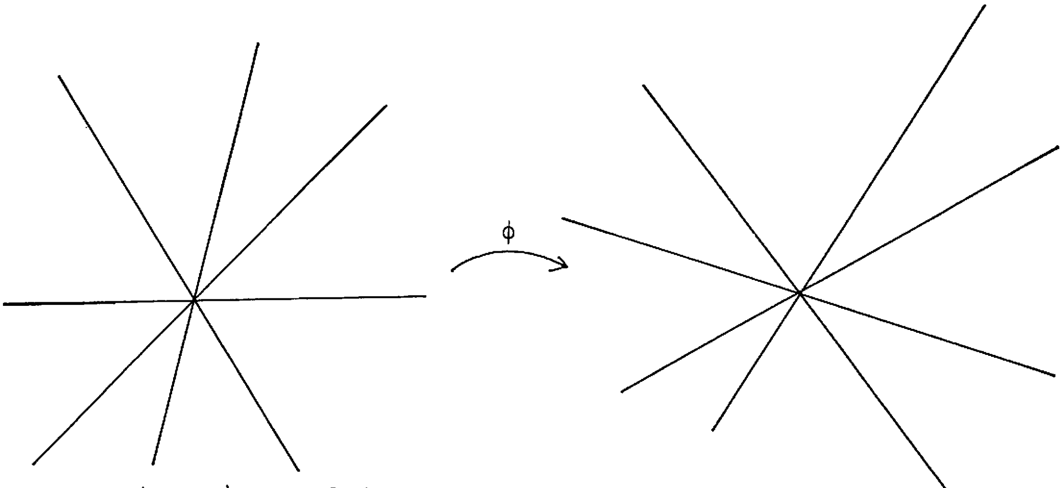
and let $\vec{p}$ be the representative of ϕ in $m_n \big/ m_n^{k+1}$.

Let $B(t)$ be the matrix with columnvectors $\vec{b}_1(t), \ldots, \vec{b}_m(t)$ and $B^r(t)$ be the matrix with columnvectors $\vec{b}_1(t), \ldots, \vec{b}_m(t), \vec{p}$. Since rank $B(t) = k =$ constant for small t and rank $B^r(t) = k + 1$ for $t = 0$ (because $\vec{p} \notin \mathbb{R}\vec{b}_1(0) + \ldots + \mathbb{R}\vec{b}_m(0)$), we have:
rank $B^r(t) \geq k + 1$ for small t; so $\vec{p} \notin \mathbb{R}\vec{b}_1(t) + \ldots + \mathbb{R}\vec{b}_m(t)$ for small t.

<u>(2.12)</u> <u>Example 1</u>: Let $f_t = x^4 + y^4 + tx^2y^2$ $(t^2 \neq 4)$.
One can show that codim $(f_t) = 8$ for all $t^2 \neq 4$.
Moreover $m^5 \subseteq m^2\Delta(f_t) + m^6$ if $t = 0$
and $\quad x^2y^2 \notin m^2\Delta(f_t) + m^5$ if $t = 0$.
So according to theorem (2.11) t is a local invariant of f_t in a neighborhood of $t = 0$.

Remark that this invariant has also a geometrical meaning. Since f_t is 4-determined we can work entirely in $J^4(2,1)$. Because f_t is homogeneous of degree 4 the orbit of f_t under $L^4(2)$ coincides with the orbit of f under $L^1(2) = GL(2)$.

Our local invariant t depends on the cross ratio of the four (complex) lines with equation $f_t = 0$ since every element of $\phi \in GL(2)$ induces a projective transformation in the pencil of lines through the origin in the (complex) x-y-plane, sending $f_t = 0$ onto $f_t\phi = 0$. Cross ratio is an invariant under complex transformations.



$f_t(x,y) = x^4 + y^4 + tx^2y^2 = 0$ $\qquad$ $f_t(\phi(x,y)) = 0$

<u>Example 2</u>: Let $f = x^3 + xy^6 + ay^9 + y^{10}$

(compare also 2.5, example 2).

In this case $y^9 \notin m\Delta(f) + f^*(m) \quad \forall a \in \mathbb{R}$.

One can deduce this from:

$$\dim_{\mathbb{R}} \frac{m}{\Delta(f) + f^*(m_1)} = 14$$

$$\dim_{\mathbb{R}} \frac{m}{\Delta(f) + f^*(m_1) + y^9} = 13.$$

So a is a local RL-invariant.

<u>(2.13)</u> <u>Definition</u>: Let D be an open connected subset of $\mathbb{R}^k$. $\tau = (\tau_1, \dots, \tau_k)$ is called a (<u>k-dimensional</u>) <u>local invariant</u> of the family $\{f_\tau\}_{\tau \in D}$ if for every $\sigma \in D$ there exist $\varepsilon > 0$ such that the germs $\{f_\tau \mid \|\sigma-\tau\| < \varepsilon\}$ are all in different orbits. We also say, that $(\tau_1, \dots, \tau_k)$ is a set of local invariants of the family.

<u>Example</u>: The family $f_{t,s} = xy(x+y)(x+ty)(x+sy)$ has the set of local invariants (s,t).

They are related to two cross ratio's in the set of 5 lines, defined by $f_{t,s} = 0$.

(2.14) Theorem: *Let* $f_\tau = f + \tau_1\phi_1 + \ldots + \tau_k\phi_k$ *be a* k*-parameterfamily of germs, defined in an open connected subset* D *of* $\mathbb{R}^k$ *and let:*

1° f_τ *be* p*-determined for all* $\tau \in D$

2° $[\mathbb{R}\phi_1 + \ldots + \mathbb{R}\phi_k] \cap [m_n\Delta(f_\tau) + m_n^{p+1}] = \{\underline{0}\}$

then τ *is a* (k*-dimensional) local invariant.*

Proof:

Because f_τ is p-determined for all $\tau \in D$, we can work entirely in $J^p(n,1)$. Let $\sigma \in D$ and let $V = j^p(f_\sigma) + \mathbb{R}j^p(\phi_1) + \ldots + \mathbb{R}j^p(\phi_k)$.
We consider $V \cap \mathrm{Orb}\,(j^p(f_\sigma))$. There are two possibilities:

a) f_σ is isolated in $V \cap \mathrm{Orb}\,(j^p(f_\sigma))$.

b) f_σ is not isolated in $V \cap \mathrm{Orb}\,(j^p(f_\sigma))$.

In case b) the curveselectionlemma (cf MILNOR[20], pag. 25) implies that there is a real analytic curve:

$$p : [0,\varepsilon) \to V$$

with $p(0) = f_\sigma$ and $p(t) \in V \cap \mathrm{Orb}\,(j^p(f_\sigma))$. In that case the intersection of the tangentspaces onto V and $\mathrm{Orb}\,(j^p(f_\sigma))$ is at least 1-dimensional, so: $[\mathbb{R}\phi_1 + \ldots + \mathbb{R}\phi_k] \cap m_n\Delta(f_\sigma) \neq \{\underline{0}\}$. This gives a contradiction; so we are in case a). Now f_σ is isolated in $V \cap \mathrm{Orb}\,(j^p(f_\sigma))$ and so there is a neighborhood of $j^p(f_\sigma)$ in V such that no f_τ in this neighborhood is equivalent to f_σ. Since σ was arbitrary in D, we are done.

(2.15) Theorem: *Let* $f_\tau = f + \tau_1\phi_1 + \ldots + \tau_k\phi_k$ *be a* k*-parameter-family of germs, defined in an open connected subset of* D *of* $\mathbb{R}^k$ *and let*

1° f_τ *be p-determined for all* $\tau \in D$

2° $[\mathbb{R}\phi_1 + \ldots + \mathbb{R}\phi_k] \cap [m_n\Delta(f_\tau) + (f_\tau)^*(m_1) + m^{p+1}] = \{\underline{0}\}$

then τ *is a local RL-invariant.*

Proof: similar to (2.14).

(2.16) Remark: Proposition (2.10) and theorem (2.11) remain valid in the case of k-parameterfamilies.

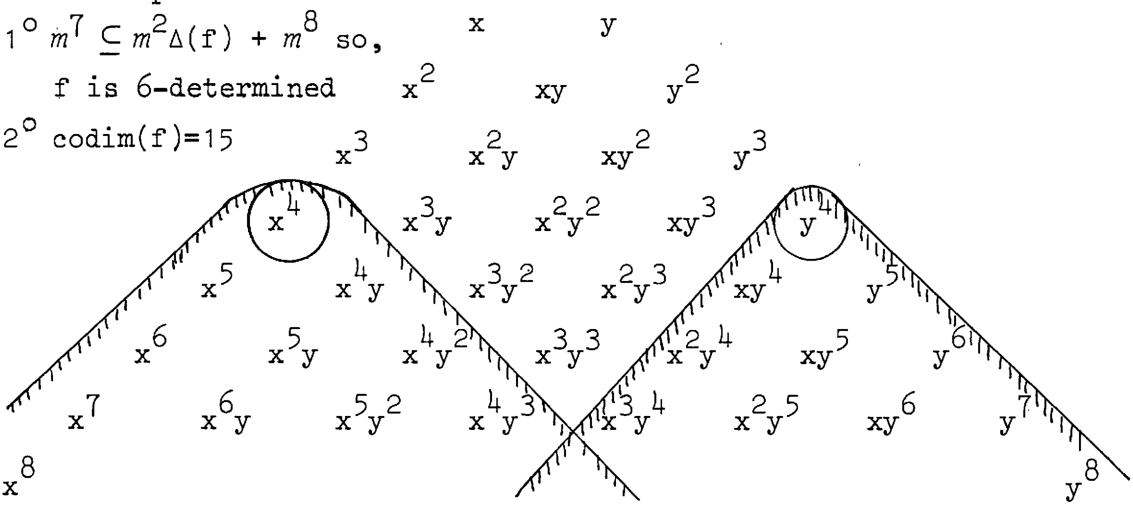
(2.17) Example: Let $f = x^5 + y^5$

From the picture we conclude

1° $m^7 \subseteq m^2\Delta(f) + m^8$ so,

f is 6-determined

2° codim(f)=15



Consider now $f_{(u,v,w)} = x^5 + y^5 + ux^3y^2 + vx^2y^3 + wx^3y^3$.

Since 15 is the minimal codimension for a germ $f \in m_2$ with $f_4 \equiv 0$, we have codim $f_{(u,v,w)} \geq 15$ for (u,v,w) small. Moreover $\mathbb{R}x^3y^2 + \mathbb{R}x^2y^3 + \mathbb{R}x^3y^3 \notin m\Delta(x^5 + y^5) + m^7$. Theorem (2.11) implies: (u,v,w) is a set of local invariants for (u,v,w) small.

Remark, that w is not a local *RL*-invariant.

§3 Splittinglemma and classificationtheorem

(3.1) We consider $f \in m_n^2$. Since $f_1 \equiv 0$ the polynomial f_2 is homogeneous of degree 2:

$$f_2 = \sum_{i,j=1}^{n} a_{ij}x_ix_j \text{ where } a_{ij} = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{(\underline{0})}.$$

The rank of the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{(\underline{0})}$ is invariant under *RL*-equivalence.

Definition: The corank of f is n minus the rank of the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{(\underline{0})}$.

Notation: corank $(f) = n - \operatorname{rank}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{(\underline{0})}$.

(3.2) Splittinglemma: *Let* $f \in m_n^2$, codim $(f) < \infty$ *and* corank $(f) = r$. *Then:* $f(x_1,\dots,x_n) \backsim g(x_1,\dots,x_r) + e_{r+1}x_{r+1}^2 + \dots + e_nx_n^2$ *where* $e_{r+1} = \pm 1,\dots, e_n = \pm 1$ *and* $g_2 \equiv 0$.

Proof:

There exists a linear isomorphism such that $f_2 \backsim e_{r+1}x_{r+1}^2 + \dots + e_nx_n^2$ where $e_{r+1} = \pm 1,\dots, e_n = \pm 1$ and r = corank (f).

So $f \overset{2}{\backsim} e_{r+1}x_{r+1}^2 + \dots + e_nx_n^2$.

We continue now by induction on k.

Let $f \overset{k}{\backsim} g_k(x_1,\dots,x_r) + e_{r+1}x_{r+1}^2 + \dots + e_nx_n^2$ with $k \geq 2$ and $g_k \in m_n^3$.

Assertion: $f \overset{k+1}{\backsim} g_{k+1}(x_1,\dots,x_r) + e_{r+1}x_{r+1}^2 + \dots + e_nx_n^2$ with $g_{k+1} \in m_n^3$.

We have $f \overset{k+1}{\sim} g_k(x_1,\ldots,x_r) + \rho(x_1,\ldots,x_n) + e_{r+1}x_{r+1}^2 + \ldots + e_n x_n^2$

where ρ is homogeneous of degree k+1.

We write ρ in the following form:

$$\rho(x_1,\ldots,x_n) = x_n h_n(x_1,\ldots,x_n) + x_{n-1}h_{n-1}(x_1,\ldots,x_{n-1}) + \ldots +$$
$$+ x_{r+1}h_{r+1}(x_1,\ldots,x_{r+1}) + p(x_1,\ldots,x_r)$$

where $h_n,\ldots,h_{r+1}$ are homogeneous of degree k and p is homogeneous of degree k+1.

Define $\phi : (\mathbb{R}^n,\underline{0}) \to (\mathbb{R}^n,\underline{0})$ by

$$(*)\ \begin{cases} \phi^1 \ (x_1,\ldots,x_n) = x_1 \\ \vdots \\ \phi^r \ (x_1,\ldots,x_n) = x_r \\ \phi^{r+1}(x_1,\ldots,x_n) = x_{r+1} + \sigma_{r+1} \\ \vdots \\ \phi^n \ (x_1,\ldots,x_n) = x_n + \sigma_n \end{cases}$$

where $\phi^1,\ldots,\phi^n$ are the components of ϕ and $\sigma_{r+1},\ldots,\sigma_n \in m_n^k$ (to be fixed later).

The Jacobianmatrix of ϕ in $\underline{0} \in \mathbb{R}^n$ is the identitymatrix. The inverse-functiontheorem implies that ϕ is a germ of diffeomorphism.

In stead of (*) we shall use the short-hand-notation:

$$(**)\ \begin{cases} x_1 \ := x_1 \\ \vdots \\ x_r \ := x_r \\ x_{r+1} := x_{r+1} + \sigma_{r+1} \\ \vdots \\ x_n \ := x_n + \sigma_n \end{cases}$$

By substitution we get (Modulo m_n^{k+2}):

$f \overset{k+1}{\sim} g_k(x_1,\ldots,x_r) + \rho(x_1,\ldots,x_n) + e_{r+1}(x_{r+1}+\sigma_{r+1})^2 + \ldots + e_n(x_n+\sigma_n$

$$\equiv g_k(x_1,\dots,x_r) + p(x_1,\dots,x_r) + x_{r+1}h_{r+1}(x_1,\dots,x_{r+1}) + \dots +$$
$$+ x_n h_n(x_1,\dots,x_n) + e_{r+1}x_{r+1}^2 + 2e_{r+1}x_{r+1}\sigma_{r+1} + \dots + e_n x_n^2 + 2e_n x_n \sigma_n$$
$$= g_k(x_1,\dots,x_r) + p(x_1,\dots,x_r) + e_{r+1}x_{r+1}^2 + \dots + e_n x_n^2 +$$
$$+ x_{r+1}[h_{r+1}(x_1,\dots,x_{r+1}) + 2e_{r+1}\sigma_{r+1}] + \dots + x_n[h_n(x_1,\dots,x_r) + 2e_n\sigma_n]$$
$$= g_{k+1}(x_1,\dots,x_r) + e_{r+1}x_{r+1}^2 + \dots + e_n x_n^2.$$

if
$$\begin{cases} \sigma_{r+1} = \dfrac{-1}{2e_{r+1}}[h_{r+1}(x_1,\dots,x_{r+1})] \in m_n^k \\ \vdots \\ \sigma_n = \dfrac{-1}{2e_n}[h_n(x_1,\dots,x_n)] \in m_n^k \end{cases}$$

and $g_{k+1}(x_1,\dots,x_r) = g_k(x_1,\dots,x_r) + p(x_1,\dots,x_r)$.

Now the assertion is proved for all $k \geq 2$.

Since codim $(f) < \infty$ there exists a s such that f is s-determined.

With (1.5)lemma there follows:

$$f(x_1,\dots x_n) \sim g(x_1,\dots,x_r) + e_{r+1}x_{r+1}^2 + \dots + e_n x_n^2.$$

(3.3) Remark:

The above proof of the splittinglemma is due to MATHER[19]. Other proofs, not requiring that codim $(f) < \infty$ are given by WASSERMANN[27] and GROMOLL-MEYER[13]. In the last case the splittinglemma is given in a Hilbertspace context. They mention also an observation of MATHER, that given any two splittings of the form $f \sim g + Q$ with $g_2 \equiv 0$ and Q a non-degenerate quadratic form, then the corresponding non-degenerate parts and degenerate parts are Right-equivalent.

(3.4) Lemma: Let $f(x_1,\dots,x_n) = g(x_1,\dots,x_r) + e_{r+1}x_{r+1}^2 + \dots + e_n x_n^2$ with $g_2 \equiv 0$; then:

1^e codim (g) = codim (f)

2^e g is k-determined $\Rightarrow$ f is k-determined.

Proof:

1° $\Delta(f) = (\partial_1 g,\dots,\partial_r g,\ x_{r+1},\dots,x_n) \in m_n$

$\Delta(g) = (\partial_1 g,\dots,\partial_r g) \in m_r$.

2° Let $\tilde{f}_k = f_k$. From the proof of the splittinglemma it follows, that:

$\tilde{f}(x_1,\dots,x_n) \backsim \tilde{g}(x_1,\dots,x_r) + e_{r+1}x_{r+1}^2 + \dots + e_n x_n^2$ with $g_k = \tilde{g}_k$.

So there is a diffeomorphism $\phi:(\mathbb{R}^r,\underline{0}) \to (\mathbb{R}^r,\underline{0})$, with $\tilde{g}\phi = g$ and this implies that $f \backsim \tilde{f}$. (extending ϕ by the identity).

Corollary: The classification of $f \in m_n^2$ follows from the classification of $g \in m_r^3$.

(3.5) Lemma: Let $f(x_1,\dots,x_n) = g(x_1,\dots,x_r) + e_{r+1}x_{r+1}^2 + \dots + e_n x_n^2$ with $g_2 \equiv 0$.

$r = 0 \Rightarrow$ codim $(f) = 0$

$r = 1 \Rightarrow$ codim $(f) \geq 1$

$r = 2 \Rightarrow$ codim $(f) \geq 3$

$r = 3 \Rightarrow$ codim $(f) \geq 7$

$r \geq 4 \Rightarrow$ codim $(f) \geq 14$

The proof is direct computation, concerning the ideal $(\partial_1 g,\dots,\partial_r g)$ for g a function of lowest degree 3.

(3.6) CLASSIFICATIONTHEOREM:

For $f \in m_n$ *with* codim $(f) < \infty$ *and* $f_1 = 0$ *we have,*

either: $f \backsim Q + g$ *where* g *is a germ of one of the polynomials in the list on the next page, and* $Q = e_{r+1}x_{r+1}^2 + \dots + e_n x_n^2$

or: codim $(f) > 9$.

r	g		type	R-codimension	RL-codimension	degree of determinacy	R-invariants	RL-invariants
r=0	$g = 0$	simple singularities	A_1	0	0	2	0	0
r=1	$g = \pm x_1^k \ (k \geq 3)$		A_{k-1}	k-2	k-2	k	0	0
r=2	$g = x_1^2x_2 \pm x_2^k \ (k \geq 4)$		D_{k+1}	k	k	k	0	0
	$g = x_1^3 \pm x_2^4$		E_6	5	5	4	0	0
	$g = x_1^3 + x_1x_2^3$		E_7	6	6	4	0	0
	$g = x_1^3 + x_2^5$		E_8	7	7	5	0	0
	$g = x_1^3 + Ax_1x_2^4 + Bx_2^6$	$\tilde{E}_8 =$	J_{10}	9	9	6	1	1
	$g = (x_1^2 \pm x_2^2)(x_1^2 + \alpha x_2^2)$	$\tilde{E}_7 =$	X_9	8	8	4	1	1
	$g = x_1^4 \pm x_1^2x_2^2 + \alpha x_2^5$		X_{10}	9	8	5	1	0
r=3	$g = x_3x_2^2 + x_1^3 + g_1x_1x_3^2 + g_2x_3^3$	$\tilde{E}_6 =$	P_8	7	7	3	1	1
	$g = x_1^3 + x_2^2x_3 \pm x_i^2x_3 + \beta x_3^4$		P_9	8	7	4	1	0
	$g = x_1^3 + x_2^2x_3 \pm x_1^2x_3 + \beta x_3^5$		P_{10}	9	8	5	1	0
	$g = x_1^3 \pm x_1x_2^2 \pm x_1x_3^2 + A[x_2^4 \pm 6x_2^2x_3^2 + x_3^4]$ $g = x_1^3 \pm x_1x_2^2 \pm x_1x_3^2 + B[4x_2^3x_3 \pm 4x_2x_3^3]$		R_{10}	9	8	4	1	0
	$g = x_1x_3^2 + x_2^3 + \alpha x_1^3x_2 \pm x_1^4$		Q_{10}	9	8	4	1	0

[Conditions: J_{10}: $4A^3 + 27B^2 \neq 0$; X_9: $\alpha \neq 0,-1,1$; X_{10}: $\alpha \neq 0$; P_8: $4g_1^3 + 27g_2^2 \neq 0$; P_9: $\beta \neq 0$; P_{10}: $\beta \neq 0$; R_{10}: $A \neq 0$ and $B \neq 0$].

The proof follows increasing corank r of f. In corank r=0 and r=1 the list is complete. In the sections on r=2 and r=3 we add remarks about some germs of codimension > 9. The proof will be given in §4.

(3.7) Remark: Two germs of different type are not equivalent. Within one type, we may have equivalent ones. An example is A_{k-1} with k odd. The parameters in the families X_9, X_{10}, P_9, P_{10}, Q_{10} and R_{10} are local invariants. The families J_{10} and P_8 have a 1-dimensional local invariant, depending on the two parameters. The equivalence in these families is discussed in (4.5) for J_{10}; in (4.10) for P_8 and in (4.13) for R_{10}. In §4 we also treat the difference between the R-classification and the RL-classification.

(3.8) Remark: We can consider the classificationproblem also in other cases: $\mathbb{R}$-analytic, $\mathbb{R}$-formal powerseries and also the $\mathbb{C}$-analytic case and $\mathbb{C}$-formal powerseries. In all these cases we have the same results as here in the C^∞-$\mathbb{R}$-case, because it turns out that classification of germs of finite codimension can be done with polynomialfunctions. In the $\mathbb{C}$-case we can replace all $\pm$-signs by +-signs and sometimes it is possible to give nicer normalforms for the orbits.(See the list I at the end).

§4 Proof of the classificationtheorem

If no confusion is possible, we use the abbrevation Δ for $\Delta(g)$ and m for m_n.

$\boxed{\text{corank} = 0}$

<u>(4.1)</u> <u>Theorem</u>: *If* $r = 0$ *then* $f(x_1,\ldots,x_n) \sim e_1x_1^2 + \ldots + e_nx_n^2$ (A_1).

<u>Proof</u>:

Since $r = 0$ we have $f(x_1,\ldots,x_n) \overset{2}{\sim} e_1x_1^2 + \ldots + e_nx_n^2$.
So $\Delta(f_2) = m$ and $m^3 \subseteq m^2\Delta(f_2) + m^4$ which implies that f_2 is 2-determined; so $f \sim f_2$.

<u>Remark</u>: We may take $e_1 \geq e_2 \geq \ldots \geq e_n$. There are n+1 equivalence-classes, corresponding to +++..+++, +++..++-, +++..+--, etc.

$\boxed{\text{corank} = 1}$

<u>(4.2)</u> <u>Theorem</u>: *If* $r = 1$ *then*

either: $f(x_1,\ldots,x_n) \sim \pm x_1^{k+1} + e_2x_2^2 + \ldots + e_nx_n^2$ (A_k) $(k \geq 2)$

or: codim $(f) = \infty$.

<u>Proof</u>:

Let codim $(f) < \infty$, then $f(x_1,\ldots,x_n) \sim g(x_1) + e_2x_2^2 + \ldots + e_nx_n^2$.

Let $g_k = ax_1^k$ with $a \neq 0$, then $\Delta(g_k) = m^{k-1}$; so $m^{k+1} \subseteq m^2\Delta(g_k) + m^{k+2}$ and so g_k is k-determined, which gives $g \sim g_k$.

If k is even and $a > 0$: $g(x_1) \sim x_1^k$; and if $a < 0$: $g(x_1) \sim -x_1^k$.

If k is odd $g(x_1) \sim x_1^k$ for all $a \neq 0$.

<u>Remark</u>: We may take $e_2 \geq e_3 \geq \ldots \geq e_n$.
If k is even we have 2n equivalenceclasses.
If k is odd we have n equivalenceclasses.
In the sequel we shall no longer mention the various quadratic cases.

$\boxed{\text{corank} = 2}$

(4.3) Proposition: *If* r = 2 *then*

$$f(x_1,\dots,x_n) \overset{3}{\backsim} g(x_1,x_2) + e_3x_3^2 + \dots + e_nx_n^2,$$

where g_3 *is in exactly one of the following four cases:*

1° $g_3(x_1,x_2) = x_1^2x_2 \pm x_2^3$

2° $g_3(x_1,x_2) = x_1^2x_2$

3° $g_3(x_1,x_2) = x_1^3$

4° $g_3(x_1,x_2) \equiv 0$

Proof:

Because $g_2 = 0$ it follows that $g_3(x_1,x_2)$ is a homogeneous polynomi[al].
We may factor g_3 into linear forms over $\mathbb{C}$. The four cases correspo[nd]
to 3, 2, 1 or 0 linear factors. By a linear map we can arrange, th[at]
g_3 gets the given form.

(4.4) Theorem: *If* r = 2 *and* $g_3 = x_1^2x_2 \pm x_2^3$ *or* $g_3 = x_1^2x_2$, *we hav[e]*

either: $f(x_1,\dots,x_n) \backsim x_1^2x_2 \pm x_2^{k-1} + e_3x_3^2 + \dots + e_nx_n^2$ (D_k) $(k \geq$

with codim (f) = k-1.

or: codim (f) = ∞.

Proof: Let codim (f) < ∞.

In case 1° we have $g_3(x_1,x_2) = x_1^2x_2 \pm x_2^3$.

$\Delta(g_3) = (2x_1x_2,\ x_1^2 \pm 3x_2^2)$

$$\left.\begin{aligned} x_1^3 &= x_1\partial_2 g_3 \mp \tfrac{3}{2}\, x_2\partial_1 g_3 \\ x_1^2x_2 &= \tfrac{1}{2}\, x_1\partial_1 g_3 \\ x_1x_2^2 &= \tfrac{1}{2}\, x_2\partial_2 g_3 \\ x_2^3 &= \pm\tfrac{1}{3}\, x_2\partial_2 g_3 \mp \tfrac{1}{6}\, x_1\partial_1 g_3 \end{aligned}\right\} \quad \text{So } m^3 \subseteq m\Delta(g_3) + m^4$$

So g_3 is 3-determined and $g \backsim g_3$.
In case 2° is $g_3(x_1,x_2) = x_1^2x_2$ and $\Delta(g_3) = (2x_1x_2, x_1^2)$, so g_3 is
finitely determined. So we have to consider higher jets than 3-je[ts]

Lemma 1: *Let* $k \geq 4$ *then*

$$x_1^2x_2 + \alpha_o x_1^k + \alpha_1 x_1^{k-1}x_2 + \ldots + \alpha_k x_2^k \overset{k}{\backsim} x_1^2x_2 + \alpha_k x_2^k \backsim x_1^2x_2 \pm x_2^k$$

Proof:

We define an element of L_2 by:

$$\begin{cases} x_1 := x_1 + \rho_1 & \text{with } \rho_1 \overset{*}{=} -\frac{1}{2}[\alpha_1 x_1^{k-2} + \ldots + \alpha_{k-1}x_2^{k-2}] \in m^{k-2} \\ x_2 := x_2 + \rho_2 & \text{with } \rho_2 \overset{*}{=} -\alpha_o x_1^{k-2} \in m^{k-2} \end{cases}$$

So we have:

$$x_1^2x_2 + \alpha_o x_1^k + \alpha_1 x_1^{k-1}x_2 + \ldots + \alpha_k x_2^k \overset{k}{\backsim}$$

$$\overset{k}{\backsim} (x_1 + \rho_1)^2(x_2 + \rho_2) + \alpha_o x_1^k + \alpha_1 x_1^{k-1}x_2 + \ldots + \alpha_{k-1}x_1x_2^{k-1} + \alpha_k x_2^k \overset{k}{\backsim}$$

$$\overset{k}{\backsim} x_1^2x_2 + x_1^2\rho_2 + 2x_1x_2\rho_1 + \alpha_o x_1^k + \alpha_1 x_1^{k-1}x_2 + \ldots + \alpha_{k-1}x_1x_2^{k-1} + \alpha_k x_2^k \overset{k}{\backsim}$$

$$\overset{k}{\backsim} x_1^2x_2 + x_1^2[\rho_2 + \alpha_o x_1^{k-2}] + x_1x_2[2\rho_1 + \alpha_1 x_1^{k-2} + \ldots + \alpha_{k-1}x_2^{k-2}] + \alpha_k x_2^k =$$

$$\overset{*}{=} x_1^2x_2 + \alpha_k x_2^k \backsim x_1^2x_2 \pm x_2^k.$$

Lemma 2: $g_k = x_1^2x_2 \pm x_2^k$ *is* k-*determined.*

Proof:

$$\Delta(g_k) = (2x_1x_2, \; x_1^2 \pm kx_2^{k-1})$$

We have $m^k \subseteq m\Delta + m^{k+1}$ since

$$m^{k-2}x_1x_2 = m^{k-2}\partial_1 g_k$$
$$x_1^k = x_1^{k-2}\partial_3 g_k \mp \frac{k}{2} x_1^{k-1}x_2^{k-2}\partial_1 g_k$$
$$x_2^k = \pm x_2\partial_2 g_k \mp \frac{1}{2k} x_1\partial_1 g_k$$

We apply Lemma 1 for $k = 4,5,\ldots,\ell$ we get

$$g_\ell \backsim x_1^2x_2 + \beta_4 x_2^4 + \ldots + \beta_{\ell-1}x_2^{\ell-1} + \beta_\ell x_2^\ell$$

Let k be the smallest integer such that $\beta_k \neq 0$.

In that case is g_k k-determined and consequently

$$g \backsim g_k \backsim x_1^2x_2 \pm x_2^k \; (D_{k+1}).$$

(4.5) Theorem: *If* $r = 2$ *and we are in the case* $g_3 = x_1^3$ *of proposition* (4.3) *then, either:* $f(x_1,\dots,x_n) \sim g(x_1,x_2) + e_3x_3^2 + \dots + e_nx_n^2$, *where*

$$g(x_1,x_2) = x_1^3 \pm x_2^4 \qquad (E_6)$$
$$g(x_1,x_2) = x_1^3 + x_1x_2^3 \qquad (E_7)$$
$$g(x_1,x_2) = x_1^3 + x_2^5 \qquad (E_8)$$
$$g(x_1,x_2) = x_1^3 + Ax_1x_2^4 + Bx_2^6 \quad (4A^3+27B^2\neq 0)(J_{10})$$

or: codim $(f) > 9$.

In J_{10} *the number* $k = \dfrac{A^3}{4A^3+27B^2}$ *is an invariant. Two germs of the family* J_{10} *are equivalent if and only if they have equal* k *and equal sign of* B. *If* $B = 0$ *they are equivalent iff the sign of* A *is the same.*

Proof: Follows from Lemma 1-5.

Remark first, that $g_3(x_1,x_2) = x_1^3$ is not finitely determined.

Lemma 1:

$$x_1^3 + \sigma_4 + \dots + \sigma_{n-1} + \tau_n \overset{n}{\sim} x_1^3 + \sigma_4 + \dots + \sigma_{n-1} + \sigma_n$$

where $\sigma_p = \alpha_p x_1x_2^{p-1} + \beta_p x_2^p$ *and* τ_n *homogeneous of degree* n.

Proof:

Define an element of L_2 by $\begin{cases} x_1: = x_1 + \rho_{n-2} & \text{with } \rho_{n-2} \in m^{n-2} \\ x_2: = x_2 \end{cases}$

$$x_1^3 + \sigma_4 + \dots + \sigma_{n-1} + \tau_n \overset{n}{\sim} x_1^3 + 3x_1^2\rho_{n-2} + \sigma_4 + \dots + \sigma_{n-1} + \tau_n =$$
$$= x_1^3 + \sigma_4 + \dots + \sigma_{n-1} + [3x_1^2\rho_{n-2} + \tau_n] =$$
$$= x_1^3 + \sigma_4 + \dots + \sigma_{n-1} + \sigma_n$$

if we choose ρ_{n-2} such that $3x_1^2\rho_{n-2} + \tau_n = \alpha_n x_1x_2^{n-1} + \beta_n x_2^n = \sigma_n$.
[Remark that the coëfficients of $x_1x_2^{n-1}$ and x_2^n have not changed].

Corollary: (Normalform):

If $g \overset{3}{\sim} x_1^3$ *then* $g \overset{n}{\sim} x_1^3 + \sigma_4 + \dots + \sigma_n$ *where* $\sigma_p = \alpha_p x_1x_2^{p-1} + \beta_p x_2^p$.

<u>Lemma 2</u>: *Let* $k \geq 4$.

a) $g_k = x_1^3 + \alpha_k x_2^k$ $(\alpha_k \neq 0)$ *is* k-*determined.*

b) $g_k = x_1^3 + \alpha_{k-1} x_1 x_2^{k-1}$ $(\alpha_{k-1} \neq 0)$ *is* (2k-3)-*determined and not* (2k-5)-*determined.*

<u>Proof</u>:

a) $\partial_1(g_k) = 3x_1^2$ and $\partial_2(g_k) = k\,\alpha_k x_2^{k-1}$ and this leads directly to: $m^k \subseteq m\Delta + m^{k+1}$.

b) $\partial_1(g_k) = 3x_1^2 + \alpha_{k-1} x_2^{k-1}$

$\partial_2(g_k) = (k-1)\alpha_{k-1} x_1 x_2^{k-2}$

We shall show: $m^{2k-3} \subseteq m\Delta + m^{2k-2}$. We compute now modulo $m\Delta + m^{2k-2}$:

1° $0 \equiv m\partial_2(g_k) = x_1 x_2^{k-2} m$

2° $0 \equiv m^{2k-5}\partial_1(g_k) = x_1^2 m^{2k-5} + \alpha_{k-1} x_2^{k-1} m^{2k-5} \equiv x_1^2 m^{2k-5}$

3° $0 \equiv x_2^{k-2}\partial_1(g_k) = 3x_1^2 x_2^{k-2} + \alpha_{k-1} x_2^{2k-3} \equiv \alpha_{k-1} x_2^{2k-3}$,

so $x_2^{2k-3} \equiv 0$ since $\alpha_{k-1} \neq 0$.

We have now all generators of m^{2k-3}, so $m^{2k-3} \subset m\Delta + m^{2k-2}$ and g_k is (2k-3)-determined.

Since $x_2^{2k-4} \notin m\Delta + m^{2k-3}$ we have that g_k is not (2k-5)-determined.

—o—

We apply Lemma 1 for n=4; so let:

$$g_4 = x_1^3 + \alpha_4 x_1 x_2^3 + \beta_4 x_2^4$$

If $\beta_4 \neq 0$ define an element of L_2 by: $\begin{cases} x_1 := x_1 \\ x_2 := x_2 - p x_1 \text{ with } p \overset{*}{=} \frac{4}{4\beta_4} \in \mathbb{R} \end{cases}$

So we have:

$$g_4 \overset{4}{\sim} x_1^3 + \alpha_4 x_1 (x_2 - p x_1)^3 + \beta_4 (x_2 - p x_1)^4 =$$

$$= x_1^3 + \gamma_1 x_1^4 + \gamma_2 x_1^3 x_2 + \gamma_3 x_1^2 x_2^2 + (\alpha_4 - 4p\beta_4) x_1 x_2^3 + \beta_4 x_2^4$$

$$\overset{*}{=} x_1^3 + \gamma_1 x_1^4 + \gamma_2 x_1^3 x_2 + \gamma_3 x_1^2 x_2^2 + \beta_4 x_2^4.$$

Next apply again Lemma 1. Since the coefficients of $x_1x_2^3$ and x_2^4 don't change we get:

$$g_4 \overset{4}{\backsim} x_1^3 + \beta_4 x_2^4 \overset{4}{\backsim} x_1^3 \pm x_2^4$$

g_4 is 4-determined (lemma 2), so

$$g \backsim x_1^3 \pm x_2^4 \quad (E_6)$$

If $\beta_4=0$ and $\alpha_4\neq 0$ then $g_4 = x_1^3 + \alpha_4 x_1 x_2^3 \overset{4}{\backsim} x_1^3 + x_1x_2^3$. Then g_4 is 5-determined (Lemma 2), so we have to consider

$$g_5 = x_1^3 + x_1x_2^3 + \gamma_0 x_1^5 + \ldots + \gamma_5 x_2^5.$$

Lemma 3:

a) $g_5 = x_1^3 + x_1x_2^3 + \gamma_0 x_1^5 + \ldots + \gamma_5 x_2^5 \overset{5}{\backsim} x_1^3 \pm x_1x_2^3$

b) $g_4 = x_1^3 + x_1x_2^3$ *is 4-determined.*

Proof: we shall give an outline of the computation:

step 1: Using $\begin{cases} x_1 := x_1 + \rho \text{ with } \rho = -\frac{1}{3}(\gamma_0 x_1^3 + \ldots + \gamma_3 x_2^3) \in m^3 \\ x_2 := x_2 \end{cases}$

we get $g_5 \overset{5}{\backsim} x_1^3 + x_1x_2^3 + \gamma_4 x_1 x_2^4 + \gamma_5 x_2^5$.

step 2: Using $\begin{cases} x_1 := x_1 \\ x_2 := x_2 + \sigma \text{ with } \sigma = -\frac{1}{3}\gamma_4 x_2^2 \in m^2 \end{cases}$

we get $g_5 \overset{5}{\backsim} x_1^3 + x_1x_2^3 + \gamma_5 x_2^5$

The coëfficient of x_2^5 is still the same as in step 1!

step 3: Using $\begin{cases} x_1 := x_1 \\ x_2 := px_1 + x_2 \text{ with } p = \gamma_5 \in \mathbb{R} \end{cases}$

we get $g_5 \overset{5}{\backsim} x_1^3 + x_1(px_1 + x_2)^3 + \beta_5(px_1 + x_2)^5$.

step 4: Using $\begin{cases} x_1 := x_1 + \rho \\ x_2 := x_2 \text{ with } \rho = -\frac{1}{3}(p^3x_1^2 + 3p^2x_1x_2 + 3px_2^2) \in m^2 \end{cases}$

we get $g_5 \overset{5}{\backsim} x_1^3 + x_1x_2^3 + x_1.[\text{degree } 4] + \rho x_2^3 + \gamma_5 x_2^5$

The coëfficient of x_2^5 is equal to $-p + \gamma_5 = 0$.

step 5: Apply again the system of step 1 and 2. The coëfficient of $x_2^{\,5}$ does not change, so we get: $g_5 \overset{5}{\backsim} x_1^{\,3} + x_1x_2^{\,3}$.

step 6: Since g_4 is 5-determined and $g_5 \overset{5}{\backsim} g_4$ for all $\gamma_o,\ldots,\gamma_5$ we get g_5 is 5-determined and so also $g_5 \backsim g_4$. Let $h_4 = g_4$ then $h_5 = g_4 + \gamma_o x_1^{\,5} +\ldots+ \gamma_5 x_2^{\,5}$ for some values of $\gamma_o,\ldots,\gamma_5$. Since the righthandside is 5-determined we have

$$h \backsim g_4 + \gamma_o x_1^{\,5} +\ldots+ \gamma_5 x_2^{\,5} \backsim g_4.$$

So g_4 is 4-determined.

Remark 1:

It is also possible to prove $x_2^{\,5} \in m\Delta(x_1^{\,3} + x_1x_2^{\,3} + \beta_5 x_2^{\,5}) + m^6$ for all β_5 and then to use proposition (2.2) to prove

$$x_1^{\,3} + x_1x_2^{\,3} + \beta_5 x_2^{\,5} \overset{5}{\backsim} x_1^{\,3} + x_1x_2^{\,3}.$$

Remark 2:

If f_k is (k+1)-determined; it is not always true that $f_k + \beta_o x_1^{\,k+1} + \beta_1 x_1^{\,k} x_2 +\ldots+ \beta_{k+1} x_2^{\,k+1}$ is also (k+1)-determined. If $m^k \subseteq m\Delta(f_k) + m^{k+1}$ this guarentees only (k+1)-determinacy for small values of $\beta_o,\ldots,\beta_{k+1}$ (compare proposition (2.10)).

We return to the case that $\beta_4 = 0$ and $\alpha_4 \neq 0$.

From the Lemma 3 it follows that $g_4 \overset{4}{\backsim} x_1^{\,3} + x_1x_2^{\,3}$ is 4-determined.

So
$$g \backsim x_1^{\,3} + x_1x_2^{\,3} \quad (E_7)$$

If $\alpha_4 = 0$ and $\beta_4 = 0$ we have $g_4 = x_1^{\,3}$ and we consider

$$g_5 = x_1^{\,3} + \alpha_5 x_1 x_2^{\,4} + \beta_5 x_2^{\,5}$$

If $\beta_5 \neq 0$ then we can derive in the same way as in the case E_6 that $g_5 \overset{5}{\backsim} x_1^{\,3} + \beta_5 x_2^{\,5} \backsim x_1^{\,3} + x_2^{\,5}$, which is 5-determined; so

$$g \backsim x_1^{\,3} + x_2^{\,5} \quad (E_8)$$

If $\beta_5 = 0$ then g_5 is 7-determined and we have to study higher jets. First we derive a normalform in a more general case.

<u>Lemma 4</u>: *For* $\mu \neq 0$ *and* $k \geq 5$ *we have:*

$$x_1^3 + \mu x_1 x_2^{k-1} + \beta_{k+1} x_2^{k+1} + \ldots + \beta_{n-1} x_2^{n-1} + \alpha_n x_1 x_2^{n-1} + \beta_n x_2^n \overset{n}{\sim}$$
$$\overset{n}{\sim} x_1^3 + \mu x_1 x_2^{k-1} + \beta_{k+1} x_2^{k+1} + \ldots + \beta_n x_2^n.$$

<u>Proof</u>:

Define an element of $\mathcal{L}_2$ by: $\begin{cases} x_1 := x_1 \\ x_2 + \rho \text{ with } \rho \in m^{n-k+1};\ \rho = \dfrac{-\alpha_n}{\mu} x_2^{n-k-1}. \end{cases}$

The left-hand side is n-equivalent to:

$$x_1^3 + \mu x_1 x_2^{k-1} + \mu x_1 x_2^{k-2}\rho + \beta_{k+1} x_2^{k+1} + \ldots + \beta_{n-1} x_2^{n-1} + \alpha_n x_1 x_2^{n-1} + \beta_n x_2^n$$
$$= x_1^3 + \mu x_1 x_2^{k-1} + \beta_{k+1} x_2^{k+1} + \ldots + \beta_{n-1} x_2^{n-1} + \beta_n x_2^n + x_1 x_2^{k-2}[\mu\rho + \alpha_n x_2^{n-k+1}]$$
$$= x_1^3 + \mu x_1 x_2^{k-1} + \beta_{k+1} x_2^{k+1} + \ldots + \beta_{n-1} x_2^{n-1} + \beta_n x_2^n.$$

<u>Corollary</u>: (<u>normalform</u>): *let* $\mu \neq 0$ *and* $n \geq k \geq 5$.

If $g \overset{k}{\sim} x_1^3 + \mu x_1 x_2^{k-1}$ *then* $g \overset{n}{\sim} x_1^3 + \mu x_1 x_2^{k-1} + \beta_{k+1} x_2^{k+1} + \ldots + \beta_n x_2^n$.

Let us return to $g_5 = x_1^3 + \alpha_5 x_1 x_2^4$.

Since g_5 is not 5-determined we study higher jets of g:

<u>Lemma 5</u>: *Let* $g_7 = x_1^3 + \alpha_5 x_1 x_2^4 + \beta_6 x_2^6 + \beta_7 x_2^7$ *and* $(4\alpha_5^3 + 27\beta_6^2) \neq 0$.

1° g_7 *is 7-determined*

2° g_6 *is in fact 6-determined.*

<u>Proof</u>:

a) If $\alpha_5 = 0$ and $\beta_6 \neq 0$ we can apply Lemma 2.

b) Let us suppose $\alpha_5 \neq 0$. We shall show that $m^7 \subseteq m\Delta + m^8$.

$$\begin{cases} \partial_1 g_7 = 3x_1^2 + \alpha_5 x_2^4 \\ \partial_2 g_7 = 4\alpha_5 x_1 x_2^3 + 6\beta_6 x_2^5 + 7\beta_7 x_2^6. \end{cases}$$

Modulo $m\Delta + m^8$ we have:

$x_1^2 m^4 \equiv 0$

$x_1 x_2^3 m^3 \equiv 0$

and
$$\begin{cases} 0 \equiv 3x_1^2 x_2^3 + \alpha_5 x_2^7 \\ 0 \equiv 4\alpha_5 x_1 x_2^5 + 6\beta_6 x_2^7 \\ 0 \equiv 4\alpha_5 x_1^2 x_2^3 + 6\beta_6 x_1 x_2^5. \end{cases}$$

Since $\begin{vmatrix} \alpha_5 & 6\beta_6 & 0 \\ 3 & 0 & 4\alpha_5 \\ 0 & 4\alpha_5 & 6\beta_6 \end{vmatrix} = -4[\alpha_5^3 + 27\beta_6^2] \neq 0$ are $\begin{cases} x_1^2 x_2^3 \equiv 0 \\ x_1 x_2^5 \equiv 0 \\ x_2^7 \equiv 0 \end{cases}$

Now 1° follows:

Because $x_2^7 \in m\Delta + m^8$ for all β_7 we have that g_7 is 6-determined.

So if $\beta_5 = 0$ we have $g \backsim x_1^3 + Ax_1x_2^4 + Bx_2^6$ (J_{10}) if $4A^3 + 27B^2 \neq 0$.

Remark 3:

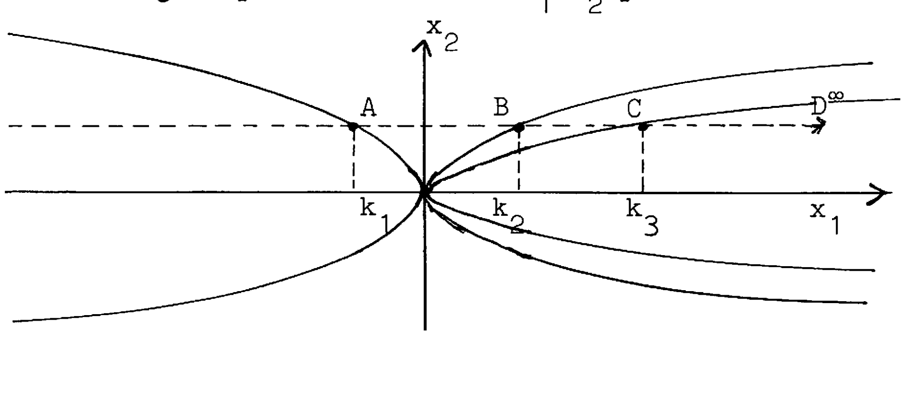
Consider the question: When are two germs $f_{(A,B)} = x_1^3 + Ax_1x_2^4 + Bx_2^6$ of type J_{10} equivalent? It turns out that the action of L_2 on the subset $x_1^3 + \mathbb{R}x_1x_2^4 + \mathbb{R}x_2^6$ of $J^6(2,1)$ coincides with the action of $GL(2)$. In fact the only possibility is a multiplication in the x_2-direction:

$$(*) \quad \begin{cases} x_1 := x_1 \\ x_2 := \lambda x_2 \quad \lambda \neq 0. \end{cases}$$

A geometrical invariant of $x_1^3 + Ax_1x_2^4 + Bx_2^6$ is constructed as follows:

$$x_1^3 + Ax_1x_2^4 + Bx_2^6 = (x_1 + k_1x_2^2)(x_1 + k_2x_2^2)(x_1 + k_3x_2^2)$$

and defines (over $\mathbb{C}$) 3 parabolas in the x_1-x_2-plane.



The line $x_2 = 1$ intersects the 3 parabolas in 3 points $A(k_1,1)$, $B(k_2,1)$ and $C(k_3,1)$. Let D^∞ be the point at infinity of $x_2 = 1$, then $(ABCD) = {}^{k_2-k_1}\!/\!_{k_3-k_1}$ is an invariant of the L_2-action.

Let now $f_{(A_1,B_1)} \backsim f_{(A_2,B_2)}$ then (*) implies:

$A_1\lambda^4 = A_2, B_1\lambda^6 = B_2$; so $A_1{}^3\lambda^{12} = A_2{}^3$, $B_1{}^2\lambda^{12} = B_2{}^2$.

Hence $\lambda^{12}(4A_1{}^3 + 27B_1{}^2) = 4A_2{}^3 + 27B_2{}^2$.

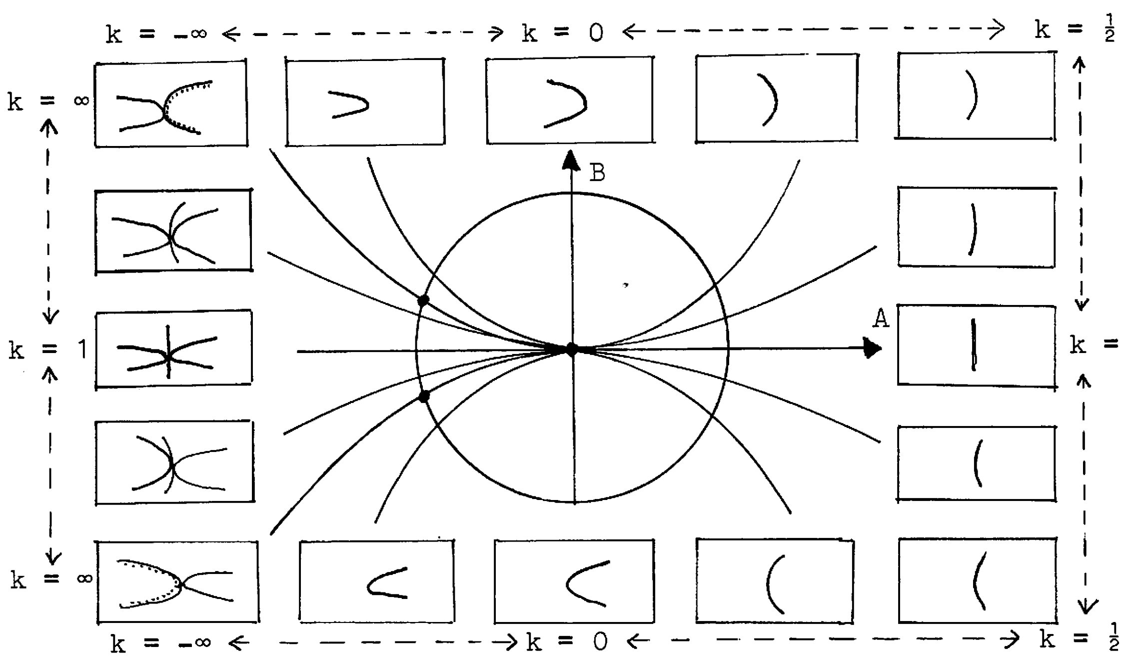
Define $k(A,B) = \dfrac{A^3}{4A^3 + 27B^2}$. Then $k(A_1,B_1) = k(A_2,B_2)$.

On the other hand, if $k(A_1,B_1) = k(A_2,B_2)$ then $f(A_1,B_1) \backsim f(A_2,B_2)$ over $\mathbb{C}$. In the real case we need the additional condition:

$$(B_1B_2 > 0) \vee (B_1B_2 = 0 \wedge A_1A_2 > 0)$$

For each $k \in \mathbb{R}$ there are two different real germs of type J_{10}. Moreover there are two different topological types; corresponding to 3 real parabolas with $k \in (-\infty,1]$ and 1 real parabola with $k \in [1,\infty)$.

The equivalenceclasses form a system of curves in the A-B-plane.



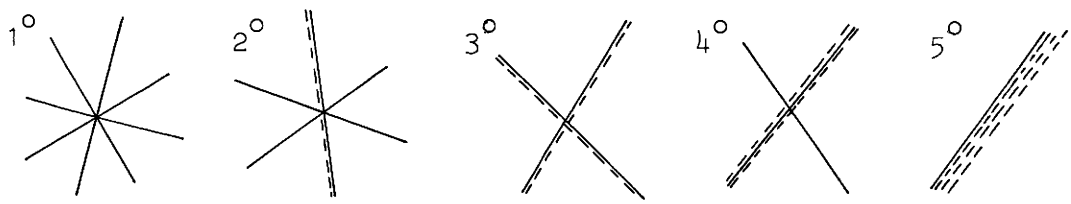
A parametrization of the family can be given by a closed curve around the origin, for example: C: $4|A|^3 + 27|B|^2 = 1$.

The two intersectionpoints of C with $4A^3 + 27B^2 = 0$ correspond to more degenerate germs. The other points of the curve are in 1-1-correspondence to the equivalenceclasses of germs of type J_{10}.

<u>(4.6)</u> <u>Proposition</u>: *If* $r = 2$ *and we are in the case* $g_3 = 0$ *of proposition* (4.3) *then* $f(x_1,\dots,x_n) \overset{4}{\sim} g(x_1,x_2) + e_3x_3^2 + \dots + e_nx_n^2$, *where* g_4 *is in exactly one of the following cases:*

- 1° $g_4 = (x_1^2 \pm x_2^2)(x_1^2 + \alpha x_2^2)$ $\alpha \neq 0,-1,1$
- 2° $g_4 = x_1^2(x_1^2 \pm x_2^2)$
- 3° $g_4 = (x_1^2 \pm x_2^2)^2$ with codim $(f) \geq 10$
- 4° $g_4 = x_1^3x_2$ with codim $(f) \geq 10$
- 5° $g_4 = x_1^4$ with codim $(f) \geq 11$
- 6° $g_4 = 0$ with codim $(f) \geq 15$

<u>Proof</u>: Because $g_3 = 0$, $g_4(x_1,x_2)$ is a homogeneous polynomial of degree 4. We may factor g_4 into linear forms over $\mathbb{C}$. The six cases correspond to 4, 3, 2, 2, 1 or 0 factors; indicated in the following pictures of the sets $g_4 = 0$.



By linear transformation one can obtain one of the given expressions for g_4. In each case one constructs first a normalform of the 5-jet of f. Then straight-forward computations show:

- 3° $g_4 = (x_1^2 \pm x_2^2)^2$ $\Rightarrow$ codim $(f) \geq 10$
- 4° $g_4 = x_1^3x_2$ $\Rightarrow$ codim $(f) \geq 10$
- 5° $g_4 = x_1^4$ $\Rightarrow$ codim $(f) \geq 11$
- 6° $g_4 = 0$ $\Rightarrow$ codim $(f) \geq 15$.

(4.7) Theorem: *If* $r = 2$ *and we are in case* 1° *of proposition* (4.6) *then* $f(x_1,\dots,x_n) \sim (x_1^2 \pm x_2^2)(x_1^2 + \alpha x_2^2)$ *where* $\alpha \neq 0,-1$ *or* 1. *The number* α *is an invariant under R-and RL-equivalence* .

Proof:

A straight-forwardcomputation shows, that $m^5 \subseteq m^2\Delta(f) + m^6$ for all $\alpha \neq 0$, -1 or 1.

The invariance of α is related to the crossratio of the four (complex) lines with equation: $(x_1^2 \pm x_2^2)(x_1^2 + \alpha x_2^2) = 0$.

Two germs of the family are equivalent if and only if their cross-ratios are equal (modulo permutation of the lines, which gives a permutation of the six possible answers: d, $\frac{1}{d}$, 1-d, $\frac{1}{1-d}$, $\frac{d}{d-1}$ and $\frac{d-1}{d}$) (Compare also (2.12) example 1).

(4.8) Theorem: *If* $r = 2$ *and we are in case* $g_4 = x_1^2(x_1^2 \pm x_2^2)$ *of proposition* (4.6) *then:*

either: $f(x_1,\dots,x_n) \sim x_1^4 \pm x_1^2x_2^2 + \alpha x_2^p + e_3x_3^2 + \dots + e_nx_n^2$

$(p > 5)$ (X_{p+5})

or: codim $(f) = \infty$

If codim $(f) < \infty$ *then* α *is a local* R-*invariant;*
for RL-*equivalence we can arrange that* $\alpha = \pm 1$.

Proof: $x^4 \pm x_1^2x_2^2$ has infinite codimension. The theorem is a consequence of the following Lemmas:

Lemma 1: (normalform): *If* $g \overset{4}{\sim} x_1^4 \pm x_1^2x_2^2$ *then*

$$g \overset{k}{\sim} x_1^4 \pm x_1^2x_2^2 + \alpha_5x_2^5 + \dots + \alpha_kx_2^k.$$

Proof: For $k = 4$ the statement is true; we proceed by introduction on k. Let $g \overset{k+1}{\sim} x_1^4 \pm x_1^2x_2^2 + \alpha_5x_2^5 + \dots + \alpha_kx_2^k + \tau_{k+1}$ where $\tau_{k+1} = \lambda_0x_1^{k+1} + \lambda_1x_1^kx_2 + \dots + \lambda_kx_1x_2^k + \lambda_{k+1}x_2^{k+1}$.

Define an element of L_2 by:

$$\begin{cases} x_1 := x_1 + \sigma_1 \text{ with } \sigma_1 \overset{*}{=} -[\frac{1}{4}\lambda_o x_1^{k-2} \pm \frac{1}{2}\lambda_k x_2^{k-2}] \in m^{k-2} \\ x_2 := x_2 \end{cases}$$

So we have:

$$\begin{aligned} g &\overset{k+1}{\smile} x_1^4 + 4x_1^3\sigma_1 \pm x_1^2x_2^2 \pm 2x_1x_2^2\sigma_1 + \alpha_5x_2^5 + \ldots + \alpha_kx_2^k + \tau_{k+1} = \\ &= x_1^4 \pm x_1^2x_2^2 + \alpha_5x_2^5 + \ldots + \alpha_kx_2^k + x_1^3[4\sigma_1 + \lambda_ox_1^{k-2}] + \\ &\quad + x_1x_2^2[\pm 2\sigma_1 + \lambda_kx_2^{k-2}] + \lambda_1x_1^kx_2 + \ldots + \lambda_{k-1}x_1^2x_2^{k-1} + \lambda_{k+1}x_2^k \\ &\overset{*}{=} x_1^4 + x_1^2x_2^2 + \alpha_5x_2^5 + \ldots + \alpha_kx_2^k + \mu_1x_1^kx_2 + \ldots + \mu_{k-1}x_1^2x_2^{k-1} + \\ &\quad + \lambda_{k+1}x_2^k. \end{aligned}$$

Next define an element of L_2 by:

$$\begin{cases} x_1 := x_1 \\ x_2 := x_2 + \sigma_2 \text{ with } \sigma_2 \overset{*}{=} \mp \frac{1}{2}[\mu_1x_1^{k-2} + \ldots + \mu_{k-1}x_2^{k-2}] \in m^{k-2} \end{cases}$$

So we have:

$$\begin{aligned} g &\overset{k+1}{\smile} x_1^4 \pm x_1^2x_2^2 + \alpha_5x_2^5 + \ldots + \alpha_kx_2^k + \\ &\quad + x_1^2x_2.[\pm 2\sigma_2 + \mu_1x_1^{k-2} + \ldots + \mu_{k-1}x_2^{k-2}] + \lambda_{k+1}x_2^{k+1} \\ &\overset{*}{=} x_1^4 \pm x_1^2x_2^2 + \alpha_5x_2^5 + \ldots + \alpha_kx_2^k + \lambda_{k+1}x_2^{k+1}. \end{aligned}$$

<u>Lemma 2</u>: $x_1^4 \pm x_1^2x_2^2 + \alpha x_2^k$ *is* k-*determined if* $\alpha \neq 0$; $(k \geq 5)$.

<u>Proof</u>: A straightforward computation shows: $m^{k+1} \subseteq m^2\Delta(f) + m^{k+2}$ for all $\alpha \neq 0$.

<u>Lemma 3</u>: α *is a local R-invariant of* $x_1^4 \pm x_1^2x_2^2 + \alpha x_2^k$; *for RL-equivalence we can arrange, that* $\alpha = \pm 1$.

<u>Proof</u>:

(i) $x_2^k \notin m\Delta(f) + m^{k+1}$ for all $\alpha \neq 0$; apply (2.9).

(ii) $x_2^k \in m\Delta(f) + f^*(m_1)$ for all $\alpha \neq 0$; apply (2.3).

Now theorem (4.8) is proved.

$\boxed{\text{corank} = 3}$

(4.9) Proposition: *If* $r = 3$ *then*

$$f(x_1,\dots x_n) \overset{3}{\sim} g_3(x_1,x_2,x_3) + e_4x_4^2 + \dots + e_nx_n^2$$

where g_3 *has one of the following expressions:*

a) $g_3(x_1,x_2,x_3) = x_3x_2^2 + x_1^3 + g_1x_1x_3^2 + g_2x_3^3$ with $4g_1^3 + 27g_2^2 \neq 0$

b) $g_3(x_1,x_2,x_3) = x_1x_3^2 + x_2^3$

c) $g_3(x_1,x_2,x_3) = x_1^3 + x_2^2x_3 \pm x_1^2x_3$

d) $g_3(x_1,x_2,x_3) = x_1(x_1^2 \pm x_2^2 \pm x_3^2)$

e) $g_3(x_1,x_2,x_3) = x_2(x_1x_2 - x_3^2)$ *with* codim $(f) \geq 10$

f) $g_3(x_1,x_2,x_3) = x_1(x_2^2 \pm x_3^2)$ *with* codim $(f) \geq 10$

g) $g_3(x_1,x_2,x_3) = x_1(x_1^2 \pm x_2^2)$ *with* codim $(f) \geq 11$

h) $g_3(x_1,x_2,x_3) = x_1^2x_2$ *with* codim $(f) \geq 11$

k) $g_3(x_1,x_2,x_3) = x_1^3$ *with* codim $(f) \geq 15$

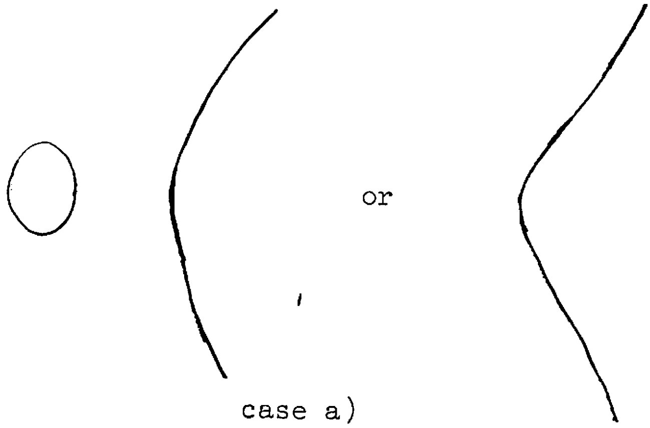
Proof:

Since $g_3 = 0$ is the equation of a cubic curve in the projective plane, we can use the projective classification of real cubic curves (cf. BURAU[8] or V.D. WAERDEN[26]).

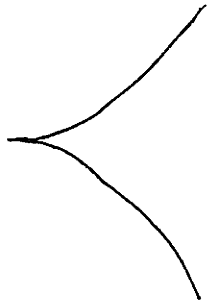
In case a), b) and c) the curves are irreducible, in the other cases the curves are reducible. Case a) is the elliptic curve (= without multiple points). Case b) is a curve with cusp-point. Case c) is the curve with double point.

By linear transformation we can arrange that g_4 gets into one of the given expressions.

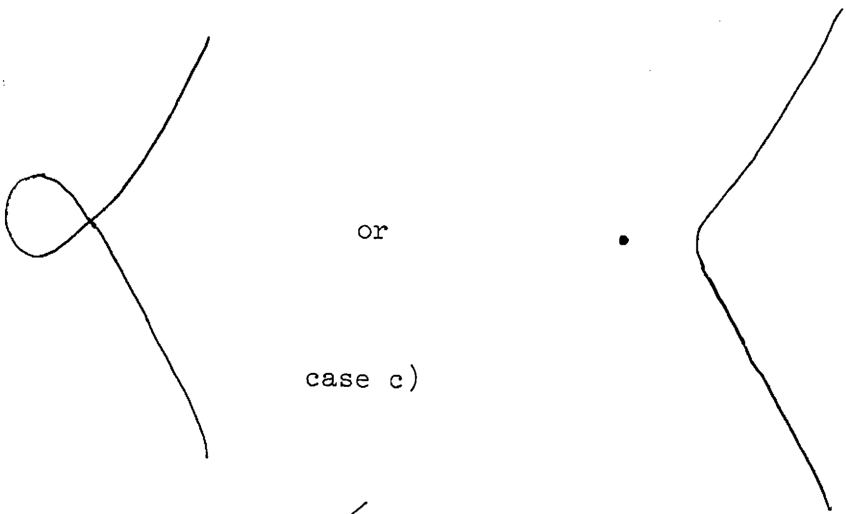
Next one constructs in the cases e-f a normalform of the 4-jet of f. Then straight-forward computations show the assertions concerning the codimension in e)-k).



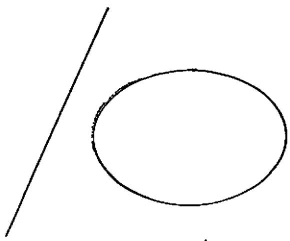
case a)



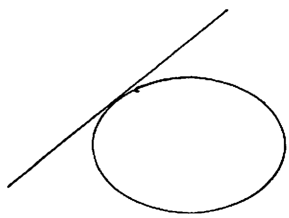
case b)



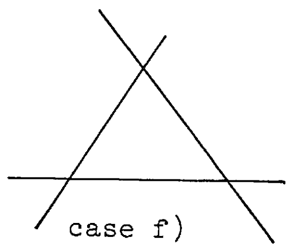
case c)



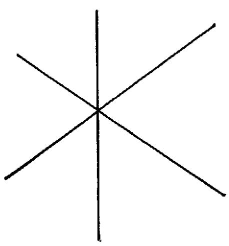
case d)



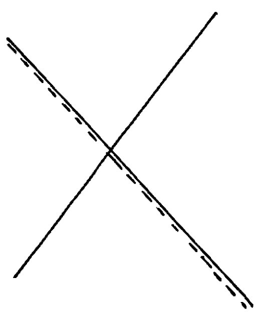
case e)



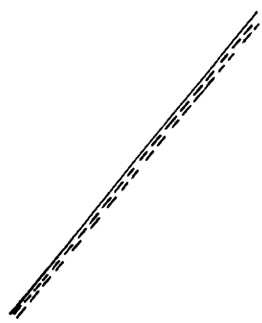
case f)



case g)



case h)



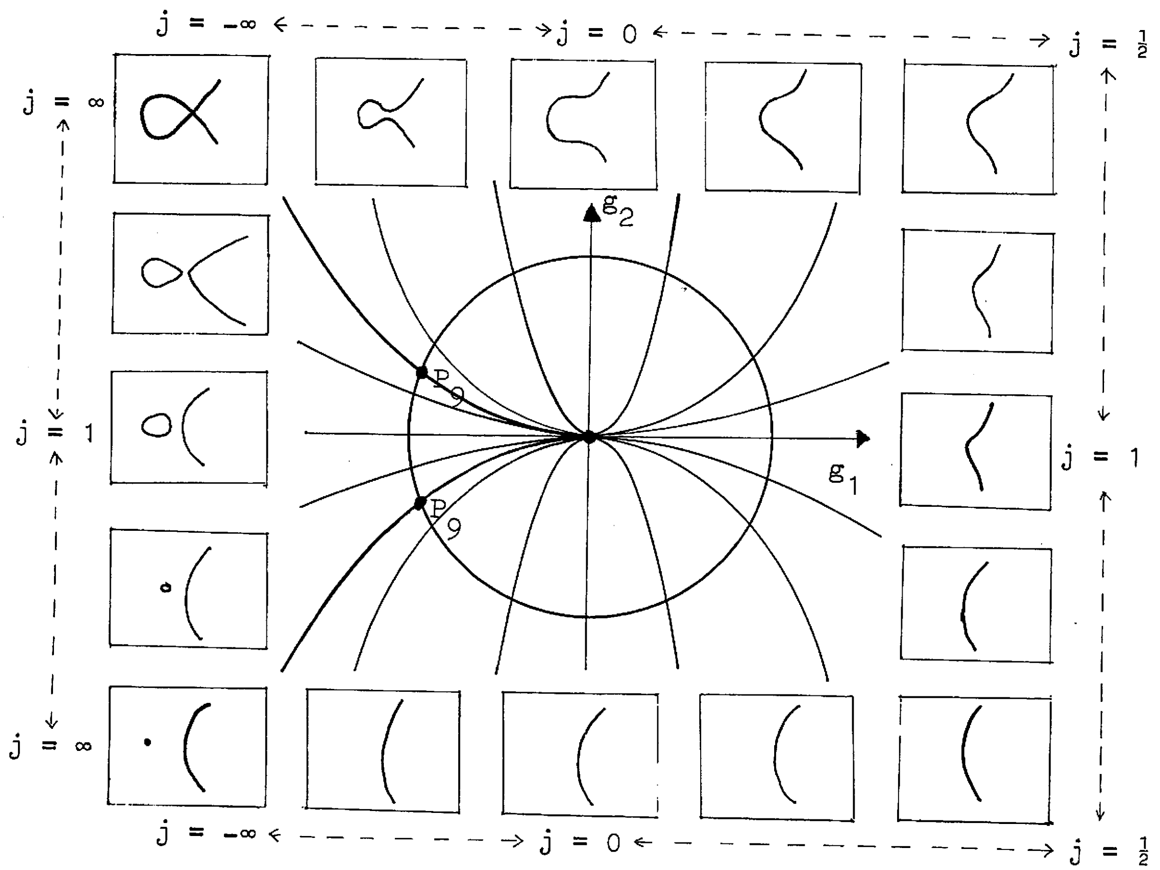
case k)

<u>(4.10)</u> <u>Theorem</u>: *If* $r = 3$ *and we are in case a) of proposition* (4.9) *then* $f(x_1,\dots,x_n) \backsim x_3x_2^2 + x_1^3 + g_1x_1x_3^2 + g_2x_3^3 + e_4x_4^2 + \dots + e_nx_n^2$ (P_8), *with* $4g_1^3 + 27g_2^2 \neq 0$ *and* codim $(f) = 7$.

The number $j = \dfrac{4g_1^3}{4g_1^3 + 27g_2^2}$ *is an invariant of* f.

Two elements of this family are equivalent iff their g_2*'s have the same sign and their* j*'s are equal. If* $g_2 = 0$ *two elements are equivalent iff their* g_1*'s have the same sign.*

<u>Proof</u>: A straight-forward computation shows that f is 3-determined. On the homogeneous polynomials of degree 3 in x_1,x_2,x_3 the action of L_3 coincides with GL(3). So j is the classical j-invariant of elliptic curves. In the complex case $j \in \mathbb{C}$ classifies the elliptic curves completely. In the real case we have for every $j \in \mathbb{R}$ two different real elliptic curves. Moreover there are 2 different topological types: unipartite with $j \in (-\infty,1]$ and bipartite with $j \in [1,\infty)$.



A parametrization of the family can be given by a circular curve C around the origin, for example

$$C: \quad 4|g_1|^3 + 27|g_2|^2 = 1$$

The two intersection points of C with $4g_1^3 + 27g_2^2 = 0$ correspond to the two types of curves with double point. The other points of the curve are in 1-1-correspondence to the equivalenceclasses of real elliptic curves.

(4.11) Theorem: *If* $r = 3$ *and we are in case c) of proposition* (4.9) *then either:* $f(x_1,\ldots,x_n) \backsim x_1^3+x_2^2x_3\pm x_1^2x_3+\beta_k x_3^k+e_4x_4^2+\ldots+e_nx_n^2 \; (P_{k+5})$

(with $\beta_k \neq 0$ *and* $k \geq 4$*)*

or: $\operatorname{codim}(f) = \infty$.

If $\operatorname{codim}(f) < \infty$ *then each* β_k *is invariant under R-equivalence only; for RL-equivalence we can arrange* $\beta_k = \pm 1$.

Proof: $g_3 = x_1^3 + x_2^2x_3 \pm x_1^2x_3$ has infinite codimension. The theorem is a consequence of the following Lemmas:

Lemma 1: *Let* τ_j *be a homogeneous polynomial of degree* j*; then*
$x_1^3 + x_2^2x_3 \pm x_1^2x_3 + \tau_j \overset{j}{\backsim} x_1^3 + x_2^2x_3 \pm x_1^2x_3 + \lambda x_3^j$.

Proof:

Let an element of L_3 be defined by $x_i := x_i + \sigma_i$ with $\sigma_i \in m^{j-2}$ then
$g_3 = x_1^3 + x_2^2x_3 \pm x_1^2x_3 + \tau_j \overset{j}{\backsim} g_3 + \sigma_1\partial_1(g_3) + \sigma_2\partial_2(g_3) + \sigma_3\partial_3(g_3) + \tau_j$

A direct computation shows that:

$$x_1m^2 + x_2m^2 \subset m\Delta(g_3)$$

so also
$$x_1m^{j-1} + x_2m^{j-1} \subset m^{j-1}\Delta(g_3).$$

This means that we can choose σ_1,σ_2 and σ_3 in such a way in m^{j-1} that the terms of $\tau_j(x_1,x_2,x_3)$, that are divisible by x_1 or x_2 vanish against $\sigma_1\partial_1(g_3) + \sigma_2\partial_2(g_3) + \sigma_3\partial_3(g_3)$. So $g_3 + \tau_j \overset{j}{\sim} g_3 + \lambda x_3^j$.

Corollary: We have the following normalform for the k-jet of $f (k \geq 4)$:

$$g \overset{k}{\sim} x_1^3 + x_2^2 x_3 \pm x_1^2 x_3 + \alpha_4 x_3^4 + \ldots + \alpha_k x_3^k.$$

Lemma 2: $g = x_1^3 + x_2^2 x_3 \pm x_1^2 x_3 + \lambda x_3^k$ $(\lambda \neq 0)$ *is* k-*determined* $(k \geq 4)$

Proof:

$$\begin{cases} \partial_1 g = 3x_1^2 \pm 2x_1 x_2 \\ \partial_2 g = 2x_2 x_3 \\ \partial_3 g = x_2^2 \pm x_1^2 + k\lambda x_3^{k-1} \end{cases}$$

We shall show: $m^{k+1} \subseteq m^2\Delta(g) + m^{k+2}$.

Since $m^3\Delta(g) + m^{k+2} = m^2\Delta(g_3) + m^{k+2}$, we find already all generators of m^5, except x_3^5. So we have only to show, that $x_3^{k+1} \in m\Delta + m^{k+2}$.

$x_3^2\partial_3 g = x_3^2 x_2^2 \pm x_1^2 x_3^2 + k\lambda x_3^{k+1}$

So $k\lambda x_3^{k+1} \equiv \pm x_1^2 x_3^2 \equiv \frac{2}{3} x_1 x_2 x_3^2 \equiv 0 \pmod{m^2\Delta + m^{k+2}}$. Since $\lambda \neq 0$ we have $x_3^{k+1} \in m^2\Delta + m^{k+2}$.

Lemma 3: *In* $x_1^3 + x_2^2 x_3 \pm x_1^2 x_3 + \lambda x_3^k$ $\lambda \neq 0$ *is a local invariant under R-equivalence.*

Proof:

Since $\dim \dfrac{m}{m\Delta + m^{k+1}} > \dim \dfrac{m}{m\Delta + m^{k+1} + x_3^k}$ we have $x_3^k \notin m\Delta + m^{k+1}$ and this implies the lemma.

Lemma 4: $x_1^3 + x_2^2 x_3 \pm x_1^2 x_3 + \lambda x_3^k \underset{RL}{\sim} x_1^3 + x_2^2 x_3 \pm x_1^2 x_3 \pm x_3^k$

Proof:

$x_3^k \in m\Delta + f^*(m_1) + m^{k+1}$ for all $\lambda \neq 0$; so x_3^k is contained in the tangentspace to the RL-orbit; so (2.3) applies and we are done.

It is possible to give explicit formulas for the diffeomorphisms:

Let $x_i := px_i$ then

$$x_1^3 + x_2^2x_3 \pm x_1^2x_3 + \lambda x_3^k \underset{R}{\backsim} p^3x_1^3 + p^3x_2^2x_3 \pm p^3x_1^2x_3 + \lambda p^kx_3^k$$
$$\underset{RL}{\backsim} x_1^3 + x_2^2x_3 \pm x_1^2x_3 + \lambda p^{k-3}x_3^k = x_1^3 + x_2^2x_3 \pm x_1^2x_3 + x_3^k$$

if $p = \sqrt[k-3]{\left|\frac{1}{\lambda}\right|}$.

(4.12) Theorem: *If* $r = 3$ *and we are in case b) of proposition* (4.9) *then either:*

$$f(x_1,\ldots,x_n) \backsim x_1x_3^2 + x_2^3 + \alpha x_1^3x_2 \pm x_1^4 + e_4x_4^2 + \ldots + e_nx_n^2 \quad (Q_{10})$$

with codim $(f) = 9$

or: codim $(f) > 9$

If codim $(f) = 9$ *then* α *is local-invariant under R-equivalence only; for RL-equivalence we can arrange that* $\alpha = -1$, 0 or 1.

Proof:

The proof is a consequence of the following lemma 1-2.

Lemma 1: *If* $g \overset{3}{\backsim} x_1x_3^2 + x_2^3$ *we have the following normalform for the n-jet:* $g \overset{n}{\backsim} x_1x_3^2 + x_2^3 + \sigma_4 + \ldots + \sigma_n$ $(n \geq 4)$ *where*
$\sigma_p = \alpha_px_1^{p-1}x_2 + \beta_px_1^p$.

Proof:

By introduction on n; for $n = 3$ is the statement true.

Let τ_n be a homogeneous polynomial of degree n and let
$g \overset{n}{\backsim} x_1x_3^2 + x_2^3 + \sigma_4 + \ldots + \sigma_{n-1} + \tau_n$.

Define an element of L_3 by
$$\begin{cases} x_1 := x_1 + \sigma_1 \text{ with } \sigma_1 \in m^{n-2} \\ x_2 := x_2 + \sigma_2 \text{ with } \sigma_2 \in m^{n-2} \\ x_3 := x_3 + \sigma_3 \text{ with } \sigma_3 \in m^{n-2} \end{cases}$$

Then: $g \overset{n}{\sim} x_1x_3^2 + x_2^3 + x_3^2\sigma_1 + 2x_1x_3\sigma_3 + 3x_2^2\sigma_2 + \sigma_4 +\ldots+ \sigma_{n-1} + \tau_n =$

$= x_1x_3^2 + x_2^3 + \sigma_4 +\ldots+ \sigma_{n-1} + [2x_1x_3\sigma_3 + x_3^2\sigma_1 + 3x_2^2\sigma_2 + \tau_n]$

$= x_1x_3^2 + x_2^3 + \sigma_4 +\ldots+ \sigma_{n-1} + \sigma_n$

by a proper choise of σ_1, σ_2 and σ_3.

Lemma 2: $x_1x_3^2 + x_2^3 + Ax_1^3x_2 + Bx_1^4$ *is 4-determined for all* $B \neq 0$.

Proof: cf (2.5) example 3.

Now we start the classification in case 3b):

Lemma 1 implies, that

$$g \overset{4}{\sim} x_1x_3^2 + x_2^3 + \alpha_4x_1^3x_2 + \beta_4x_1^4.$$

If $\beta_4 \neq 0$ g is 4-determined (Lemma 2) and we can arrange that:

$$g \sim x_1x_3^2 + x_2^3 + Ax_1^3x_2 \pm x_1^4 \quad (Q_{10})$$

Since $x_1^3x_2 \notin m\Delta + m^5$ for all A; A is a local R-invariant. Moreover A is not a local RL-invariant, since

$$x_1^3x_2 \in m\Delta + f^*(m_1) + m^5 \text{ for all } A \neq 0.$$

With RL-action we can arrange that $\alpha = 0, +1$ or -1.

Next we have to consider the cases $\beta_4 = 0$ and $\alpha_4 \neq 0$; but this gives already codim (g) > 9. A classification in higher codimension is possible, using Lemma 1, but becomes more and more complicated.

(4.13) Theorem: *If* r = 3 *and we are in case* d) *of proposition* (4.9) *then* codim (f) $\geq$ 9. *If* codim (f) = 9 *then*

$f(x_1,\ldots,x_n) \sim g(x_1,x_2,x_3) + e_4x_4^2 +\ldots+ e_nx_n^2$ *with*

$$\left.\begin{array}{ll} g = x_1^3 + e_2x_1x_2^2 + e_3x_1x_3^2 + A[x_2^4 - 6e_2e_3x_2^2x_3^2 + x_3^4] & (A\neq 0) \\ \text{or } g = x_1^3 + e_2x_1x_2^2 + e_3x_1x_3^2 + B[4x_2^3x_3 - 4e_2e_3x_2x_3^3] & (B\neq 0) \end{array}\right\}(R_{10})$$

A *and* B *are local R-invariants.*

Proof: $g_3 = x_1^3 + e_2x_1x_2^2 + e_3x_1x_3^2$ has infinite codimension. The theorem is a consequence of the following lemmas:

We use the abbrevations:

$$p(x_2,x_3) = x_2^4 - 6e_2e_3x_2^2x_3^2 + x_3^4$$
$$q(x_2,x_3) = 4x_2^3x_3 - 4e_2e_3x_2x_3^3$$

Lemma 1: $g \overset{4}{\sim} g_3 + Ap(x_2,x_3) + Bq(x_2,x_3)$.

Proof: Define an element of L_3 by $x_i := x_i + \sigma_i$ with $\sigma_i \in m^2$ $(i = 1,2,3)$. Let $g_4 = g_3 + \tau_4$; where τ_4 is a homogeneous polynomial of degree 4. Then $g \overset{4}{\sim} g_3 + \sigma_1\partial_1 g + \sigma_2\partial_2 g + \sigma_3\partial_3 g + \tau_4$.
A straight-forward computation of $m^2\Delta + m^5$ shows, that we can choose σ_1,σ_2 and σ_3 in such a way that $g \overset{4}{\sim} g_3 + Ap(x_2,x_3) + Bq(x_2,x_3)$.

Lemma 2: *If* $e_2A^2 + e_3B^2 \neq 0$ *then* $g_3 + Ap(x_2,x_3) + Bq(x_2,x_3)$ *is 4-determined and* codim (f) = 9. *If* $e_2A^2 + e_3B^2 = 0$ *then* codim (f) > 9.

Proof:

$m^5 \subset m^2\Delta + m^6$ for all values of A and B with $e_2A^2 + e_3B^2 \neq 0$.

Lemma 3: *If* $e_2 = e_3$ *then* $g \sim g_3 + A'p(x_2,x_3)$ $(A' \neq 0)$

and $g \sim g_3 + B'q(x_2,x_3)$ $(B' \neq 0)$

Proof:

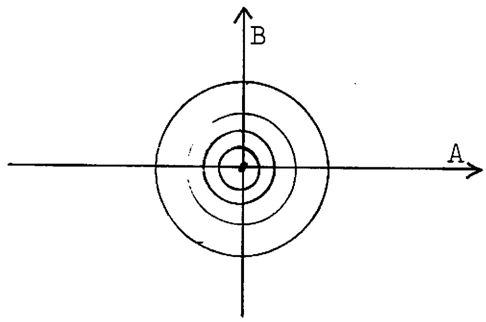
The substitution
$$\begin{cases} x_2 := x_2\cos\phi + x_3\sin\phi \\ x_3 := -x_2\sin\phi + x_3\cos\phi \end{cases}$$
implies that

$$g \sim x_1^3 + e_2x_1x_2^2 + e_2x_1x_3^2 + [A\cos 4\phi - B\sin 4\phi]p(x_2,x_3) + $$
$$+ [A\sin 4\phi - B\cos 4\phi]q(x_2,x_3).$$

If $\phi = \operatorname{arctg}\frac{A}{B}$ then the coëfficient of $p(x_2,x_3)$ vanishes.

If $\phi = \operatorname{arctg}\frac{B}{A}$ then the coëfficient of $q(x_2,x_3)$ vanishes.

We remark that orbits intersect A-B-plane in circles:



<u>Lemma 4</u>: *If* $e_2 \neq e_3$ *then* $g \backsim g_3 + A'p(x_2,x_3)$ *if* $\frac{A-B}{A+B} < 0$

and $g \backsim g_3 + B'q(x_2,x_3)$ *if* $\frac{A-B}{A+B} > 0$

<u>Proof</u>:

The substitution $\begin{cases} x_2 := \frac{1}{2}(\lambda + \frac{1}{\lambda})x_2 + \frac{1}{2}(\lambda - \frac{1}{\lambda})x_3 \\ x_3 := \frac{1}{2}(\lambda - \frac{1}{\lambda})x_2 + \frac{1}{2}(\lambda + \frac{1}{\lambda})x_3 \end{cases}$

implies that $g \backsim x_1^3 + e_2x_1x_2^2 - e_2x_1x_3^2 +$

$\frac{1}{2}[(\lambda^4 + \frac{1}{\lambda^4})A + (\lambda^4 - \frac{1}{\lambda^4})B]p(x_2,x_3) + \frac{1}{2}[(\lambda^4 - \frac{1}{\lambda^4})A + (\lambda^4 + \frac{1}{\lambda^4})B]q(x_2,x_3)$

a) Let $(\lambda^4 + \frac{1}{\lambda^4})A + (\lambda^4 - \frac{1}{\lambda^4})B = 0$ then $\lambda^8A + A + \lambda^8B - B = 0$

$\lambda^8(A + B) = B - A$ and $\lambda^8 = \frac{B-A}{A+B}$.

If $\frac{A-B}{A+B} < 0$ there are real solutions, so the coëfficients of $p(x_2,x_3)$ can vanish.

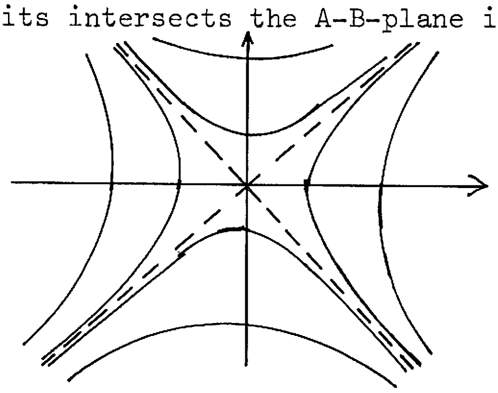
b) Let $(\lambda^4 - \frac{1}{\lambda^4})A + (\lambda^4 + \frac{1}{\lambda^4})B = 0$

$\lambda^8A - A + \lambda^8B + B = 0$

$\lambda^8(A+B) = A - B$ and $\lambda^8 = \frac{A-B}{A+B}$.

If $\frac{A-B}{A+B} > 0$ there are real solutions, so the coëfficient of $q(x_2,x_3)$ can vanish.

We remark, that orbits intersects the A-B-plane in hyperbolas:



Remark:

In the case $e_2 \neq e_3$ we can also use the normalform $x_1^3 + x_1x_2x_3$ for the 3-jet of g. This form is easier for the computations.

If codim (f) = 9 then one can show:

$$g \backsim x_1^3 + x_1x_2x_3 + Cx_2^4 + Dx_3^4 \text{ with } C.D \neq 0$$

which can be transformed such that C' = 1 and D' $\neq$ 0 or such that C' $\neq$ 0 and D = 1.

§5 Remarks

A. A conjecture of Zeeman and strong equivalence.

(5.1) In (2.1) I already mentioned, that the theorems (1.7) and (1.8) don't determine the degree of determinacy completely. Zeeman conjectured in a lecture at the IHES (Bûres-sur-Yvette) that:

$$f \text{ is k-determined} \Leftrightarrow m_n^{k+1} \subset m_n^{2}\Delta(f) + m_n^{k+2}.$$

In the following example I show that this conjecture is not true.

(5.2) Counterexample: $f = x_1^3 + x_1x_2^3$ (E_7).
This example is also treated in (1.11).
f has the following properties:

1° $m_2^5 \subset m_2\Delta(f)$

2° $m_2^5 \not\subset m_2^2\Delta(f)$

3° f is 4-determined.

In (1.11) I showed 1° and in (4.5) lemma 3 I showed 3°. Since $\partial_1 f = 3x_1^2 + x_2^3$ and $\partial_2 f = 3x_1x_2^2$ it is impossible that $x_2^5 \in m_2^2\Delta(f)$. So $m_2^5 \not\subset m_2^2\Delta(f)$.

(5.3) Although the conjecture is not true, the algebraic condition $m_n^{k+1} \subset m_n^2\Delta(f) + m_n^{k+2}$ of theorem (1.7) can still be translated in terms of determinacy. This is the reason for the following two definitions.

<u>(5.4) Definition</u>: Two germs f and g are called <u>strong-(right)-equivalent</u> if there is a $\phi \in L_n$ such that $g = f\phi$ and the derivative $d\phi(\underline{0})$ is the identity. Notation $g \underset{R}{\approx} f$ or $g \approx f$.
The germs $\phi \in L_n$ with $d\phi(\underline{0}) = 1$ form a subgroup of L_n, which acts on $\mathcal{E}_n$ and induces an algebraic action on $J^k(n,1)$.

<u>(5.5) Definition</u>: A germ $f \in \mathcal{E}(n,1)$ is called <u>strong-k-determined</u> if for all $g \in \mathcal{E}_n$ with $g_k = f_k$ we have g is strong-equivalent with f.

<u>(5.6) Theorem</u>: $m_n^{k+1} \subseteq m_n^2 \Delta(f) + m_n^{k+2} \Leftrightarrow$ f *is strong-k-determined.*
<u>Proof</u>:
1[e]) The part $\Rightarrow$ of the proof is similar to that of theorem (1.7) and follows from two lemma's:
<u>Lemma 1</u>: *Let* $m_n^{k+1} \subseteq m_n^2 \Delta(f) + m_n^{k+2}$. *Then there exists for all* $t \in \mathbb{R}$ *a mapgerm* $\vec{\xi}: \mathbb{R}^{n+1} \to \mathbb{R}^n$ *defined in a neighborhood* U *of* $(o,t_o) \in \mathbb{R}^{n+1}$ *which satisfies:*

(i) $\vec{\xi}(o,t) = 0$ *for all* $(o,t) \in U$

(ii) $(d\vec{\xi})(o,t) = 0$ *for all* $(o,t) \in U$

(iii) $\nabla F(x,t).\vec{\xi}(x,t) + g(x) - f(x) = 0$ *for all* $(x,t) \in U$.

<u>Proof</u>: to satisfy (i), (ii) and (iii) we need only to show that

$$m_n^{k+1} \subseteq \Delta^*(F) m_n^2$$

and this follows direct from the proof of (1.7) lemma 1.

<u>Lemma 2</u>: *For each* $t_o \in \mathbb{R}$ *there is* $\varepsilon > 0$ *such that* $F_t \underset{R}{\approx} F_{t_o}$ *for all* t *with* $|t-t_o| < \varepsilon$.

<u>Proof</u>: We consider as in (1.7) lemma 2 the differentialequation

a) $\frac{\partial h}{\partial t}(x,t) = \vec{\xi}(h(x,t),t)$

with the initial condition:

b) $h(x,t_o) = x$

Since $\xi_j \in m_n^2$ we have $\frac{\partial}{\partial x_i}\left(\frac{\partial h^j}{\partial t}\right) = 0$ for all t and $j = 1,\ldots,n$.

So $\frac{\partial}{\partial t}\left(\frac{\partial h}{\partial x_i}\right) = 0$, hence $\frac{\partial h}{\partial x_i}$ = constant and dh_t is constant.

So: $dh_t = dh_{t_o} = 1$

which proves that h_t is a strong-equivalence.

The rest of the proof is the same as in (1.7) lemma 2.

2[e]) The part $\Leftarrow$ of the proof is similar to that of theorem (1.8). Replace in the proof of theorem (1.8) the set V by $W = \{g \in \mathcal{E}_n | g \underset{s}{\sim} f\}$. In order to prove the theorem is sufficient to show that:

b) $\tau(W_{k+1}) \equiv m_n{}^2\Delta(f)$ (Mod. $m_n{}^{k+2}$).

Let $h_t: (\mathbb{R}^n,\underline{0}) \to (\mathbb{R}^n,\underline{0})$ a germ of diffeomorphism with $h_o = 1$ and $dh_t(\underline{o}) = 1$. Then we have:

$$\frac{d}{dt}(fh_t)\Big|_{t=o} = \nabla f \bullet \frac{dh_t}{dt}\Big|_{t=o} \in \mathcal{E}_n(\partial_1 f,\ldots,\partial_n f).$$

Let $\vec{\xi} = \frac{\partial h_t}{\partial t}$ then $\vec{\xi}(\underline{0}) = \frac{\partial h_t(\underline{0})}{\partial t} = \underline{0}$ since $h_t(\underline{0}) = \underline{0}$ and

and $\frac{\partial \xi_i}{\partial x_j}(\underline{0}) = \frac{\partial^2 h^i}{\partial x_j \partial t}(\underline{0}) = \frac{\partial}{\partial t}\frac{\partial h_t{}^i(\underline{0})}{\partial x_j} = \frac{\partial}{\partial t}\delta_{ij} = 0$ since $dh_t(\underline{0}) = 1$

so $\xi_i \in m_n{}^2$ ($i = 1,\ldots,n$).

This means $\frac{d}{dt}(fh_t)\Big|_{t=0} \in m_n\Delta(f)$ which proves $\tau(W_{k+1}) \subset m_n\Delta(f)$ mod. m^{k+2}.

Moreover let $\alpha \in m_n{}^2\Delta(f)$; $\alpha(x) = \nabla f(x)\bullet\vec{\xi}(x)$ with $\xi_i \in m_n{}^2$, and $h_t(x) = x + t\vec{\xi}(x)$; then $h_t \in L_n$ and $dh_t(0) = 1$ and $\frac{d}{dt} fh_t\Big|_{t=0} = \alpha$ which proves $\tau(W_{k+1}) \supset m_n\Delta(f)$ mod. m^{k+2}.

<u>(5.7)</u> <u>Example</u>: $f_\lambda(x_1,x_2) = x_1{}^3 + x_1x_2{}^3 + \lambda x_2{}^5$ has the property $m^5 \subseteq m\Delta + m^6$ but it is not true that $m^5 \subset m^2\Delta + m^6$.

We found already: $f_\lambda(x_1,x_2)$ is (ordinary) 4-determined. Since $m^6 \subseteq m^2\Delta + m^7$, f_λ is strong-5-determined for every $\lambda \in \mathbb{R}$ (but not strong-4-determined!).

Since $x_2{}^5 \notin m^2\Delta(f) + m^6$ for all $\lambda \in \mathbb{R}$ we can conclude that λ is a local invariant under strong-equivalence. [using an extended version of theorem (2.9)].

(5.8) Remark: The classification of germs under strong-equivalence is not so interesting since there are already a lot of invariants in the case of a non-degenerate critical point.

Although an arbitrary f with Rank (d^2f) is maximal is strong 2-determined, it is not possible to bring the germ to the normalform

$$e_1x_1^2 + \ldots + e_nx_n^2 \text{ with } e_i = \pm 1.$$

B. Left-multiplication

(5.9) In the list of singularities with codimension ≤ 9 are some families with one local R-invariant, which is not a local RL-invariant:

$$
\begin{array}{lll}
X_{10}: & x_1^4 \pm x_1^2x_2^2 + \alpha x_2^5 & \alpha \neq 0 \\
P_9: & x_1^3 + x_2^2x_3 \pm x_1^2x_3 + \beta x_3^4 & \beta \neq 0 \\
P_{10}: & x_1^3 + x_2^2x_3 \pm x_1^2x_3 + \beta x_3^5 & \beta \neq 0 \\
R_{10}: & x_1^3 \pm x_1x_2x_3 \pm x_2^4 + \beta x_3^4 & \beta \neq 0 \\
Q_{10}: & x_1x_3^2 + x_2^3 + \ x_1^3x_2 \pm x_1^4 & \alpha \neq 0
\end{array}
$$

In most of the cases we used theorem (2.3) for the proof of the RL-equivalence of the germs in the family.

Sometimes it is also possible to see this in the following way. We treat as example X_{10}:

$$
\begin{aligned}
g &= x_1^4 \pm x_1^2x_2^2 + \alpha x_2^5 \qquad \alpha \neq 0 \\
\alpha^4 g &= \alpha^4 x_1^4 \pm \alpha^4 x_1^2x_2^2 + \alpha^5 x_2^5 \\
\alpha^4 g &= (\alpha x_1)^4 \pm (\alpha x_1)^2(\alpha x_2)^2 + (\alpha x_2)^5.
\end{aligned}
$$

So
$$g \underset{RL}{\frown} \alpha^4 g \underset{R}{\frown} x_1^4 \pm x_1^2x_2^2 + x_2^5$$

The only left-action we used in this computation is scalarmultiplication with α^4 and this is an element of $GL(1)$.

It is possible to treat the other cases of the list in the same way.

This raises a more general question: In which cases does the action of $L_n \times GL(1)$ coincide with the action of $L_n \times L_1$?

<u>(5.10)</u> <u>Theorem</u>: *Let* $n = 2$ *and let* $f_t = f + t\phi$.
If $f_t \underset{RL}{\sim} f_{t_o}$ *for all* t *in a connected interval* I *of* $\mathbb{R}$, *then* f_t *and* f_{t_o} *are also equivalent under the action of* $L_2 \times GL(1)$ *for all* $t \in I$.

<u>Proof</u>:

Since $f_t = f + t\phi$ for $t \in I$ are all in the same *RL*-orbit, ϕ has to lie in the tangentspace to the *RL*-orbit; so:

$$\phi \in m_2\Delta(f_t) + f_t{}^*(m_1) \quad \forall t \in I.$$

This implies $\phi = \sigma_1\partial_1 f_t + \sigma_2\partial_2 f_t + \sum_j \alpha_j(t) f_t{}^j$ with $\sigma_1, \sigma_2 \in m_2$.
BRIANCON ([5] and [6]) proved that:

$$f_t{}^2 \in \&_2(x_1\frac{\partial f_t}{\partial x_1}, x_2\frac{\partial f_t}{\partial x_2}) \subset m_2\Delta(f_t).$$

So we can find $\tau_1, \tau_2 \in m_2$ such that

$$\phi = \tau_1\partial_1 f_t + \sigma_2\partial_2 f_t + \alpha_1(t) f_t \qquad (*).$$

We now return to the situation described in theorem (2.3), where 1-parameterfamilies of diffeomorphisms of sourcespace and targetspace were constructed.
Our equation (*) implies that the diffeomorphism k of the targetspace has to satisfy the differentialequation:

$$\begin{cases} \frac{\partial k}{\partial t}(y,t) = -\alpha_1(t).k(y,t) \\ k(y,t_o) = x \end{cases}$$

The solution goes as follows:

$$\frac{dk(y,t)}{k(y,t)} = -\alpha_1(t)dt$$

$$\ln k\ (y,t) = \beta(t) + C(y)$$

$$k(y,t) = e^{\beta(t)+C(y)} = e^{C(y)}.\gamma(t)$$

The initial condition gives: $y = k(y,t_o) = e^{C(y)}.\gamma(t_o)$

so: $$k(y,t) = y.\frac{\gamma(t)}{\gamma(t_o)}\ .$$

So k is a scalarmultiplication, so we are done.

(5.11) Remark: The proof of theorem (5.10) shows that the condition $f_t^{\,2} \in m\Delta$ is enough to get the result.

A similar theorem for $n \geq 3$ doesn't exist, since BRIANCON gives an example with $f^{n-1} \notin \mathcal{E}_n(x_1\partial_1 f,\ldots,x_n\partial_n f)$, namely

$$f = (x_1.x_2.x_3)^3 + [x_1^{\,3n-1} + x_2^{\,3n-1} +\ldots+ x_n^{\,3n-1}].$$

In the cases P_9, P_{10}, R_{10} and Q_{10} we proceed as follows. Since those germs are 4-determined or 5-determined; we have $m^5 \subseteq m\Delta$.
Because $f_2 \equiv 0$ we have $f^2 \in m^6$ and this implies $f^2 \in m\Delta$. So we can apply the proof of theorem (5.10), which shows that the RL-equivalence of the family can already be done by $L_n \times GL(1)$-action.

PART II: DEFORMATION OF SINGULARITIES AND ADJACENCY

Introduction:

After the classificationproblem in part I, I treat in part II the adjacency problem and study approximations of a functiongerm in its universal deformation.

In §6 and §7 I introduce some known topological invariants of a singularity, namely Milnor number, the intersectionform and the monodromygroup, and investigate the relation between the invariants of the germ and its approximations.

In §8 I use this in order to explain and prove in a new way some results on adjacency which were partly known already.

In §9 I treat the new notion of μ-adjacency, which describes a relation between families of germs with constant Milnor number.

In §10 I introduce a topology in the orbitspace and study it for the set of germs with Milnor number ≤ 10. In their relative topology we get copies of $\mathbb{C}$ or $\mathbb{C}-\{\underline{0}\}$ for the 1-parameter families in the orbitspace. We illustrate the relations adjacency and μ-adjacency in the list III at the end.

This research while in progress was in a later stage to a large extend covered and then influenced by published and unpublished work of Arnold, Lamotke, Saito and Gabrielov. We indicate those references, but we believe that our presentation and survey and some of the proofs still have an independent interest.

§6 Milnorfibration and vanishing cycles.

(6.1) We consider a holomorphic mapping $f : (U,\underline{0}) \to (\mathbb{C},0)$, where U is an open subset of $\mathbb{C}^{n+1}$ and $\underline{0}$ is the only critical point of f. This situation is studied by MILNOR [20] and others.

There exist $\varepsilon > 0$ and $\delta > 0$ such that S_ε is transversal to $f^{-1}(t)$ for all $|t| \leq \delta$. Notation: $S_\varepsilon \pitchfork f^{-1}(t)$ for all $|t| \leq \delta$.

We write $B = B_\varepsilon^{2n+1}$ and $D = D_\delta^{2}$ and define:

$$E_f = f^{-1}(\partial D) \cap B$$

E_f is a compact oriented manifold with boundary and $f : E_f \to \partial D$ is the projection of a fibrebundle with typical fibre $X_f = f^{-1}(\delta) \cap B$. (see also (6.2)). It is well-known that X_f has the homotopytype of $S^n \vee \ldots \vee S^n$; hence $H_n(X_f) \cong \mathbb{Z}^{\mu(f)}$ for some $\mu(f) \in \mathbb{N}$. $\mu(f)$ is called Milnor's number. The intersectionform $<-,->$ on $H_n(X_f)$ is a bilinear form, which is symmetric if n is even and antisymmetric if n is odd.

(6.2) Lemma: *As before let* $U \subset \mathbb{C}^{n+1}$ *and let* $g : (U,\underline{0}) \to (\mathbb{C},0)$ *be a holomorphic mapping such that:*

a) g *has no critical points on* ∂B

b) g *has no critical values on* ∂D

c) $S_\varepsilon \pitchfork g^{-1}(t)$ *for all* $t \in D$

and let Σ *be the set of critical values of* g,

then:

1^o *The map* $g : g^{-1}(D \setminus \Sigma) \cap B \to D \setminus \Sigma$ *is the projection of a (locally trivial) fibrebundle.*

2^o *The map* $g : g^{-1}(D) \cap \partial B \to D$ *is the projection of a trivial fibrebundle.*

3^o *Every path* $v : [0,1] \to D - \Sigma$, *connecting points* a *and* b, *induces for any connection in the bundle* g *a diffeomorphism* $v_* : f^{-1}(a) \cap B \to f^{-1}(b) \cap B$. *The isotopyclass of this diffeomorphism is unique, that is independent of the connection.*

4^o *This connection can be chosen such that for every closed path the restriction* $v_* : f^{-1}(a) \cap \partial B \to f^{-1}(a) \cap \partial B$ *is the identity.*

Proof: We use Ehresmann's fibrationtheorem [11]:
Let E and B be smooth manifolds, B connected and $p : E \to B$ a smooth surjective mapping, with the property that for all $x \in B$ the rank of the differential of p in x equals the dimension of B and $p^{-1}(x)$ is compact and connected. Then $p : E \to B$ is a smooth fibrebundle and so all fibres $p^{-1}(x)$ are diffeomorphic.
Our g has maximal rank on $g^{-1}(D \setminus \Sigma) \cap B$ and by the transversality-condition also on $g^{-1}(D) \cap \partial B$; so we can apply this theorem to obtain 1^o and 2^o.
As moreover every fibrebundle over a contractible space, like a disc is trivial, we have 2^o.
Using a suitable connection, we find the required diffeomorphism v_* and as we have a product structure on the boundary $f^{-1}(D) \cap \partial B$, we can arrange that v_* is the identity on the boundary of the fibre $f^{-1}(a) \cap B$.

Remark: From the above lemma it follows also that $E_f \to \partial D$ is a fibrebundleprojection.

(6.3) Definitions: In the following we shall use deformations and approximations of f. A deformation of f is a holomorphic mapping $F : U \times W \to \mathbb{C}$ with $\underline{0} \in U \subset \mathbb{C}^{n+1}$ and $\underline{0} \in W \in \mathbb{C}^k$ and the property $F(x,\underline{0}) = f(x)$.

A deformation F of f is called <u>infinitesimally versal</u> if

$$\mathcal{E}_{n+1} = (\partial_o f,\dots,\partial_n f) + \mathbb{C}[\phi_1,\dots,\phi_k].$$

Here $\mathcal{E}_{n+1}$ denotes the ring of germs at $\underline{0} \in \mathbb{C}^{n+1}$ of holomorphic functions from $\mathbb{C}^{n+1}$ to $\mathbb{C}$; $(\partial_o f,\dots,\partial_n f)$ the ideal in $\mathcal{E}_{n+1}$, spanned by the partial derivatives of f and $\mathbb{C}[\phi_1,\dots,\phi_k]$ the $\mathbb{C}$-vectorspace, spanned by $\phi_1,\dots,\phi_k$, where ϕ_i is defined by $\phi_i = (\frac{\partial f}{\partial y_i})_{y=o}$ $(i=1,\dots,k)$ and $y_1,\dots,y_k$ are coördinates in $\mathbb{C}^k$.
If f has an isolated critical point in $\underline{0}$ then inf. versal deformations exist and can be written in the form $F(x,w) = f(x) + \Sigma_{i=1}^{k} w_i\phi_i$.
A deformation $F : U \times W \to \mathbb{C}$ of f is called <u>versal</u> if for any other deformation $G : U \times W' \to \mathbb{C}$ there exist analytic maps:

$$\phi : U \times W' \to U \text{ with } \phi(x,0) = x$$
$$\psi : W' \to W \qquad \text{with } \phi(0) = 0$$

such that $G(x,u) = F(\phi(x,u),\psi(u))$.

An important theorem of MATHER [19] says that the properties versal (for W small enough) and inf. versal are equivalent.

<u>(6.4)</u> Let $F : U \times W \to \mathbb{C}$ be a deformation of f. For $w \in W$ the mapping $F_w : U \to \mathbb{C}$, defined by $F_w(x) = F(x,w)$ is called <u>approximation</u> of f.

As in (6.1) we can consider the corresponding fibrebundle projection:

$$F_w : E_{F_w} = F_w^{-1}(\partial D) \cap B \to \partial D.$$

<u>(6.5)</u> <u>Lemma</u>: *There exists* $\eta > 0$ *so that we have for* $\|w\| < \eta$:

a) *all critical points of* F_w *are inside* B.
b) *all critical values of* F_w *are inside* D.
c) $S_\varepsilon \pitchfork F_w^{-1}(t)$ *for all* $t \in D$.
d) *the fibrations* $E_{F_w} \to \partial D$ *and* $E_f \to \partial D$ *are diffeomorphic.*

<u>Proof</u>: If we use the continuity of f and ∇f, transversalityarguments and an extended version of Ehresmann's fibrationtheorem, we can in each of the cases a)-d) define an open neighborhood in which the assertion is fulfilled.

(6.6) Lemma: *Let F be a versal deformation of f. There exist* $w \in W$ *arbitrarily close to any* $w \in W$ *such that:*

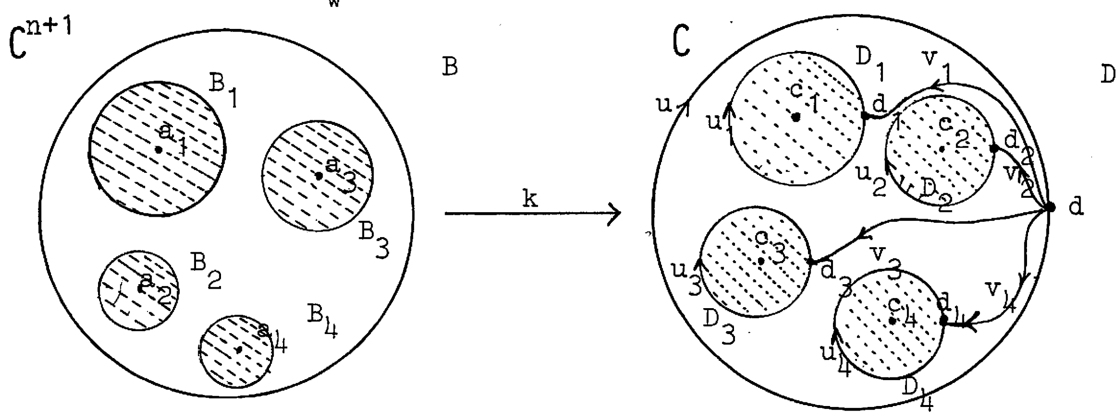
e) *all critical points of* F_w *are non-degenerate.*
f) *all critical values of* F_w *are different.*

Proof: The points $w \in W$ such that F_w has not $\mu(f)$ (= Milnor's number) distinct critical values from an algebraic variety, the so-called bifurcationvariety Bif(f) (cf. LOOIJENGA [18]).

If F_w has $\mu(f)$ distinct critical values, then all its critical points are non-degenerate (cf. MILNOR [20], appendix B).

Since f is a versal deformation, $w \notin \mathrm{Bif}(f)$ for generic $w \in W$. So $W \setminus \mathrm{Bif}(f)$ is dense in W.

(6.7) Let now $w \in W$ be chosen in such a way that the approximation satisfies properties a),...,f) of lemma (6.5) and (6.6). In that case F_w is called a generic approximation of f. We next recall the construction of the vanishing cycles as given by BRIESKORN [7] or LAMOTKE [16]. Call $F_w = k$.



Let $a_1,\dots,a_q$ be the critical points of k, and $c_1,\dots,c_q$ the corresponding critical values. Let $B_1,\dots,B_q$ be disjoint (2n+1)-balls around $a_1,\dots,a_q$ and inside B. Let $D_1,\dots,D_q$ be disjoint 2-discs around $c_1,\dots,c_q$ and inside D, chosen in such a way that we get local fibrations:

$$k : B_i - \{a_i\} \to D_i \quad (i=1,\dots,q)$$

satisfying the usual transversality-conditions:

$$\partial B_i \pitchfork k^{-1}(t) \text{ if } t \in D_i \setminus \{c_i\} \quad (i=1,\dots,q)$$

Take points $d_1,\dots,d_q$ on $\partial D_1,\dots,\partial D_q$ and let $d \in \partial D$. We next consider paths v_i in $D \setminus \bigcup_{i=1}^{q} D_i$ from d to d_i. These paths induce the following maps $(i=1,\dots,q)$:

$$Q_i = k^{-1}(d_i) \cap B_i \hookrightarrow k^{-1}(d_i) \cap B \xrightarrow{(v_i)_*} k^{-1}(d) \cap B \cong X_f,$$

which give in the homology

$$\gamma_i : H_n(Q_i) \xrightarrow{(v_i)_{**}} H_n(X_f)$$

Since S^n is a deformationretract of Q_i we have $H_n(Q_i) \cong \mathbb{Z}$. Let $s_i \in H_n(Q_i)$ be the cycle represented by S^n.

Define $\ell_{v_i} \in H_n(X_f)$ by $\gamma_i(s_i) = \ell_{v_i}$.

We set $L_f = \bigcup_{i=1}^{q} \{\ell_v \mid v$ path from d to d_i in $\overline{D \setminus \bigcup_{i=1}^{q} D_i}\}$.

The elements of L_f are called the vanishing cycles of f.

(6.8) Let $u_1,\dots,u_q$ and u be closed paths along $D_1,\dots,D_q$ and D. An arbitrary path v from d to d_i in $D - \bigcup_{i=1}^{q} D_i$ induces a map

$\sigma_{\ell_v} := (v^{-1}u_i v)_{**} : H_n(X_f) \to H_n(X_f)$.

The Picard-Lefschetzformula [21] implies

$$\sigma_{\ell_v}(x) = x - (-1)^{\frac{n(n+1)}{2}} \langle \ell_v, x\rangle \ell_v .$$

From now on we only consider the case, that n is even, in that case the intersectionform is symmetric. The selfintersectionnumber for $\alpha \in L_f$ is given by $\langle\alpha,\alpha\rangle = 2(-1)^{\frac{n}{2}} = \begin{cases} +2 & n \equiv 0 \pmod 4 \\ -2 & n \equiv 2 \pmod 4 \end{cases}$ (cf. [21])

So $\sigma_\alpha(x) = x - 2\,\frac{\langle\alpha,x\rangle}{\langle\alpha,\alpha\rangle}\,\alpha$ and we see that σ_α is just a reflection in the direction of the vanishing cycle α. Note that $\sigma_\alpha^2 = 1$ and that σ_α preserves the intersectionform. In the sequel we shall restrict the treatment to the case $n \equiv 2 \pmod 4$; the case $n \equiv 0 \pmod 4$ is similar.

(6.9) Consider the mapping $\Psi : \pi_1(D \setminus \overline{\bigcup_{i=1}^{q} D_i}, d) \to \mathrm{Aut}[H_n(X_f;\mathbb{Z})]$ that assigns to every closed path v the induced map $v_{**} : H_n(X_f) \to H_n(X_f)$.

Definition: The image of Ψ is called the monodromygroup W_f of f. Clearly W_f contains also the reflections in the direction of a vanishing cycle. Also $\sigma := u_{**} : H_n(X_f) \to H_n(X_f)$ is an element of W_f. σ is called the monodromy-operator.

(6.10) Now we choose $v_1,\dots,v_q$ in such a way that:

1° they intersect only in d and have no selfintersections.

2° $(v_1^{-1}u_1v_1) \cdot (v_2^{-1}u_2v_2) \cdot \ldots \cdot (v_q^{-1}u_qv_q) \overset{h}{\simeq} u$.

In this case the set of vanishing cycles $\ell_{v_1},\dots,\ell_{v_q}$ and the set of the reflections $\sigma_{v_1},\dots,\sigma_{v_q}$ are called fundamental; and we use the notations $\ell_1,\dots,\ell_q$ and $\sigma_1,\dots,\sigma_q$.

With these notations we state:

Theorem: (LAMOTKE [16])

a) $\{\ell_1,\dots,\ell_q\}$ *is a basis of* $H_n(X_f)$

b) $W_f(L_f) = L_f$

c) W_f *is generated by* $\{\sigma_1,\dots,\sigma_q\}$

d) $W_f\{\ell_1,\dots,\ell_q\} = L_f$

e) $\sigma_q \cdot \sigma_{q-1} \cdot \ldots \cdot \sigma_1 = \sigma$.

(6.11) Remarks on the basis.

A basis of vanishing cycles $\{\ell_1,\dots,\ell_q\} \in H_n(X_f)$ induced by paths $v_1,\dots,v_q$ from d to $d_1,\dots,d_q$, having the property:

(P1) The paths $v_1,\dots,v_q$ intersect only in d and have no selfintersections is called a <u>weak distinguished</u> basis (LAZZERI[17] calls it a <u>geometric basis</u>). In fact every set of vanishing cycles, having property (P1) is a basis. If moreover the property

(P2) $(v_1^{-1}u_1v_1) \cdot (v_2^{-1}u_2v_2) \cdot \ldots \cdot (v_q^{-1}u_qv_q) \overset{h}{\simeq} u$

is satisfied, then the basis is called <u>distinguished</u>. The basis $\{\ell_1,\ldots,\ell_q\}$ in theorem (6.10) can always be chosen in such a way, that the basis is distinguished.

§7 Topological properties of an approximation.

(7.1) We now return to the situation, that $w \in W$ is chosen in such a way, that approximation F_w satisfies the properties of lemma (6.5), but not necessarily those of (6.6).

Let $\{a_1,\ldots,a_p\}$ be the critical points of g, not necessarily non-degenerate. For every critical point a_i we can consider the mappings $g_i : (U_i,\underline{0}) \to (\mathbb{C},0)$ with $U_i \subset \mathbb{C}^{n+1}$, locally defined by:

$$g_i(z) = g(z-a_i) - g(a_i) \quad (i=1,\ldots,p),$$

each g_i having an isolated critical point in $\underline{0}$.

For each g_i we can repeat the construction of the vanishing cycles L_{g_i}, the monodromy σ_{g_i} and the groups W_{g_i}. We will compare them with the corresponding notions of f.

(7.2) As in (6.7) let $B_1,\ldots,B_p$ be disjoint $(2n+1)$-balls around $a_1,\ldots,a_p$ and inside B. Let $D_1,\ldots,D_p$ be small disjoint 2-discs around $g(b_1),\ldots,g(b_p)$ and inside D, chosen in such a way, that the transversality condition $\partial B_i \pitchfork g^{-1}(t)$ for all $t \in D_i \setminus \{g(b_i)\}$ is satisfied and such that we have local fibrations

$$g : E_{g_i} = g^{-1}(\partial D_i) \cap B_i \to \partial D_i .$$

We consider again the points d, $d_1,\ldots,d_p$ and the paths $u_1,\ldots,u_p$ and $v_1,\ldots,v_p$. If $g(b_i) = g(b_j)$ we choose $D_i = D_j$; $d_i = d_j$, $u_i = u_j$ and $v_i = v_j$.

(7.3) We next define a generic approximation h of f, near to g, which can also be used to obtain generic approximations for the mappings g_i $(i=1,\dots,p)$. We consider also the corresponding fibrations

$$h : E_{h_i} = h^{-1}(\partial D_i) \cap B_i \to \partial D_i .$$

(7.4) Lemma: *For all* $g = F_w$ *with* $\|w\| < \eta$ *(where* η *is defined as in lemma* (6.5)) *we can find* $s \in W$ *with* $\|\vec{s}\| < \eta$ *arbitrarily close to* w, *such that* $h = F_s$ *satisfies:*

a) *all critical points of* h *are inside* $B_1 \cup \dots \cup B_p$.

b) *all critical values of* h *are inside* $D_1 \cup \dots \cup D_p$.

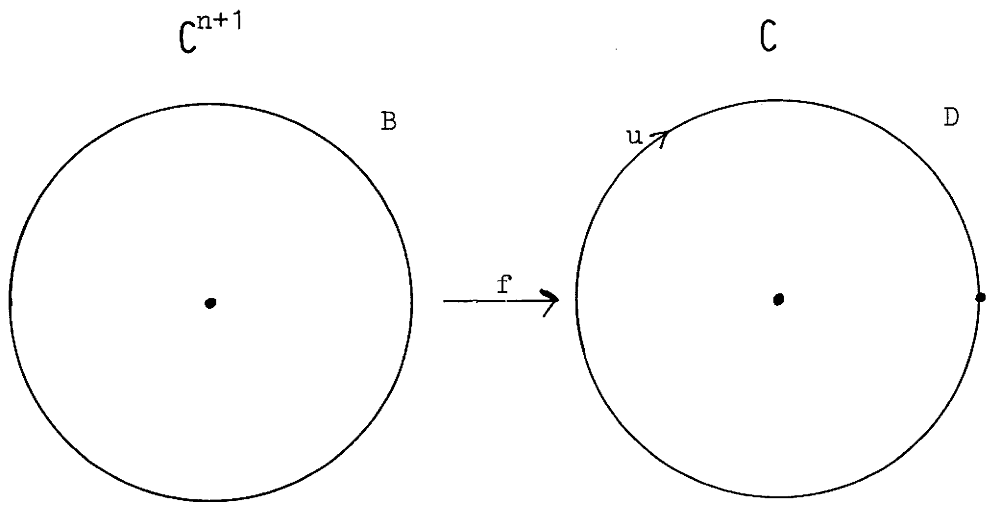
c) $\partial B_i \cap h^{-1}(t)$ *for all* $t \in \partial D_i$.

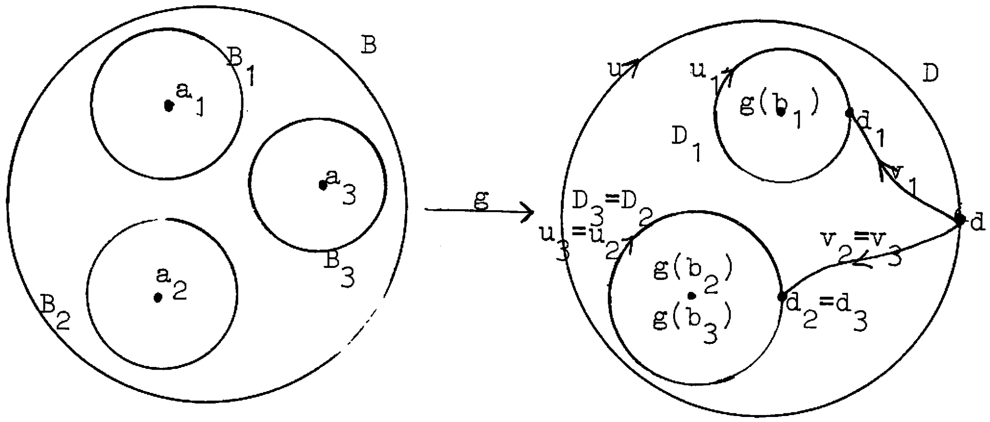
d) *the fibrations* $E_{g_i} \to \partial D_i$ *and* $E_{h_i} \to \partial D_i$ *are diffeomorphic.*

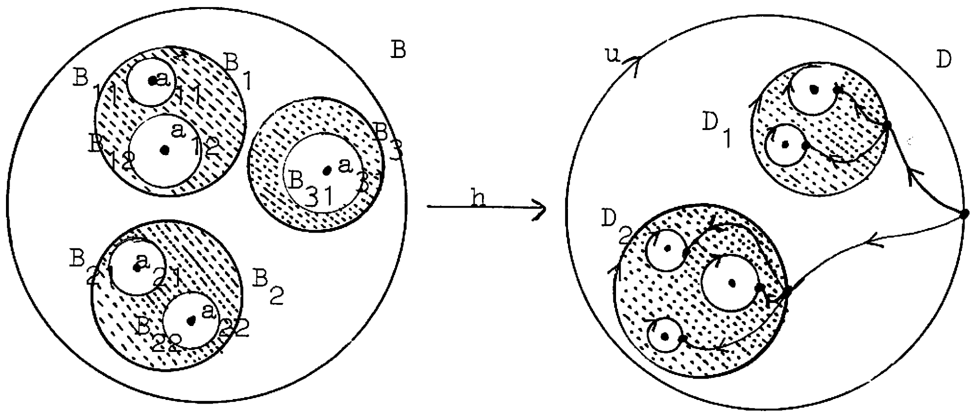
e) *all critical points of* h *are non-degenerate.*

f) *all critical values of* h *are different.*

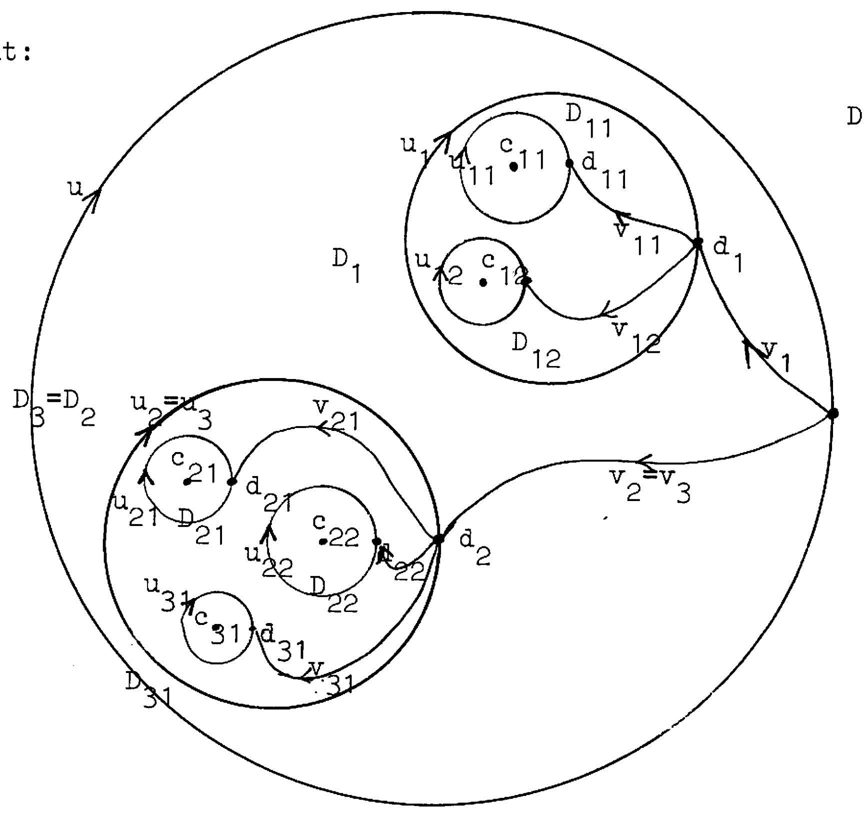
The proof is a specialization of (6.5) and (6.6) and will be ommitted.







enlargement:



(7.5) We now repeat the construction of the vanishing cycles with an approximation h, satisfying a)-f) of lemma (7.4). We use a notation with double-indices. Let $\{a_{i1},\dots,a_{ir_i}\}$ be the critical points of h inside B_i.

The following is defined in the obvious way:

- balls B_{ij} with $a_{ij} \in B_{ij} \subset B_i \subset B$
- discs D_{ij} with $D_{ij} \subset D_i \subset D$
- points d_{ij} with $d_{ij} \in \partial D_{ij}$
- paths v_{ij} from d_i to d_{ij} inside D_i
- paths u_{ij} around D_{ij}

Consider the following diagram:

$$\begin{tikzcd}
Q_{ij} = h^{-1}(d_{ij}) \cap B_{ij} \arrow[r, "(v_{ij})_*"] \arrow[dr, "(v_i v_{ij})_*"'] & h^{-1}(d_i) \cap B_i \cong X_{g_i} \arrow[d, "(v_i)_*"] \\
& h^{-1}(d) \cap B \cong X_f
\end{tikzcd}$$

inducing in the homology:

$$\begin{tikzcd}
\mathbb{Z} \cong H_n(Q_{ij}) \arrow[rr, "(v_{ij})_{**}"] \arrow[ddrr] & & H_n(X_{g_i}) \arrow[dd, "(v_i)_{**}"] \\
s_{ij} \arrow[r, mapsto, dashed] \arrow[dr, mapsto, dashed] & \hat{\ell}_{ij} \arrow[d, mapsto, dashed] & \\
& \ell_{ij} & H_n(X_f)
\end{tikzcd}$$

Let s_{ij} be a generator of $H_n(Q_{ij})$. Define $\hat{L}_{g_i} \subset H_n(X_{g_i})$ as the set of vanishing cycles with respect to g_i and $L_f \subset H_n(X_f)$ as the set of vanishing cycles with respect to f.

Choosing paths v_i and v_{ij} in such a way that

1° The paths don't intersect each other and are not-selfintersecting

2° $(v_1^{-1}u_1v_1) \cdot (v_2^{-1}u_2v_2) \cdot \ldots \cdot (v_p^{-1}u_pv_p) \overset{h}{\simeq} u$

3° $(v_{i1}^{-1}u_{i1}v_{i1}) \cdot (v_{i2}^{-1}u_{i2}v_{i2}) \cdot \ldots \cdot (v_{ir_i}^{-1}u_{ir_i}v_{ir_i}) \overset{h}{\simeq} u_i$ $(i=1,\ldots,p)$

the <u>fundamental vanishing cycles</u> $\hat{\ell}_{ij} \in \hat{L}_{g_i} \subset H_n(Xg_i)$ and $\ell_{ij} \in L_f \subset H_n(X_f)$ are defined by: $\hat{\ell}_{ij} = (v_{ij})_{**}s_{ij}$ and $\ell_{ij} = (v_iv_{ij})_{**}s_{ij}$.

<u>(7.6)</u> <u>Theorem</u>:

a) $\{\hat{\ell}_{i1},\ldots,\hat{\ell}_{ir}\}$ *is a basis of* $H_n(Xg_i) \cong \mathbb{Z}^{\mu(g_i)}$ $(i=1,\ldots p)$

b) $\{\ell_{11},\ldots,\ell_{1r},\ldots,\ell_{p1},\ldots,\ell_{pr_p}\}$ *is a basis of* $H_n(X_f) \cong \mathbb{Z}^{\mu(f)}$

c) *The bases in* a) *and* b) *are distinguished.*

The proof is a consequence of (6.10).

<u>(7.7)</u> <u>Corollary</u>:

The map $(v_i)_{**} : H_n(Xg_i) \to H_n(X_f)$ *is injective and so we can identify* $H_n(Xg_i)$ *with a subspace of* $H_n(X_f)$ $(i=1,\ldots,p)$ *and* $H_n(X_f) = H_n(Xg_1) \oplus \ldots \oplus H_n(Xg_p)$ *over* $\mathbb{Z}$.

<u>(7.8)</u> The mappings $g_1,\ldots,g_p$ and f define <u>monodromy-groups</u> $\hat{W}g_1,\ldots,\hat{W}g_p$ and W_f in resp. $\mathrm{Aut}[H_n(Xg_1)],\ldots,\mathrm{Aut}[H_n(Xg_p)]$ and $\mathrm{Aut}[H_n(X_f)]$. We shall "extend" the above injections, and also identify $\hat{W}g_1,\ldots,\hat{W}g_p$ with subgroups of W_f.

Set $\Sigma_i = \mathrm{Int} \bigcup_{j=1}^{r_i} D_{ij}$ and $\Sigma = \mathrm{Int} \bigcup_{i=1}^{p} \bigcup_{j=1}^{r_i} D_{ij}$.

Let Wg_i be the image of the composed map:

$\pi_1(D_i-\Sigma_i,d_i) \to \pi_1(D-\Sigma,d) \overset{\Psi}{\to} \mathrm{Aut}[H_n(X_f)]$

given by $[w] \mapsto [v_i^{-1}wv_i] \mapsto (v_i^{-1}wv_i)_{**}$.

<u>Proposition</u>: *$\hat{W}g_i$ and Wg_i are isomorphic.*

<u>Proof</u>: Set $X_i = Xg_i = h^{-1}(d_i) \cap B_i$; $X = h^{-1}(d_i) \cap B$ and $X' = \overline{X \setminus X_i}$.

We have the following situation:

$$\begin{tikzcd}
\pi_1(D_i-\Sigma_i,d_i) \arrow[r, "\Psi_i"] \arrow[d, hook] & \mathrm{Aut}[H_n(X_i)] \supset \hat{W}_i \\
\pi_1(D-\Sigma,d_i) \arrow[r, "\tilde{\Psi}"] \arrow[d, "\cong"', "v_i"] & \mathrm{Aut}[H_n(X)] \supset \hat{W}_i \\
\pi_1(D-\Sigma,d\ ) \arrow[r, "\Psi"] & \mathrm{Aut}[H_n(X_f)] \supset W
\end{tikzcd}$$

Define: $\hat{W}_i = \Psi_i[\pi_1(D_i-\Sigma_i,d_i)] = \hat{W}\ _i$

$\tilde{W}_i = \tilde{\Psi}\ [\pi_1(D_i-\Sigma_i,d_i)]$

$W = \tilde{\Psi}\ [\pi_1(D-\Sigma,d_i)]$

First we shall show $\hat{W}_i \cong \tilde{W}_i$.

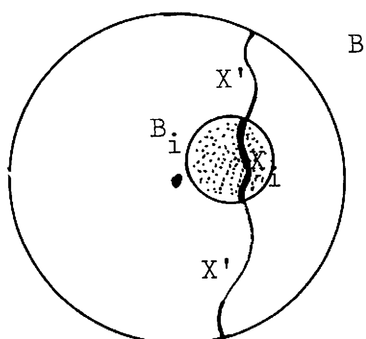
Let in general $h : Y \to Y$ be a map with $h|A = 1$ then the <u>variationmap</u> $\mathrm{var}_h : H_n(Y,A) \to H_n(Y)$ is defined by $\mathrm{var}_h[x] = [x - h(x)]$. Considering the composed map

$$H_n(Y) \xrightarrow{i_*} H_n(Y,A) \xrightarrow{\mathrm{var}_h} H_n(Y)$$

we see that $h_* = 1 + i_*\mathrm{var}_h$. Moreover var is a natural transformation (cf. [16]).

Let $[w] \in \pi_1(D_i-\Sigma_i,d_i)$. We consider the following commutative diagram:

$$\begin{tikzcd}
H_q(X,\partial X) \arrow[r, "\mathrm{var}_w"] \arrow[d, "i_*"] & H_q(X) \arrow[d, equal] \\
H_q(X,X') \arrow[r, "\mathrm{var}''_w"] & H_q(X) \\
H_q(X_i,\partial X_i) \arrow[r, "\mathrm{var}'_w"] \arrow[u, "(\mathrm{exc})_*"] & H_q(X_i) \arrow[u]
\end{tikzcd}$$



The definitions of var_w, var'_w and var''_w are justified, because it is possible to choose w such that $w_*|X' = 1$. From the above diagram follows:

<u>Lemma 1</u>: *If* $w \in \pi_1(D_i-\Sigma_i,d_i)$ *then* $\mathrm{var}'_w = 0 \Rightarrow \mathrm{var}_w = 0$.

<u>Lemma 2</u>: *$\hat{W}_i$ and $\tilde{W}_i$ are isomorphic.*

<u>Proof</u>: Let $\mathrm{Aut}[H_n(X);\ H_n(X_i)]$ be the subset of $\mathrm{Aut}[H_n(X)]$ consisting of the automorphisms that map $H_n(X_i)$ into itself. Then

$\tilde{W}_i \subset \mathrm{Aut}[H_n(X);\ H_n(X_i)]$.

The natural map: $\mathrm{Aut}[H_n(X);\ H_n(X_i)] \to \mathrm{Aut}[H_n(X_i)]$ defines a surjective morphism:

$$\tilde{W}_i \to \hat{W}_i .$$

We next show the injectivity:

For $[w] \in \pi_1(D_i - \Sigma_i, d_i)$ we have

$$\Psi_i[w] = 1 + (i_1)_* \mathrm{var}'_w$$

$$\tilde{\Psi}[w] = 1 + (i_2)_* \mathrm{var}_w .$$

Let $\Psi_i[w] = 1$ on $H_n(X_i)$. Then $(i_1)_* \mathrm{var}'_w = 0$ and so $\mathrm{var}'_w = 0$. Lemma 1 implies $\mathrm{var}_w = 0$ and so $\tilde{\Psi}[w] = 1$.

<u>Lemma 3</u>: *There is an isomorphism $\Phi : W \to W_f$ mapping $\tilde{W}_i$ onto W_{g_i}.*

<u>Proof</u>: The path v_i from d to d_i induces a diffeomorphism $(v_i)_* : X_f \to X_i$. By conjugation with $(v_i)_{**}$ we get an isomorphism $\mathrm{Aut}[H_n(X)] \to \mathrm{Aut}[H_n(X_f)]$. Clearly the following diagram is commutative:

$$\begin{array}{ccc}
\pi_1(D-\Sigma, d_i) & \xrightarrow{\tilde{\Psi}} & \mathrm{Aut}[H_n(X)] \\
\cong \downarrow & & \cong \downarrow \\
\pi_1(D-\Sigma, d) & \xrightarrow{\Psi} & \mathrm{Aut}[H_n(X_f)]
\end{array}$$

and this proves the lemma.

We have now proved our proposition.

<u>(7.9)</u> We denote by σ_{ij} the reflections in the direction of the fundamental vanishing cycles ℓ_{ij}; by $\sigma = u_{**} : H_n(X_f) \to H_n(X_f)$ the

monodromy operator of f and by $\sigma_i = (v_i^{-1} u_i v_i)_{**} : H_n(X_f) \to H_n(X_f)$ the transported monodromy operators of g_i $(i=1,\ldots,p)$.

From the above statements follow:

Theorem:

a) $H_n(X_f) = H_n(Xg_1) \oplus \ldots \oplus H_n(Xg_p)$

b) W_f *is generated by* $Wg_1,\ldots,Wg_p$

c) $\sigma = \sigma_p \circ \ldots \circ \sigma_1$ *and* $\sigma_i = \sigma_{ir_i} \circ \ldots \circ \sigma_{i1}$ $(i=1,\ldots,p)$

d) $W_f(Lg_1 \cup \ldots \cup Lg_p) = L_f$

(7.10) Corollary: *With respect to the chosen basis of fundamental vanishing cycles, the matrix* M_f *of the intersection form on* $H_n(X_f)$ *is given by*

$$M_f = \begin{pmatrix} Mg_1 & A_{12} & \cdots & A_{1p} \\ A_{21} & Mg_2 & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & Mg_p \end{pmatrix}$$

where Mg_i *are the matrices of the intersectionforms on* $H_n(Xg_i)$ *and* $A_{ij}{}^T = A_{ji}$. *By a proper choise of the basis all the matrices* $Mg_1,\ldots,Mg_p$ *and* M_f *are simultaneously given with respect to a distinguished basis.*

Examples will be given in §8.

§8 Adjacency of singularities

(8.1) In this paragraph we apply the theory of intersectionforms of §6 and §7 on adjacency of germs. In this way we explain some results, which were in a different way obtained by ARNOLD and SAITO. In the first part of this paragraph we give definitions and report on their results.
In this paragraph we assume $n \equiv 2 \pmod{4}$.

(8.2) Definition: A germ $\hat{f} \in m$ is called simple if there exists an open neighborhood U of $\hat{f}$ in m such that U intersects only a finite number of orbits (under the action of biholomorphic mapgerms).
The orbit of a simple $\hat{f}$ is also called simple.
We remark that $\hat{f}$ being simple implies that codim $(\hat{f})$ is finite and that this definition is equivalent to the definition of ARNOLD [1].

(8.3) Definition: A germ $\hat{f} \in m$ is called simple elliptic (or mildly non-simple) if codim $(\hat{f}) < \infty$ and there exists an open neighborhood U of $\hat{f} \in m$, intersecting only a finite number of orbits with codimension smaller than codim $(\hat{f})$.
The orbit of a simple elliptic $\hat{f}$ is also called simple elliptic.

SAITO [22] proved that the exceptional curve of the resolution of the hypersurface $f = 0$ is an elliptic curve without singularities if and only if $\hat{f}$ is a simple elliptic germ. This explains the chosen name.

The following two classificationtheorems (8.4) and (8.5) were essentially obtained by ARNOLD [1]. In our list [23] we had already all simple singularities and among other non-simple singularities we had two of the three simple elliptic families. For simple elliptic singularities see also DUISTERMAAT [10].

<u>(8.4)</u> <u>Theorem</u>: $\hat{f}$ *is simple if and only if* $\hat{f}$ *is of type* A_k, D_k, E_k, *where:*

$$
\begin{array}{llll}
A_k : & z_o^{k+1} + z_1^2 + z_2^2 + \ldots + z_n^2 & (k \geq 1); & \text{codim } A_k = k-1 \\
D_k : & z_o^2 z_1 + z_1^{k-1} + z_2^2 + \ldots + z_n^2 & (k \geq 4); & \text{codim } D_k = k-1 \\
E_6 : & z_o^3 + z_1^4 + z_2^2 + \ldots + z_n^2 & & \text{codim } E_6 = 5 \\
E_7 : & z_o^3 + z_o z_1^3 + z_2^2 + \ldots + z_n^2 & & \text{codim } E_7 = 6 \\
E_8 : & z_o^3 + z_1^4 + z_2^2 + \ldots + z_n^2 & & \text{codim } E_8 = 7
\end{array}
$$

<u>(8.5)</u> <u>Theorem</u>: $\hat{f}$ *is simple elliptic if and only if* $\hat{f}$ *is of type* P_8, X_9 *or* J_{10} *(or in Saito's notation* $\tilde{E}_6$, $\tilde{E}_7$ *or* $\tilde{E}_8$*), where:*

$$
\begin{array}{l}
\tilde{E}_6 = P_8 : z_o^3 + z_1^3 + z_2^3 + \mu z_o z_1 z_2 + z_3^2 + \ldots + z_n^2 \\
\tilde{E}_7 = X_9 : z_o^4 + z_1^4 + \mu z_o^2 z_1^2 + z_2^2 + z_3^2 + \ldots + z_n^2 \\
\tilde{E}_8 = J_{10} : z_o^6 + z_1^3 + z_2^2 + \mu z_o^4 z_1 + z_2^2 + z_3^2 + \ldots + z_n^2
\end{array}
$$

codim $P_8 = 7$; codim $X_9 = 8$; codim $J_{10} = 9$.

<u>(8.6)</u> <u>Remark on intersectionmatrices</u>.

PHAM [21] and recently GABRIELOV [12] computed intersectionmatrices for singularities of the form:

$$z_o^{a_o} + z_1^{a_1} + \ldots + z_n^{a_n}.$$

We refer for the general form for these intersectionmatrices to their papers, and also to HIRZEBRUCH-MAYER [14] p.88 and give here only a few examples for $n = 2$.

An easy way to describe intersectionmatrices is by a diagram. The correspondance between matrix and diagram is as follows:

1^o always $a_{ii} = -2$

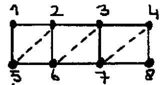
2^o $\overset{i}{\bullet}\quad\overset{j}{\bullet} \Leftrightarrow a_{ij} = a_{ji} = 0$

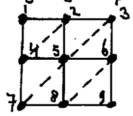
$\overset{i}{\bullet}\text{———}\overset{j}{\bullet} \Leftrightarrow a_{ij} = a_{ji} = 1$ $\qquad$ $\overset{i}{\bullet}\text{- - - -}\overset{j}{\bullet} \Leftrightarrow a_{ij} = a_{ji} = -1$

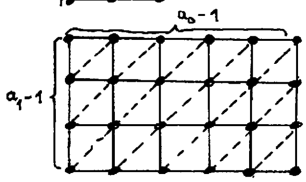
$\overset{i}{\bullet}\text{═══}\overset{j}{\bullet} \Leftrightarrow a_{ij} = a_{ji} = 2$ $\qquad$ $\overset{i}{\bullet}\text{= = = =}\overset{j}{\bullet} \Leftrightarrow a_{ij} = a_{ji} = -2$

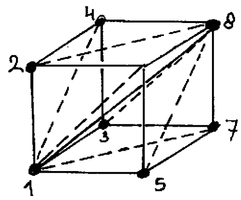
Examples: (taken from GABRIELOV):

(i) $z_o^4 + z_1^2 + z_2^2$ (A_3) has diagram 

(ii) $z_o^5 + z_1^3 + z_2^2$ (E_8) has diagram 

(iii) $z_o^4 + z_1^4 + z_2^2$ (X_9) has diagram 

(iv) $z_o^{a_o} + z_1^{a_1} + z_2^2$ has diagram 

(v) $z_o^3 + z_1^3 + z_2^3$ has diagram 

With the given ordering each basis is distinguished.

HIRZEBRUCH-MAYER [14] showed that

the intersectionform of $z_o^{a_o} + z_1^{a_1} + z_2^{a_2}$ in case $a_o \geq a_1 \geq a_2$ is:

negative definite $\Leftrightarrow \frac{1}{a_o} + \frac{1}{a_1} + \frac{1}{a_2} > 1 \Leftrightarrow$

$\Leftrightarrow (a_o, a_1, a_2) = (n,2,2), (3,3,2), (4,3,2)$ or $(5,3,2)$; $(n \geq 2)$.

negative semi-definite $\Leftrightarrow \frac{1}{a_o} + \frac{1}{a_1} + \frac{1}{a_2} = 1 \Leftrightarrow$

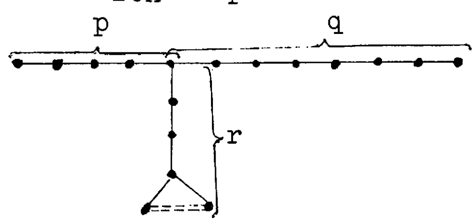
$\Leftrightarrow (a_o, a_1, a_2) = (6,3,2), (4,4,2)$ or $(3,3,3)$.

In the case of simple singularities one can also apply the following: There is a 1-1-correspondence between simple germs $\hat{f}$ and algebraic varieties X, given by f = 0, having in $\underline{0}$ a rational doublepoint. In that case we can use the minimal resolution $\pi : \tilde{X} \to X$ of this singular variety. TJURINA [24] and Brieskorn showed that if f is of type A_k, D_k or E_k then $\pi^{-1}(\underline{0})$ is diffeomorphic with the typical fibre X_f of the Milnorfibration. The corresponding intersectionforms have all been computed; their matrices can be given with respect to a distinguished basis by diagrams as before, and these diagrams happen to be the usual Dynkindiagrams for A_k, D_k, E_k and their intersectionforms are all negative definite.

Recently ARNOLD [3] announced that GABRIELOV had also computed intersectionmatrices in other cases. For the singularity

$$z_o^p + z_1^q + z_2^r + \lambda z_o z_1 z_2$$

with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, the intersectionform with respect to a weak distinguished basis is given by:



and $\mu = p + q + q - 1.$

(8.7) Change of basis.

The intersectionmatrix can change if we use another basis. The question arises if there is a nice form for these matrices over $\mathbb{Z}$. One can ask this question with respect to:

a) a basis of cycles in $H_n(X_f)$

b) a weak distinguished basis in $H_n(X_f)$

c) a distinguished basis in $H_n(X_f)$

Moreover one has to say, what one likes to call a "nice" matrix. In the case of simple singularities one can arrange that with respect to a distinguished basis, the diagram is just the corresponding Dynkindiagram. This diagram has the form of a tree; and the matrix has the properties $a_{ii} = 2$ and $a_{ij} \leq 0$ if $i \neq j$.

So a definition of "nice" could be: $a_{ii} = 2 \ \wedge \ a_{ij} \leq 0$ if $i \neq j$.

But already in the case of the simple elliptic singularities it is impossible to obtain this nice situation, even if we allow a basis of type a).

Namely let $(-q_{ij})$ be the matrix of the intersectionform of a simple elliptic singularity. Then:

1° the quadratic form $\Sigma q_{ij} x_i x_j$ on $\mathbb{R}^n$ is positive with kernel-dimension 2.

2° there is no partition of $\{1,\dots,n\}$ into two non-empty sets I and J such that $(i,j) \in I \times J$ implies $q_{ij} = 0$ (cf. LAZZERI [17]; proposition 2).

If now $q_{ij} \leq 0$ for all $i \neq j$, then BOURBAKI [4] (p.78) implies that the kerneldimension of the quadratic form is 0 or 1. This gives a contradiction.

(8.8) Definition: A germ $\hat{g}$ is called adjacent to $\hat{f}$ if in any neighborhood of $\hat{f}$ there are germs of the orbit of $\hat{g}$.
Notation: $\hat{g} \leq \hat{f}$.
If $\hat{g} \leq \hat{f}$ the orbit of $\hat{g}$ is also called adjacent to the orbit of $\hat{f}$.

Examples:

1° $f_t(x) = x^8 + tx^7$ is for $t = 0$ of type A_7 and if $t \neq 0$ of type A_6 so $A_6 \leq A_7$.

2° $f_t(x,y) = x^2y + y^8 + tx^2$ is for $t = 0$ of type D_9 and if $t \neq 0$ of type A_7 so $A_7 \leq D_9$.

3° $f_t(x,y) = \phi(x,y) + tx^2 + ty^2$ is for $t \neq 0$ of type A_1. This shows $A_1 \leq \hat{\phi}$ for all .

(8.9) Proposition: *If* $\hat{g} \leq \hat{f}$ *then there exists an injection* $H_n(X_g) \to H_n(X_f)$ *preserving the intersectionform* $\langle -,- \rangle$ *and mapping a distinguished basis of* $H_n(X_g)$ *into a distinguished basis of vanishing cycles of* $H_n(X_f)$, *such that the intersectionmatrix of* $\hat{g}$ *can be identified with a diagonal submatrix of the intersectionmatrix of* $\hat{f}$.

Proof:

From the definition of adjacency follows that with respect to a versal deformation $F : U \times W \to \mathbb{C}$ of f there exist $w \in W$ arbitrarily close to $\underline{0} \in W$ such that $\hat{g}$ is equivalent to the approximation $\hat{F}_w$. Then we can apply theorem (7.6). So we can consider $H_n(X_g)$ as a subset of $H_n(X_f)$ and there is a basis of vanishing cycles $\{\ell_1,\ldots,\ell_q\}$ of $H_n(X_f)$ such that $\{\ell_{p+1},\ldots,\ell_q\}$ is a basis of vanishing cycles of $H_n(X_g)$.

The following two theorems characterize simple and simple elliptic singularities by properties of the intersectionform. They were stated in a letter of ARNOLD to the international mathematical conference on manifolds and related topics in Tokyo (1973).

(8.10) Theorem: $\hat{f}$ *is simple if and only if the intersectionform on* $H_n(X_f)$ *is negative definite.*

Proof:

Remark (8.6) shows that a simple singularity has a negative definite intersectionform.

If g is not simple, then some germ in at least one of the following three families is adjacent to $\hat{g}$. (cf. ARNOLD [3])

$$\tilde{E}_6 = P_8: z_o^3 + z_1^3 + z_2^3 + \mu z_o z_1 z_2 + z_3^2 + \ldots + z_n^2$$
$$\tilde{E}_7 = X_9: z_o^4 + z_1^4 + z_2^2 + \mu z_o^2 z_1^2 + z_3^2 + \ldots + z_n^2$$
$$\tilde{E}_8 = J_{10}: z_o^6 + z_1^3 + z_2^2 + \mu z_o^4 z_1 + z_3^2 + \ldots + z_n^2$$

In those families of germs with constant Milnornumber, the intersectionform is also constant and can be computed from $z_o^3 + z_1^3 + z_2^3$, $z_o^4 + z_1^4 + z_2^2$ and $z_o^6 + z_1^3 + z_2^2$.
These are negative semi-definite with a 2-dimensional kernel. The intersectionmatrix of $\hat{g}$ contains a negative semi-definite matrix as diagonal submatrix and cannot be negative definite.

(8.11) Theorem: $\hat{f}$ *is simple elliptic if and only if the intersectionfrom on* $H_n(X_f)$ *is negative semi-definite.*

Proof:

If $\hat{f}$ is simple elliptic it follows from (8.5) and (8.6) that the intersectionform is negative semi-definite.

Let $\hat{g}$ be not simple elliptic. We already know from (8.10) that a simple germ has a negative definite intersectionform. So let us assume, that $\hat{g}$ is not simple or simple elliptic.

In the same way is in (8.10) one shows now that some germ in at least one of the following three families is adjacent to $\hat{g}$:

P_9 : $az_0z_1z_2 + z_0^3 + z_1^3 + z_2^4 + z_3^2 + \ldots + z_n^2$

X_{10}: $az_0z_1z_2 + z_0^4 + z_1^5 + z_2^2 + z_3^2 + \ldots + z_n^2$

J_{11}: $az_0z_1z_2 + z_0^3 + z_1^7 + z_2^2 + z_3^2 + \ldots + z_n^2$

ARNOLD announced in [3] (see also DEMAZURE [9]) that GABRIELOV had computed the intersectionforms and that in all these cases there is a vector with positive value. So $\hat{g}$ cannot have a negative semi-definite intersectionform.

(8.12) Theorem:

For simple singularities we have:

$$\hat{g} \leq \hat{f} \Leftrightarrow \text{Dynkindiagram } (\hat{g}) \subset \text{Dynkindiagram } (\hat{f})$$

where the Dynkindiagram of a germ of type A_k, D_k *or* E_k *equals the usual Dynkindiagram of* A_k, D_k, E_k *in the theory of semi-simple Lie-algebra's:*

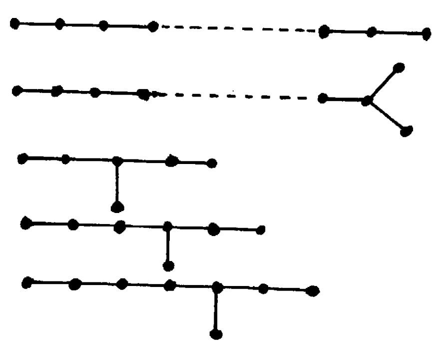
A_k : (k points)

D_k : (k points)

E_6 :

E_7 :

E_8 :



Proof: ARNOLD [1] proved the theorem by direct computations, using the definitions of adjacency and by comparison of the results with the possible subdiagrams of the corresponding Dynkindiagrams.
We next show, that it is possible to prove that the adjacency implies the inclusion of Dynkindiagrams, using the theory developed in §7.
In this alternative proof the relation with the theory of the intersectionforms becomes clearer.

Let $\hat{g} \leq \hat{f}$. In (8.9) we found that we can consider $H_n(X_g)$ as a subset of $H_n(X_f)$ and that there exists a basis of vanishing cycles of $H_n(X_f)$ such that $\{\ell_{p+1},\ldots,\ell_q\}$ is a basis of vanishing cycles of $H_n(X_g)$.
If $\hat{g}$ is simple, then it is always possible to choose a distinguished basis $\{\ell_{p+1},\ldots,\ell_q\}$ in such a way, that the intersectionmatrix of $\hat{g}$

is in the normalform, given by the Dynkindiagram. The intersection-matrix of $\hat{f}$ contains this matrix as submatrix, but it is not necessarily in the normalform.

Since $\hat{f}$ is simple, the intersectionform is negative definite and the set L_f and the bilineair form $-\langle -,- \rangle$ satisfy the definition of <u>rootsystem</u>. We shall apply now a customary argument in the classificationtheory of rootsystems (cf. BOURBAKI [4]).

The ordered basis $\{\ell_1,\dots,\ell_p, \ell_{p+1},\dots,\ell_q\}$ defines an ordering of the roots of L_g and L_f. Because the intersectionmatrix on $H_n(F_g)$ has Cartanform with respect to $\{\ell_{p+1},\dots,\ell_q\}$ these roots are fundamental (with respect to L_g). Using the ordering we can now select fundamental roots $\{m_1,\dots,m_p\}$ (with respect to L_f), such that:

$$m_1 < \dots < m_p < \ell_{p+1} < \dots < \ell_q.$$

Morerover we write $m_i = \ell_i$ if $p+1 \leq i \leq q$.

We shall prove that $\{m_1,\dots,m_p, m_{p+1},\dots,m_q\}$ is fundamental with respect to L_f.

<u>Lemma</u>: $\langle m_i,m_j \rangle \geq 0$ *for* $i \neq j$.

<u>Proof</u>:

(i) if $i \leq p \wedge j \leq p$: then $\langle m_i,m_j \rangle \geq 0$ because m_i and m_j are fundamental.

(ii) if $i > p \wedge j > p$: then $\langle m_i,m_j \rangle = \langle \ell_i,\ell_j \rangle \geq 0$.

(iii) if $i \leq p \wedge j > p$ (or resp. $i > p$ and $j \leq p$) : We have that m_i-m_j is not a root, for otherwise $m_i = (m_i-m_j) + m_j$ with $m_i-m_j > 0$, so m_i is not fundamental.

The m_j-chain through m_i : $m_i + sm_j,\dots,m_i + tm_j$ starts with m_i; so $s = 0$.

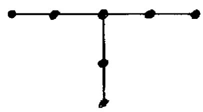
The formula: $\dfrac{-\langle m_i,m_j \rangle}{-\langle m_i,m_i \rangle} = \dfrac{s-t}{2}$ gives $\langle m_i,m_j \rangle \geq 0$ since $\langle m_i,m_i \rangle = -2$.

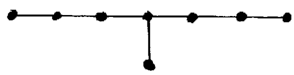
So with respect to the basis $\{m_1,\dots,m_p, m_{p+1},\dots,m_q\}$, resp. $\{m_1,\dots,m_p\}$ the intersectionmatrices of $\hat{f}$ and $\hat{g}$ are simultaneously in Cartanform. So the Dynkindiagram of f is a subdiagram of the Dynkindiagram of g.

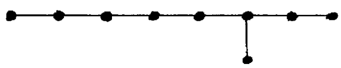
(8.13) Theorem:

a) *If* $\hat{g} \leq \hat{f}$ *and* $\hat{f}$ *is a simple elliptic singularity, then*
either: $\hat{g}$ *is equivalent to* $\hat{f}$
or: $\hat{g}$ *is simple.*

b) *Moreover if* $\hat{g}$ *is simple we have:*
$\hat{g} \leq \hat{f} \Leftrightarrow$ Dynkindiagram $(\hat{g}) \subset$ Dynkindiagram $(\tilde{E}_\ell)$
where: $\ell = 6$ *if* $\hat{f}$ *has type* P_8
$\ell = 7$ *if* $\hat{f}$ *has type* X_9
$\ell = 8$ *if* $\hat{f}$ *has type* J_{10}
and the Dynkindiagram of $\tilde{E}_\ell$ *are the so-called extended Dynkindiagrams for* E_ℓ *in the theory of semi-simple Liegroups* (see BOURBAKI [4], p.199):

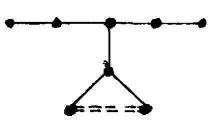
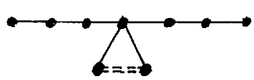
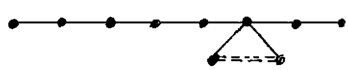
$\tilde{E}_6$:  (case P_8)

$\tilde{E}_7$:  (case X_9)

$\tilde{E}_8$:  (case J_{10})

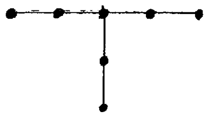
Proof: Our proof and calculation that the inclusion implies the adjacency was more complicated than that given in the paper of SAITO [22], which appeared recently. Therefore we show only that adjacency implies inclusion. SAITO proved that part by direct computations, using the definition of adjacency. Our proof shall use the intersectionform and the monodromy group.

The intersectionforms of the simple elliptic singularities can be given by the following diagrams (with respect to a weak distinguished basis; compare GABRIELOV):

P_8  X_9  J_{10} 

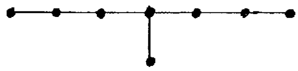
The kerneldimension is 2. If i and j are such that $\overset{i}{\bullet}=\!=\!=\!=\overset{j}{\bullet}$ (so $<\ell_i,\ell_j> = -2$), then $<\ell_i-\ell_j,\ \ell_i-\ell_j> = 0$, so e_i-e_j is a kernelvector. After dividing out by the subspace, spanned by e_i-e_j, the intersectionform is given by the following diagrams:

P_8



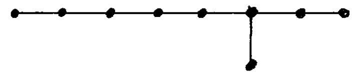
$\tilde{E}_6$

X_9



$\tilde{E}_7$

J_{10}



$\tilde{E}_8$

These diagrams correspond with negative quadratic forms with a 1-dimensional kernel.
Let $g \leq f$ then $W_g \subset W_f$ and since $H_n(X_g) \cap \mathbb{R}[e_i-e_j] = \{\underline{0}\}$ this implies $W_g \subset W(\tilde{E}_\ell)$, where $W(\tilde{E}_\ell)$ is the Weylgroup of $\tilde{E}_\ell$.

With the Weylgroups $W(\tilde{E}_\ell)$ there correspond a (infinite) set $\mathcal{H}$ of hyperplanes in a vectorspace V, which divides V into chambers. The reflections in the hyperplanes generate $W(\tilde{E}_\ell)$. The reflections in the walls of one Weylchamber already generate $W(\tilde{E}_\ell)$.

A vertex P of a Weylchamber is called a <u>special</u> <u>vertex</u> if for every hyperplane $H \in \mathcal{H}$ there is a parallel hyperplane in $\mathcal{H}$ through P. The reflections in the hyperplanes through a special vertex P generate the group $W(E_\ell)$. Any finite subgroup of $W(\tilde{E}_\ell)$ is also a subgroup of $W(E_\ell)$.

In general the subgroup of $W(\tilde{X})$ fixing a vertex Q has a Coxetergraph, that can be derived from the Coxetergraph $\tilde{X}$ by removing one of the nodes:

$\tilde{E}_6$ gives E_6 and not A_6 and D_6.

$\tilde{E}_7$ gives E_7 and A_7 and not D_7.

$\tilde{E}_8$ gives E_7, D_7 and A_7.

So we know already that:

1^{o} $W(E_7)$ has subgroup $W(A_7)$

2^{o} $W(E_8)$ has subgroups $W(A_8)$ and $W(D_7)$.

Assertion 1: $W(A_6)$ *and* $W(D_6)$ *are not subgroups of* $W(E_6)$.

Proof:

Order of $W(E_6) = 2^7.3^4.5$

Order of $W(D_6) = 2^5.6!$

Order of $W(A_6) = 7!$

The assertion follows now from the Lagrange-theorem on the order of a subgroup.

Assertion 2: $W(D_7)$ *is not a subgroup of* $W(E_7)$.

Proof:

Order $W(D_7) = 2^6.7!$ divides on order $W(E_7) = 2^{10}.3^4.5.7$ so we cannot apply the arguments of assertion 1. A.M. Cohen (Utrecht) pointed out to me, that the (following) straightforward computation shows, that it is impossible to find within $\mathbb{R}^7$ an extension of the rootsystem D_7, containing only vectors of length $\sqrt{2}$ and with innerproducts -1, 0 or 1 with the vectors of D_7:

We proceed as follows:

D_7 has a realization in $\mathbb{R}^7$ by the following combinations of basis-vectors: $\pm e_i \pm e_j$ $(1 \leq i < j \leq 7)$. Also E_7 can be realized in $\mathbb{R}^7$.

Extend the system with $x = \sum_{i=1}^{7} \alpha_i e_i$ (with $\sum_{i=1}^{7} \alpha_i^2 = 2$) and such that $\langle x, \pm e_i \pm e_j\rangle \in \{-1,0,1\}$.

Then we must have: $\pm \alpha_i \pm \alpha_j \in \{-1,0,1\}$ $1 \leq i < j \leq 7$.

This implies $\alpha_i \in \{-1,-\frac{1}{2},0,\frac{1}{2},1\}$.

When $\alpha_i^2 = 1$ then $\exists j \neq i$ with also $\alpha_j^2 = 1$ and $\alpha_k = 0$ if $k \neq i,j$.

This gives just the elements of D_7.

If $x \notin D_7$ then $\alpha_i \in \{-\frac{1}{2},0,\frac{1}{2}\}$ and consequently:

$\|x\|^2 \leq \frac{7}{4} < 2$ and this is not possible since $\|x\|^2 = 2$.

<u>Lemma</u>: *Let $\hat{g} \leq \hat{f}$. If* f *is simple elliptic and* g *is simple then* $\mu(g) \leq \mu(f) - 2$.

<u>Proof</u>:

Let K be the kernel of $\langle -,- \rangle$ on $H_n(X_f;\mathbb{Q}) \cong \mathbb{Q}^{\mu(f)}$.

dim $H_n(X_g;\mathbb{Q})$ + dim K = dim$[K + H_n(X_g;\mathbb{Q})]$ + dim$[K \cap H_n(X_g;\mathbb{Q})]$

so : $\mu(g) + 2 \leq \mu(f) + 0$

and: $\mu(g) \leq \mu(f) - 2$

We now consider various cases:

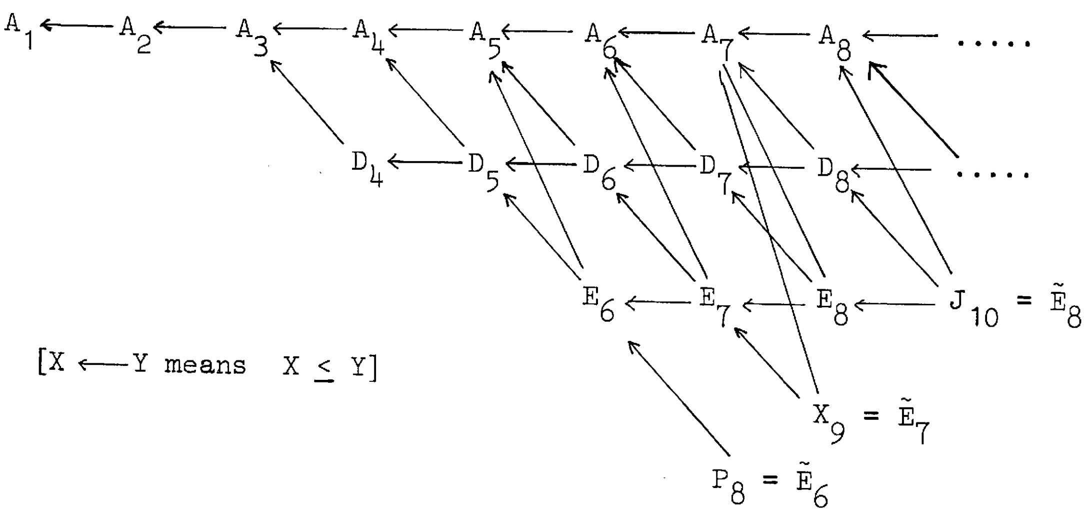
a) If f is of type P_8, then $\mu(g) \leq 6$ and so g is of type $D_k (k \leq 6)$, $A_k (k \leq 6)$ or E_6. Assertion 1 gives that g is not of type A_6, D_6; so the only possibilities are the connected subgraphs of $\tilde{E}_6$.

b) If f is of type X_9, then $\mu(g) \leq 7$ and so g is of type $D_k (k \leq 7)$, $A_k (k \leq 7)$, E_6 or E_7. Assertion 2 gives that g is not of type D_7; so the only possibilities are the connected subgraphs of $\tilde{E}_7$.

c) If f is of type J_{10}, then $\mu(g) \leq 8$ and so g is of type $D_k (k \leq 8)$, $A_k (k \leq 8)$, $E_k (k \leq 8)$ and they correspond just with the connected subgraphs of $\tilde{E}_8$.

Now we are done.

<u>(8.14)</u> <u>Corollary</u>: *Adjacency diagram for simple and simple elliptic germs.*



[X ⟵ Y means X ≤ Y]

(8.15) Definition: The germs $\hat{g}_1,\dots,\hat{g}_p$ are called simultaneously adjacent to $\hat{f}$ if there exists a (germ of) deformation of $\hat{f}$ such that for every neighborhood U of $\underline{0} \in \mathbb{C}^k$ there is $\lambda \in U$ such that f_λ has exactly p critical points $a_1,\dots,a_p$ and the germs at $\underline{0} \in \mathbb{C}^m$ of $g(x-a_i) - g(a_i)$ are equivalent to the germs $\hat{g}_i$. A similar definition holds for orbits.

Corollary: *If $\hat{g}_1,\dots,\hat{g}_p$ are simultaneously adjacent to $\hat{f}$ then the conclusions of theorem (7.9) and remark (7.10) are valid.*

(8.16) Problem: *Let $\hat{f}$ be a simple germ and let $\hat{g}_1,\dots,\hat{g}_p$ be simultaneously adjacent to $\hat{f}$. Can one construct the Dynkindiagram of $\hat{f}$ from the disjoint union of the Dynkindiagrams of $\hat{g}_1,\dots,\hat{g}_p$ by adding branches between differnt components?*

The answer is no. We give the following counterexample:

Let $f_t = x_1^3 + x_2^4 + tx_1^2$

Then: $\partial_1 f_t = 3x_1^2 + 2tx_1 = 0 \longrightarrow x_1 = 0 \ \vee \ x_1 = -\frac{2t}{3}$

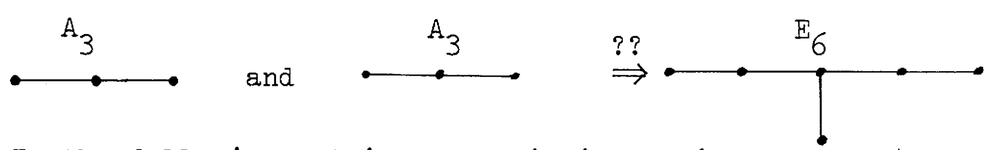
$\qquad \partial_2 f_t = 4x_2^3 = 0 \qquad \longrightarrow x_2 = 0$

So we have critical points: $(0,0)$ and $(-\frac{2t}{3},0)$.

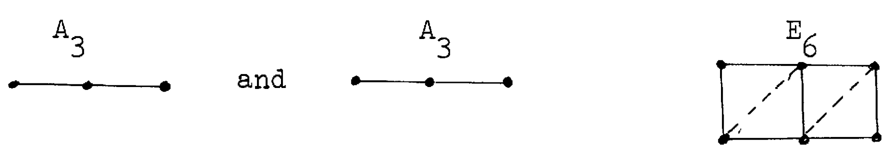
In $(0,0)$ we have for $t \neq 0$ a germ of type A_3. In $(-\frac{2t}{3},0)$ we have for $t \neq 0$ a singularity with Milnornumber equal 3. So the singularity must be of type A_3.

If $t = 0$ $f_t = x_1^3 + x_2^4$ is of type E_6.

So two germs of type A_3 are simultaneously adjacent to a germ of type E_6.



Remark: In the following matrix for E_6 it is possible to see two submatrices, equivalent to A_3:



§9 μ-homotopic germs

Throughout this paragraph we study germs with isolated singularity at $\underline{0}$.

(9.1) Definition: Two germs $\hat{g}_a$ and $\hat{g}_b$ are called μ-homotopic, if there exists a continuous 1-parameter family g_t, $t \in [a,b] \subset \mathbb{R}$ connecting $\hat{g}_a$ and $\hat{g}_b$ and such that $\mu(g_t)$ is constant for all $t \in [a,b]$.

Examples:

a) $g_t = z_1^3 + z_2^3 + tz_1^{31} z_2^{47}$ contains μ-homotopic germs for all $t \in \mathbb{C}$; this is not surprising as all g_t are in one and the same orbit.

b) $g_t = z_1^4 + z_2^4 + tz_1^2 z_2^2$ contains μ-homotopic germs for all $t^2 \neq 4$.

(9.2) Proposition: *μ-homotopy is an equivalencerelation.*
The proof is a straightforward verification of the definition of equivalence relation.

The equivalenceclasses are called μ-homotopyclasses or μ-classes.

(9.3) Proposition: *If $\hat{g}_a$ and $\hat{g}_b$ are μ-homotopic, then there exists an isomorphism $H_n(X_{g_a}) \to H_n(X_{g_b})$ preserving the intersectionform* $\langle -,- \rangle$.

Proof:

Define $G(t,x) = g_t(x)$.

The set $\{(t,x) \mid \frac{\partial G}{\partial x_0}(t,x) = \dots = \frac{\partial G}{\partial x_n}(t,x) = 0\} \subset [a,b] \times \mathbb{C}^{n+1}$ contains $[a,b] \times \underline{0}$ as isolated component. This follows from the fact, that for any g_c with $c \in [a,b]$ every small deformation g_t of g_c has only one critical point inside a small ball B, with radius depending on c. It is possible to find $\varepsilon > 0$ such that g_t has an isolated critical point at $\underline{0} \in \mathbb{C}^{n+1}$ and no other critical points inside a ball of radius ε for all $t \in [a,b]$. Next we can apply the proof of theorem 1 of TJURINA [24] and our proposition follows .

(9.4) Definition:

$$Z\,(p) = \{f \in \mathcal{E}_n \mid \mu(f) \geq p\}$$

$$\Sigma\,(p) = \{f \in \mathcal{E}_n \mid \mu(f) = p\}$$

$$Z^k(p) = \{f \in J^k(n,1) \mid \mu(f) \geq p\}$$

$$\Sigma^k(p) = \{f \in J^k(n,1) \mid \mu(f) = p\}$$

Remark: The sets $Z(p)$, $\Sigma(p)$, $Z^k(p)$ and $\Sigma^k(p)$ are invariant under the right-action of biholomorphic mappings. Moreover they are unions of μ-classes.

(9.5) Proposition:

a) $Z^k(p)$ is an algebraic subset of $J^k(n,1)$

b) $\Sigma^k(p)$ is a difference of two algebraic subsets in $J^k(n,1)$

c) $Z^k(p)$ and $\Sigma^k(p)$ have only a finite number of topological components.

Proof:

a) Remember: $\mu(f) = \dim \frac{\mathcal{E}}{\Delta(f)}$, where $\Delta(f) = (\partial_1 f, \ldots, \partial_n f)$.

Assertion: $\dim \frac{\mathcal{E}}{\Delta(f)} \geq p \Leftrightarrow \dim \frac{\mathcal{E}}{\Delta(f) + m^p} \geq p$

$\Leftarrow$ is trivial

$\Rightarrow$ (following MATHER [19]):

Let $\dim \frac{\mathcal{E}}{\Delta(f) + m^p} < p$; consider the following increasing sequence of (p+1) ideals:

$$\Delta \subset \Delta + m \subset \ldots \subset \Delta + m^k \subset \ldots \subset \Delta + m^p$$

since $\dim \frac{\mathcal{E}}{\Delta(f)} \geq 0$ and $\dim \frac{\mathcal{E}}{\Delta(f) + m^p} < p$, there exists a $k < p$ such that $\dim \frac{\mathcal{E}}{\Delta(f) + m^k} = \dim \frac{\mathcal{E}}{\Delta(f) + m^{k+1}}$.

So $\Delta(f) + m^k = \Delta(f) + m^{k+1}$ and $m^k \subseteq \Delta(f) + m^{k+1}$.

From the Nakayamalemma it follows that: $m^k \subseteq \Delta(f)$.

So $\dim \frac{\mathcal{E}}{\Delta(f)} = \dim \frac{\mathcal{E}}{\Delta(f) + m^k} \leq \dim \frac{\mathcal{E}}{\Delta(f) + m^p} < p$.

Now the assertion is proved.

The condition $\dim \frac{\mathcal{E}}{\Delta(f) + m^p} \geq p$ is clearly algebraic, since it is a rank-condition on a subspace of the finite dimensional vectorspace $\frac{\mathcal{E}}{m^p}$ and gives rise to determinants in the coördinates of $J^k(n,1)$.

b) follows from the fact that $\Sigma^k(p) = S^k(p) \setminus S^{k+1}(p)$.

c) A theorem of Whitney says, that for any pair of algebraic sets, the difference has at most a finite number of topological components (cf. MILNOR [20]).

Corollary: *Every topological component of* $Z^k(p)$ *coïncides with a* μ-class.

(9.6) List of μ-classes with $\mu \leq 10$.

$\Sigma(1)$:	A_1						
$\Sigma(2)$:	A_2						
$\Sigma(3)$:	A_3						
$\Sigma(4)$:	A_4	D_4					
$\Sigma(5)$:	A_5	D_5					
$\Sigma(6)$:	A_6	D_6	E_6				
$\Sigma(7)$:	A_7	D_7	E_7				
$\Sigma(8)$:	A_8	D_8	E_8		P_8		
$\Sigma(9)$:	A_9	D_9		X_9	P_9		
$\Sigma(10)$:	A_{10}	D_{10}	J_{10}	X_{10}	P_{10}	Q_{10}	R_{10}

The symbols correspond to those in §3. The complex normalforms are given in list I at the end.

<u>(9.7)</u> <u>Proposition</u>: *The classes of the list are in different topological components of* $\Sigma(p)$.

<u>Proof</u>:

The intersectionforms are different, so by proposition (9.3) there is no μ-homotopy, joining any two different classes in the list.

<u>(9.8)</u> <u>Definition</u>: $\hat{g}$ is called μ-adjacent to $\hat{f}$ if every neighborhood of $\hat{f}$ contains an element, that is μ-homotopic to $\hat{g}$.

Since g and f have isolated singular point at $\underline{0}$, we can work entirely in $J^k(n,1)$ for k large enough. The following lemma shows, that the definition of μ-adjacency depends only on the μ-class of $\hat{g}$ and $\hat{f}$.

<u>(9.9)</u> <u>Lemma</u>: *Let* $A^k(p)$ *be a topological component of* $\Sigma^k(p)$ *and* $B^k(p)$ *be a topological component of* $\Sigma^k(q)$ $(q \leq p)$.

Then either: $A^k(p) \cap \overline{B^k(q)} = \emptyset$

or: $A^k(p) \subset \overline{B^k(q)}$.

<u>Proof</u>:

Let $C^k(q)$ be the top.component of $S^k(q)$ such that $A^k(q) \subset C^k(q)$. The sets $\Sigma^k(q) = S^k(q) \setminus S^k(q+1)$ and $S^k(q)$ have the same number of topological components; so

either 1° $B^k(q) \subset C^k(q)$

or 2° $B^k(q) \cap C^k(q) = \emptyset$

1° gives $A^k(p) \subset C^k(q) = \overline{B^k(q)}$

2° gives $\overline{B^k(q)} \cap C^k(q) \neq \emptyset$ and so $\overline{B^k(q)} \cap A^k(q) = \emptyset$.

<u>(9.10)</u> <u>Theorem</u>: *If* $\hat{g}$ *is* μ*-adjacent to* $\hat{f}$ *then there exists an injection* $H_n(X_g) \to H_n(X_f)$ *preserving* $<-,->$ *and a distinguished basis of vanishing cycles such that the intersectionmatrix of* $\hat{g}$ *is a diagonal submatrix of the intersectionmatrix of* $\hat{f}$.

<u>Proof</u>:

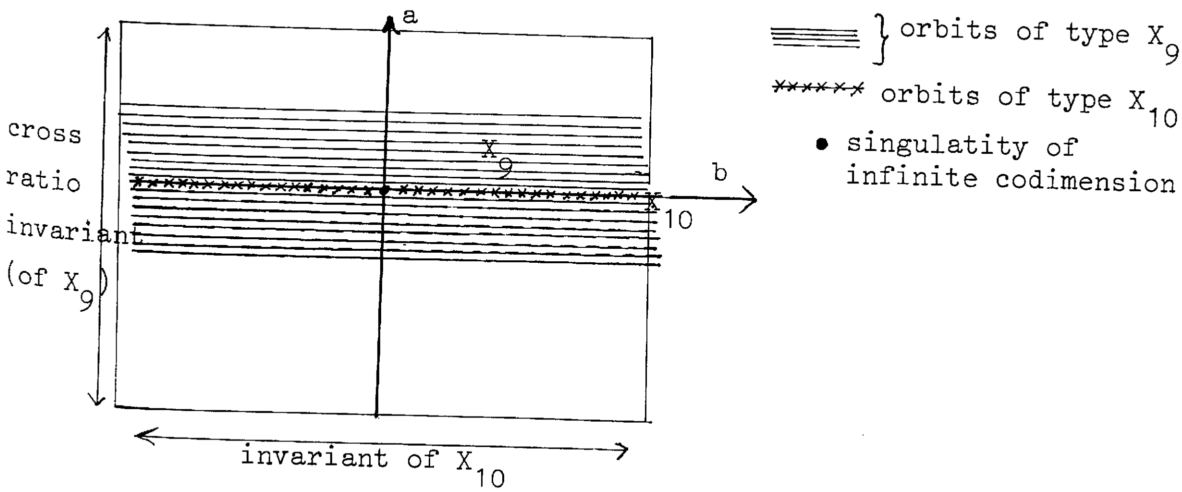
similar to (8.9).

(9.11) Theorem: *If* g *is a simple singularity and* f_t *a 1-parameter family with* μ *constant.*
If $g \leq f_{t_o}$ *then also* $g \leq f_t$.
Proof:
The μ-homotopyclass of g is μ-adjacent to the μ-class of f_{t_o} (and so also of f_t).
So in every neighborhood of f_t, there are germs μ-homotopic with $\hat{g}$. Since $\hat{g}$, is simple μ-homotopy implies eqivalence.

Corollary: If a simple singularity g is μ-adjacent to f, then g is (ordinary) adjacent to f.

(9.12) Remark: *(difference between adjacency and* μ*-adjacency).*
Arnold gave a complete graph of the adjacency relation between simple singularities. Saito also considered P_8, X_9 and J_{10} (the simple elliptic singularities $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$). In these cases adjacency and μ-adjacency between the different classes coincide. This is in general not the case. As an example we have the following result.

Let g be of type X_9: $g = z_o^4 + 2z_o^2 z_1^2 + az_1^4 \quad (a \neq 0,1)$
and f of type X_{10} : $f = z_o^4 + 2z_o^2 z_1^2 + bz_1^5 \quad (b \neq 0)$
then g is not adjacent to f for no (fixed) values of the parameters a and b. Also here the crossratio gives the obstruction. Indeed g is μ-adjacent to f. The following picture illustrates this situation:
Consider the 2-parameter family: $z_o^4 + 2z_o^2 z_1^2 + az_1^4 + bz_1^5$



(9.13) Remark: It is possible to extend the graph of the adjacency-relation of simple and simple elliptic singularities (cf. (8.14)) with the other singularities of the list. Further computations are then needed. We treat this in §10. As an example: consider the path of germs:

$$f_t = t^2 z_o^2 z_1 + z_o^4 + 2z_o^2 z_1^2 + 2tz_o z_1^3 + z_1^5$$

If $t \neq 0$ f_t is of type D_7 and if $t = 0$ f_t is of type X_{10}.
So D_7 is adjacent to X_{10} (for all $b \neq 0$).
SAITO proved that D_7 is not adjacent to X_9. So in the family considered in (9.12) the polynomials of type X_{10} differ from those of type X_9 by the property, that X_{10} is in the closure of D_7 and X_9 is not.

§10 On the topology of the orbitspace.

(10.1) Let G_k be the set of germs of holomorphic mappings $(\mathbb{C}^n,\underline{0}) \to (\mathbb{C},0)$ with codimension $\leq k$, having in $\underline{0}$ a critical point. Remark, that all germs in G_k are $(k+2)$-determined. We define in G_k a topology in the following way: An open set is $\{g \in G_k \mid g_{k+2}$ lies in an open set of $\mathbb{C}^{N(k+2)}\}$, where $N(k)$ is the number of coëfficients of polynomials of degree k in n variables.
The natural injection $G_k \to G_{k+1}$ is a continuous map with respect to this topology. We define $G = \bigcup_{k=1}^{\infty} G_k$ and derive the topology of G from the topology of the spaces G_k.

(10.2) Let W be the set of orbits in G under the Rightaction of biholomorphic mappings: We give W the quotient topology; the projection $\pi : G \to W$ is then a continuous mapping. So a set U in W is open if and only if $\pi^{-1}(U)$ is open in G. We use the symbol Y_ℓ of an orbit to denote also its projection in W. So in fact $\pi(Y_\ell)$ is denoted by Y_ℓ. Each simple singularity defines one point in W. The projections of some non-simple singularities will be discussed in (10.6).

(10.3) Theorem: *The topology of* W *is not Hausdorff, even not* T_1.
Proof:
Since the orbit of type A_1 (non-degenerate quadratic form) is dense in G_k for every $k \geq 1$, every open neighborhood of $w \in W$ contains the point A_1.
So there is no open neighborhood of w avoiding A_1.

(10.4) We now consider μ-classes and denote by $\Theta(w)$ the codimension of the μ-class, containing $w \in W$. Since μ-classes are topological components of differences of algebraic sets and so a finite union of manifolds, this number is well-defined.
Let U_k be the union of the μ-classes with $\Theta(w) \leq k$, so

$$U_k = \{w \in W \mid \Theta(w) \leq k\}$$

List of μ-classes in U_8:

U_0 :	A_1						
$U_1 \setminus U_0$:	A_2						
$U_2 \setminus U_1$:	A_3						
$U_3 \setminus U_2$:	A_4	D_4					
$U_4 \setminus U_3$:	A_5	D_5					
$U_5 \setminus U_4$:	A_6	D_6	E_6				
$U_6 \setminus U_5$:	A_7	D_7	E_7		P_8		
$U_7 \setminus U_6$:	A_8	D_8	E_8	X_9	P_9		
$U_8 \setminus U_7$:	A_9	D_9	J_{10}	X_{10}	P_{10}	Q_{10}	R_{10}

In the case of simple singularities there exist normal forms without local invariants, so A_k $(k \geq 1)$, D_k $(k \geq 4)$, E_6, E_7 and E_8 are points in W. The orbitspaces for the families J_{10}, X_9, X_{10}, P_8, P_9, P_{10}, Q_{10} and R_{10} will be described next.

(10.5) We gave normalforms for these families in the real case, already in (3.6). These forms can also be used in the complex case, but somtimes other normalforms are more practical. They are mentioned in the proof of (10.6) and also in the list at the end.

We shall investigate in these eight cases those values of the parameters for which the germs are equivalent. Next we take the quotient-space to this equivalence. We get a topological space (even a complex space), which can be identified with the corresponding subset in W.

In each of these 8 cases f is finitely determined, say by its k-jet. So we can work entirely in $J^k(n,1)$ and have only to consider k-jets of mappings.

Let f_t be a k-parameterfamily of germs. The condition $f_t(\phi(z)) = f_s(z)$ gives restrictions on the coëfficients of $j^k(\phi)$. It can be verified in each case seperately that ϕ has to be an element of GL(n). This is left to the reader. Even in most of the cases the only possible action is multiplication by a scalar of each coördinate:

$$\begin{cases} z_1 := \alpha z_1 & (\alpha \neq 0) \\ z_2 := \beta z_2 & (\beta \neq 0) \\ z_3 := \gamma z_3 & (\gamma \neq 0) \end{cases}$$

We call this a <u>diagonal isomorphism</u>.

<u>(10.6)</u> <u>Theorem</u>:

a) *The orbitspaces of* P_8, X_9, J_{10} *and* Q_{10} *are complex isomorphic with* $\mathbb{C}$.

b) *The orbitspaces of* P_9, P_{10}, R_{10} *and* X_{10} *are complex isomorphic with* $\mathbb{C}-\{\underline{0}\}$.

<u>Proof</u>:

<u>case P_8</u>: $f_{(A,B)} = z_1^3 + z_2^2 z_3 + Az_1 z_3^2 + Bz_3^3$ with $4A^3 + 27B^2 \neq 0$.

If $f_{(A,B)}(\phi(z)) = f_{(A',B')}$ then ϕ must be a diagonal isomorphism and we get:

$$f_{(A',B')} = \alpha^3 z_1^3 + \beta^2\gamma z_2^2 z_3 + A\alpha\gamma^2 z_1 z_3^2 + B\gamma^3 z_3^3$$

So $f_{(A,B)}$ and $f_{(A',B')}$ are equivalent $\Leftrightarrow$

$\Leftrightarrow \exists\, \alpha,\beta,\gamma \in \mathbb{C}-\{\underline{0}\}$ with $\alpha^3 = 1 \wedge \beta^2\gamma = 1 \wedge \alpha\gamma^2 A = A' \wedge \gamma^3 B = B' \Leftrightarrow$

$\Leftrightarrow \exists\, \alpha,\beta \in \mathbb{C}-\{\underline{0}\}$ with $\alpha^3 = 1 \wedge \alpha\beta^{-4}A = A' \wedge \beta^{-6}B = B' \Leftrightarrow$

$\Leftrightarrow \exists\, \beta \in \mathbb{C}-\{\underline{0}\}$ with $\beta^4 A = A' \wedge \beta^6 B = B'$.

Hence: $f_{(A,B)} \sim f_{(A',B')} \Leftrightarrow j(A,B) = j(A',B')$

where $j(A,B) = \dfrac{A^3}{4A^3 + 27B^2}$; the so-called j-invariant.

The orbits are characterized by $j \in \mathbb{C}$; so the orbitspace of P_8 is $\mathbb{C}$.

<u>case J_{10}</u>: $f_{(A,B)} = z_1{}^3 + Az_1z_2{}^4 + Bz_2{}^6$ with $4A^3 + 27B^2 \neq 0$.

If $f_{(A,B)}(\phi(z)) = f_{(A',B')}(z)$ then ϕ can only be a diagonal isomorphism

So $f_{(A,B)} \backsim f_{(A',B')} \Leftrightarrow \exists \alpha,\beta \in \mathbb{C}-\{0\}$ with $\alpha^3 = 1 \wedge \alpha\beta^4 A = A' \wedge \beta^6 B = B'$

This case is similar to P_8 and the orbitspace is $\mathbb{C}$.

The orbits can be characterized by $k(A,B) = \dfrac{A^3}{4A^3 + 27B^2} \in \mathbb{C}$.

<u>case X_9</u>: $f_d = z_1z_2(z_1-z_2)(z_1-dz_2)$ with $d \neq 0,1$.

d is the cross ratio of the four complex lines $f_d = 0$.

$f_d \backsim f_{d'} \Leftrightarrow$ crossratio of $f_d = 0$ and $f_{d'} = 0$ are equal $\Leftrightarrow$

$\Leftrightarrow \quad d' \in \{d, \frac{1}{d}, 1-d, \frac{1}{1-d}, \frac{d}{d-1}, \frac{d-1}{d}\}$.

Define $c : \mathbb{C}-\{0,1\} \to \mathbb{C}$ by:

$c(d) = d^2 + (\frac{1}{d})^2 + (1-d)^2 + (\frac{1}{1-d})^2 + (\frac{d}{d-1})^2 + (\frac{d-1}{d})^2$, then

$$c(d) = \frac{2d^6 - 6d^5 + 9d^4 - 8d^3 + 9d^2 - 6d + 2}{(d-1)^2 d^2}$$

The map $c : \mathbb{C}-\{0,1\} \to \mathbb{C}$ is surjective and for every $q \in \mathbb{C}$ there are at most six solutions of $c(d) = q$. The definition of c implies, that with any solution d also $\frac{1}{d}$, $1-d$, $\frac{1}{1-d}$, $\frac{d}{d-1}$, $\frac{d-1}{d}$ are solutions.

This shows that the orbitspace of X_9 is $\mathbb{C}$.

<u>case P_9</u> : $f_A = z_1z_2z_3 + z_1{}^3 + z_2{}^3 + Az_3{}^4$ with $A \neq 0$.

If $f_A(\phi(z)) = f_{A'}(z)$ then either ϕ is a diagonal isomorphism or ϕ is defined by: $\phi(z_1) = \beta z_2$; $\phi(z_2) = \alpha z_1$; $\phi(z_3) = \gamma z_3$.

In both cases:

$f_A \backsim f_{A'} \Leftrightarrow \exists \alpha,\beta,\gamma \in \mathbb{C}-\{\underline{0}\}$ with $\alpha^3 = \beta^3 = \alpha\beta\gamma = 1 \wedge \gamma^4 A = A' \Leftrightarrow$

$\Leftrightarrow \exists \gamma \in \mathbb{C}-\{\underline{0}\}$ with $\gamma^3 = 1 \wedge \gamma A = A'$.

We get the orbitspace of P_9 if we divide $\mathbb{C}-\{\underline{0}\}$ by the $\mathbb{Z}_3$-action of multiplication by 3^{rd} root of unity. This gives $\mathbb{C}-\{\underline{0}\}$.

<u>case P_{10}</u>: $f_A = z_1z_2z_3 + z_1^3 + z_2^3 + Az_3^5$ with $A \neq 0$.
This case is similar to P_9, we get again $\mathbb{Z}_3$-action on $\mathbb{C}-\{0\}$.

<u>case Q_{10}</u>: $f_A = z_1^3 + z_2^2z_3 + Az_1z_3^3 + z_3^4$.
This case is similar to P_9, we get now $\mathbb{Z}_{12}$-action on $\mathbb{C}$.

<u>case R_{10}</u>: $f_A = z_1z_2z_3 + z_1^3 + z_2^4 + Az_3^4$ with $A \neq 0$.
This case is similar to P_9, we get $\mathbb{Z}_3$-action on $\mathbb{C}-\{\underline{0}\}$.

<u>case X_{10}</u>: $f_A = z_1^4 + z_1^2z_2^2 + Az_2^5$ with $A \neq 0$.
This case is similar to P_9, we get $\mathbb{Z}_4$-action on $\mathbb{C}-\{\underline{0}\}$.

<u>(10.7)</u> We define $K(w) = \{w' \in W \mid w' \in U$ for every open set U in W, containing $w\}$.

<u>Lemma</u>: w_1 *adjacent to* w_2 *if and only if* $K(w_1) \subset K(w_2)$.
This is clear from the definitions of adjacency and of K.

<u>Examples</u>:

$$\left.\begin{aligned}
K(A_s) &= A_s \cup A_{s-1} \cup\ldots\cup A_1 \\
K(D_s) &= D_s \cup D_{s-1} \cup\ldots\cup D_4 \cup A_{s-1} \cup A_{s-2} \cup\ldots\cup A_1 \\
K(E_6) &= E_6 \cup D_5 \cup D_4 \cup A_5 \cup A_4 \cup\ldots\cup A_1 \\
K(E_7) &= E_7 \cup E_6 \cup D_6 \cup D_5 \cup D_4 \cup A_6 \cup A_5 \cup\ldots\cup A_1 \\
K(E_8) &= E_8 \cup E_7 \cup E_6 \cup D_7 \cup\ldots\cup D_4 \cup A_7 \cup\ldots\cup A_1
\end{aligned}\right\}\ \text{open sets}$$

$K(P_8(j_o)) = P_8(j_o) \cup K(E_6)$ (not open)

<u>(10.8)</u> We are interested in the orbits that occur, when we perturb a given orbit w a "little". This means that we have to study small open neighborhoods of w in W. Those open sets certainly contain $K(w)$. In the case of simple singularity $K(w)$ is the smallest open set containing w.

In the sequel we try to describe some of the open neighborhoods of w if $w \in U_8$. We remark that U_8 consists of a finite number of points and a finite number of copies of $\mathbb{C}$ and $\mathbb{C}-\{0\}$, each in itself having induced the usual Hausdorff topology. So if w is not-simple in U_8 every neighborhood of w contains at least an open neighborhood of w in $\mathbb{C}$ or $\mathbb{C}-\{0\}$. $\mathbb{C}$ can be embedded in S^2 by adding one point (call it ∞). Then open neighborhoods of ∞ in $\mathbb{C}$ are defined in the usual way.

<u>(10.9)</u> <u>Adjacency in corank 3</u>.

We consider now $V_3 = P_8 \cup P_9 \cup P_{10} \cup Q_{10} \cup R_{10}$ in the relative topology (see figure).

<u>case P_8</u>: A point $j \in P_8 = \mathbb{C}$ corresponds with a $_3$germ

$z_1^3 + z_2^2 z_3 + g_1 z_1 z_3^2 + g_2 z_3^3$ such that $\dfrac{g_1}{4g_1^3 + 27g_2^2} = j$.

An open set of j in V_3 is an open neighborhood of $\overset{\circ}{j} \in P_8 = \mathbb{C}$ in the usual topology of $\mathbb{C}$.

<u>case P_9</u>: Points w of $P_9 = \mathbb{C}-\{\underline{0}\}$ can be given by:

$z_1^3 + z_2^2 z_3 + z_1^2 z_3 + Az_3^4$ with $A \neq 0$, or also by:

$z_1^3 + z_2^2 z_3 + g_1 z_1 z_3^2 + g_2 z_3^3 + A' z_3^4$ with $A' \neq 0$, where g_1 and g_2 satisfy $4g_1^3 + 27g_2^2 = 0$ and $(g_1, g_2) \neq (0,0)$.

So an open neighborhood of $w \in P_9 = \mathbb{C}-\{\underline{0}\}$ in V_3 consists of:

1° an open neighborhood of w in $P_9 = \mathbb{C}-\{\underline{0}\}$

2° an open neighborhood of ∞ in $\mathbb{C} = P_8$.

P_8 is μ-adjacent to P_9, but not adjacent.

<u>case P_{10}</u>: In a similar way as in case P_9 one concludes that an open neighborhood of $w \in P_{10} = \mathbb{C}-\{\underline{0}\}$ in V_3 consists of:

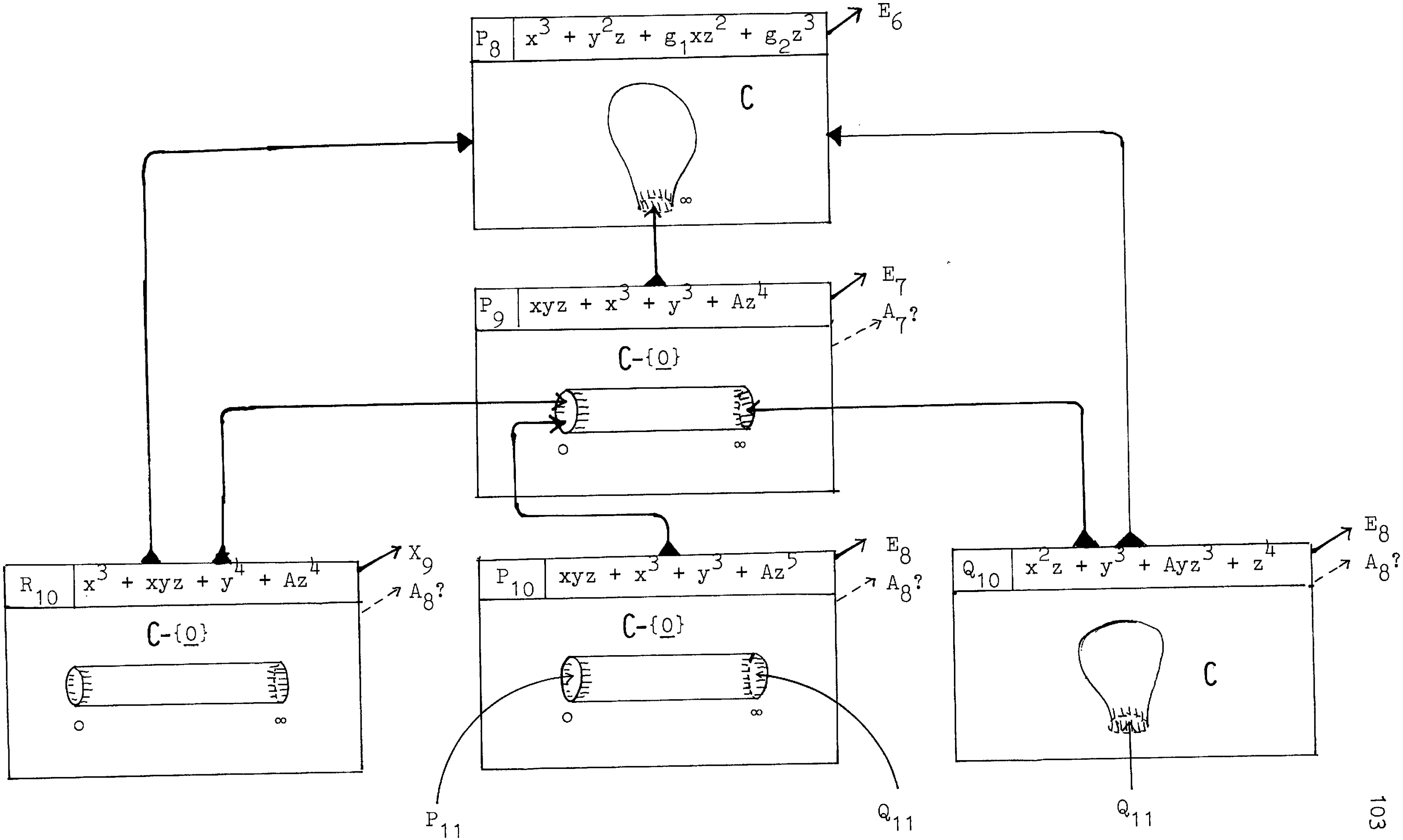
1° an open neighborhood of w in $\mathbb{C}-\{\underline{0}\} = P_{10}$

2° an open neighborhood of 0 in $\mathbb{C}-\{\underline{0}\} = P_9$

3° an open neighborhood of ∞ in $\mathbb{C} = P_8$.

P_8 is μ-adjacent to P_{10}, but not adjacent.

P_9 is μ-adjacent to P_{10}, but not adjacent.



P_8
$x^3 + y^2z + g_1xz^2 + g_2z^3$
E_6
C
∞
P_9
$xyz + x^3 + y^3 + Az^4$
E_7
A_7?
C-{0}
o
∞
R_{10}
$x^3 + xyz + y^4 + Az^4$
X_9
A_8?
C-{0}
o
∞
P_{10}
$xyz + x^3 + y^3 + Az^5$
E_8
A_8?
C-{0}
o
∞
Q_{10}
$x^2z + y^3 + Ayz^3 + z^4$
E_8
A_8?
C
P_{11}
Q_{11}
Q_{11}

<u>case Q_{10}</u>: A point $w \in Q_{10} = \mathbb{C}$ can be given by $z_1^3 + z_2^2 z_3 + Az_1 z_3^3 + z_3^4$.
We consider its universal deformation and ommit terms of degree ≤ 2:
$z_1^3 + z_2^2 z_3 + \lambda_1 z_1 z_3^2 + \lambda_2 z_3^3 + (A + \lambda_3) z_1 z_3^3 + z_3^4$.
Let $4\lambda_1^3 + 27\lambda_2^2 \neq 0$. The equation $\lambda_1^3 / 4\lambda_1^3 + 27\lambda_2^2 = j$ has for every $j \in \mathbb{C}$ solutions (λ_1, λ_2) arbitrarily close to $(0,0)$. So we get all the members of the family P_8 in the deformation.
If $4\lambda_1^3 + 27\lambda_2^2 = 0$ and $(\lambda_1, \lambda_2) \neq (0,0)$ we get germs of type P_9. We next change coordinates and transform the germ in the normalform $z_1^3 + z_2^2 z_3 + z_1^2 z_3 + \mu z_3^4$ of P_9. Then the coefficient of z_3^4 goes to ∞ if $(\lambda_1, \lambda_2, \lambda_3) \to (0,0,0)$.
So an open neighborhood of $w \in Q_{10} = \mathbb{C}$ in V_3 consists of:

1° an open neighborhood of w in $\mathbb{C} = Q_{10}$

2° an open neighborhood of ∞ in $\mathbb{C}-\{\underline{0}\} = P_9$

3° the whole set P_8

This discussion shows:

P_8 is adjacent to every germ in the family Q_{10}.

P_9 is μ-adjacent to P_{10}, but not adjacent.

<u>case R_{10}</u>: A point $w \in R_{10}$ can be given by $z_1^3 + z_1 z_2 z_3 + z_2^4 + Az_3^4$.
We consider its universal deformation and ommit terms of degree ≤ 2:
$z_1^3 + z_1 z_2 z_3 + \lambda_1 z_2^3 + \lambda_2 z_3^3 + z_2^4 + (A+\lambda_3) z_3^4$.
The 3-jet is of type P_8 if $\lambda_1 \lambda_2 \neq 0$. The j-invariant tends to ∞ if $(\lambda_1, \lambda_2) \to (0,0)$ since R_{10} has doublepoints. If $\lambda_2 = 0 \wedge \lambda_1 \neq 0$ we have a germ of type P_9. A coordinatechange to the normalform $z_1^3 + z_1 z_2 z_3 + z_2^3 + \mu z_3^4$ shows that $\mu \to 0$ if $(\lambda_1, \lambda_2) \to (0,0)$.
So an open neighborhood of $w \in R_{10} = \mathbb{C}-\{\underline{0}\}$ in V_3 consists of

1° an open neighborhood of w in $\mathbb{C}-\{\underline{0}\} = R_{10}$

2° an open neighborhood of 0 in $\mathbb{C}-\{\underline{0}\} = P_9$

3° an open neighborhood of ∞ in $\mathbb{C} = P_8$

P_8 is μ-adjacent to R_{10}, but not adjacent.

P_9 is μ-adjacent to R_{10}, but not adjacent.

(10.10) Adjacencyrelations of X_9.

The adjacency of X_9 to X_{10} is already discussed in (9.12). Every open neighborhood of $w \in X_9 = \mathbb{C}$ contains an open neighborhood of w in $\mathbb{C} = X_9$. Every open neighborhood of $w \in X_{10} = \mathbb{C}-\{\underline{0}\}$ contains:

1° an open neighborhood of w in $\mathbb{C}-\{\underline{0}\} = X_{10}$

2° an open neighborhood of ∞ in $\mathbb{C} = X_9$.

Next we study the adjacency of X_9 with P_{10}, Q_{10} and R_{10}.
Let $f_t(z_1,z_2,z_3)$ be of type X_9 if $t \neq 0$ and of type P_{10}, Q_{10} or R_{10} if $t = 0$. After change of coordinates we can arrange, that the 3-jet of f_t has the form:

$$g_t = tz_3^2 + z_3\phi_t(z_1,z_2,z_3) + \sigma_t(z_1,z_2).$$

For $t \neq 0$ holds:

$$g_t = t(z_3 + \frac{\phi_t(z_1,z_2,z_3)}{2t})^2 + \sigma_t(z_1,z_2) - \frac{[\phi_t(z_1,z_2,z_3)]^2}{4t}.$$

Since f_t is of type X_9 we have $\sigma_t(z_1,z_2) = 0$ for $t \neq 0$. The continuity of f_t implies $\sigma_t(z_1,z_2) \equiv 0$. Since $g_o = 0$ is a reducible curve in $P^2(\mathbb{C})$, f is not of type P_{10} or Q_{10}.

Remark: The same reasoning shows that there are no adjacencyrelations between:

$$X_k\colon z_1^4 + z_1^2z_2^2 + Az_2^{k-5} + z_3^2 \qquad (k \geq 9)$$

and P_ℓ: $z_1z_2z_3 + z_1^3 + z_2^3 + Az_3^{\ell-5} \qquad (\ell \geq 9)$

This is the first example of such a situation.

The following curve shows that X_9 is μ-adjacent to R_{10} and that every open neighborhood of $w \in X_{10} = \mathbb{C}-\{\underline{0}\}$ contains a neighborhood of ∞ in $\mathbb{C} = X_9$:

$$f_t = tz_3^2 + z_1z_2z_3 + z_3^3 + Az_1^4 + z_2^4 \sim tz_3^2 + Az_1^4 - \frac{2z_1^2z_2^2}{4t} + z_2^4.$$

This shows that X_9 is μ-adjacent to R_{10}. Wether X_9 is adjacent to X_{10} is unknown to me.

(10.11) Theorem:

a) E_7 *is adjacent to* P_9
b) E_8 *is adjacent to* P_{10}
c) E_8 *is adjacent to* Q_{10}
d) E_7 *is adjacent to* R_{10} *and* E_8 *is not adjacent to* R_{10}
e) E_8 *is adjacent to* X_{10}.

Proof:

a) $f_t = t^2z_3^2 + z_3(z_1z_2 + 2tz_2^2 + z_3^2) + z_1^3 + z_2^4$

$f_o = z_1z_2z_3 + z_1^3 + z_3^3 + z_2^4$ has type P_9

For $t \neq 0$ we can transform f_t in the normalform of E_7.

b) $f_t = tz_3^2 + z_3(z_1z_2 + 2t^2z_2^2 + z_3^2) + z_1^3 + tz_1z_2^3 + t^3z_2^4 + z_2^5$

$f_o = z_1z_2z_3 + z_1^3 + z_3^3 + z_2^5$ has type P_{10}

For $t \neq 0$ we can transform f_t in the normalform of E_8.

c) $f_t = t^2z_3^2 + z_3(2tz_2^2 + tz_1z_2 + z_2z_3) + z_1^3 + z_1z_2^3 + z_2^4 + tz_2^5.$

$f_o = z_2^2z_3 + z_1^3 + z_1z_2^3 + z_2^4$ has type Q_{10}

For $t \neq 0$ we can transform f_t in the normalform of E_8.

d) We shall show in (10.12) that D_7 is not adjacent to R_{10}. Since D_7 is adjacent to E_8, this shows that E_8 is not adjacent to R_{10}. Since P_9 is μ-adjacent to R_{10} and E_7 is adjacent to P_9 also E_7 is adjacent to R_{10}.

e) $f_t = tz_1^3 + z_1^4 + z_1^2 z_2^2 + Az_2^5$

$f_o = z_1^4 + z_1^2 z_2^2 + Az_2^5$ is of type X_{10}

If $t \neq 0$ we can transform f_t in the normalform of E_8.

(10.12) Theorem:

a) D_7 *is not adjacent to* P_9

b) D_7 *is not adjacent to* R_{10}

c) D_8 *is not adjacent to* Q_{10}

d) D_8 *is not adjacent to* P_{10}

e) D_8 *is not adjacent to* X_{10}.

Proof:

a) and b):

Let $\phi_t(z_1,z_2,z_3)$ be of type D_7 if $t \neq 0$. After change of coördinates we can assume, that the 4-jet of ϕ_t has the form:

$$t^2 z_3^2 + z_3 f(z_1,z_2,z_3) + z_1^2 z_2 + z_3 g(z_1,z_2,z_3) + \sigma(z_1,z_2)$$

where:

$$f(z_1,z_2,z_3) = \gamma_{11}z_1^2 + \gamma_{22}z_2^2 + \gamma_{33}z_3^2 + \gamma_{12}z_1z_2 + \gamma_{13}z_1z_3 + \gamma_{23}z_2z_3$$

$$\sigma_4(z_1,z_2) = p_o z_1^4 + p_1 z_1^3 z_2 + p_2 z_1^2 z_2^2 + p_3 z_1 z_2^3 + p_4 z_2^4$$

$$g(z_1,z_2,z_3) = \sum_{i\leq j\leq k} \partial_{ijk} z_i z_j z_k.$$

Assume that ϕ_o is of type P_9 or R_{10} then the universal deformation shows that it is sufficient to study only the 4-jet of ϕ_t.

For $t \neq 0$ we apply the substitution $z_3 := z_3 - \frac{1}{2t^2} f(z_1,z_2,z_3)$.

Then the coëfficient of z_2^4 becomes $-\frac{\gamma_{22}}{4t^2} + p_4$.

The coëfficient of $z_1z_2^3$ becomes $A = -\frac{\gamma_{12}\gamma_{22}}{2t^2} + p_3$.

The coëfficient of z_2^5 becomes $B = \frac{\gamma_{23}\gamma_{22}^2}{4t^4} - \frac{\gamma_{22}\partial_{222}}{2t^2}$.

If $-\frac{\gamma_{22}}{4t^2} + p_4 \neq 0$ we can transform ϕ_t in the normalform of D_5.

Let now $\gamma_{22} = 2t\sqrt{p_4}$, so the coëfficient of z_2^4 vanishes. Then ϕ can be transformed in $tz_3^2 + z_1^2 z_2 + Az_1z_2^3 + Bz_2^5$.and next in

$$tz_3^2 + z_1^2 z_2 + (B - \frac{A^2}{4})z_2^5.$$

If $B - \frac{A^2}{4} \neq 0$ we have an orbit of type D_6.

If $B - \frac{A^2}{4} = 0$ then:

$$\frac{\gamma_{23}p_4}{t^2} - \frac{\sqrt{p_4}}{t}\,\partial_{222}t = \tfrac{1}{4}\left(-\frac{\gamma_{12}\sqrt{p_4}}{t} + p_3\right)^2$$

$$\gamma_{23}p_4 - \sqrt{p_4}\cdot\partial_{222} = \tfrac{1}{4}(-\gamma_{12}\sqrt{p_4} + p_3t)^2.$$

When we take the limit for $t \to 0$ we get:

$$\gamma_{23}p_4 = \tfrac{1}{4}\gamma_{12}^{\;2}p_4.$$

1^o If $\gamma_{23} = \frac{1}{4}\gamma_{12}^{\;2}$, then the point (0:1:0) is a multiplepoint of the cubic curve:

$$\gamma_{11}z_1^{\;2}z_3 + \gamma_{33}z_3^{\;3} + \gamma_{12}z_1z_2z_3 + \gamma_{23}z_2z_3^{\;2} + \gamma_{13}z_1z_3^{\;2} + z_1^{\;2}z_2.$$

The tangents in this point satisfy:

$$z_1^{\;2} + \gamma_{12}z_1z_3 + \gamma_{23}z_3^{\;2}.$$

So if $\gamma_{12}^{\;2} - 4\gamma_{23} = 0$ the tangents coincide. Hence the point (0:1:0) is no doublepoint and so ϕ_o is not of type P_9 or R_{10}.

2^o If $\lim_{t\to o} p_4 = 0$ we proceed as follows. We can assume $4\gamma_{23} - \gamma_{12}^{\;2} \neq 0$. A coordinatechange in the z_1-z_3-space takes the 3-jet in the form:

$$z_1z_2z_3 + \alpha z_1^{\;3} + \beta z_3^{\;3}.$$

Since p_4 is still the coefficient of $z_3^{\;4}$ the singularity is not of type P_9 or R_{10}.

This shows part a) and b).

The proof of part c) and d) is similar, although longer and more complicated. It will be omitted.

e) Let $\phi_t(z_1,z_2)$ be of type D_8 if $t \neq 0$; after change of coördinates we can assume that 5-jet of ϕ_t has the form:

$$tz_1^{\;2}z_2 + G_4(z_1,z_2) + G_5(z_1,z_2)$$

where: $G_4(z_1,z_2) = p_oz_1^{\;4} + p_1z_1^{\;3}z_2 + p_2z_1^{\;2}z_2^{\;2} + p_3z_1z_2^{\;3} + p_4z_2^{\;4}$

$$G_5(z_1,z_2) = q_oz_1^{\;5} + \ldots + q_4z_1z_2^{\;4} + q_5z_2^{\;5}.$$

If ϕ_o is of type X_{10} then it is sufficient to study the 4-jet.

If $p_4 \neq 0$ then ϕ_t is of type D_6 if $t \neq 0$. Suppose $p_4 = 0$.

After change of coördinates in the following way:

$$\begin{cases} z_1 := z_1 - \frac{1}{2t} p_3 z_2^2 \\ z_2 := z_2 - \frac{1}{t}[p_o z_1^2 + p_1 z_1 z_2 + p_2 z_2^2] \end{cases}$$

the 4-jet of $\phi_t (t \neq 0)$ is given by: $t z_1^2 z_2$.

The coëfficients of z_2^5 is now: $q_5 - \frac{1}{4t} p_3^2$.

The coëfficient of z_2^6 is now: $\frac{3}{2t^2} p_2 p_3^2 - \frac{1}{2t} q_4 p_3$.

If ϕ_t is of type D_8 then these two coëfficients must vanish.

(modifications of the 5-jet give no contributions on terms of degree 6).

So:

1° $p_3^2 = 4tq_5$. Hence $\lim_{t \to o} p_3 = o$.

Since $p_4 = 0$ this implies that $\lim_{t \to o} q_5 \neq 0$, for otherwise ϕ_o is not of type X_{10}.

2° $3p_2 p_3^2 = tq_4 p_3$. Now is $p_3 \neq 0$ since otherwise $q_5 = 0$.

So: $3p_2 = tq_4 p_3$ and $\lim_{t \to o} p_2 \neq 0$ since otherwise ϕ_o is not of type X_{10}.

So $p_3 = \frac{tq_4}{3p_2}$; hence $q_5 = \frac{tq_4^2}{36p_2^4}$.

There follows that $\lim_{t \to o} q_5 = 0$ and this contradicts the fact that ϕ_o is of type X_{10}.

<u>(10.13) Remark</u>

I didn't succeed in computing the adjacency of A_7 to P_9 and A_8 to P_{10}, Q_{10} and R_{10}. For the adjacency of A_8 to X_{10} see (10.15).
All the other (μ)-adjacency relations are given in the list at the end. This list gives also information about the partial ordening of open sets in the part U_8 of the orbitspace W.

The graph of simple and simple elliptic singularities is extended to U_8. If we add dimensionarguments, semi-continuity of corank, et the proof is given in the sections (10.9) to (10.12).

(10.14) Remark

A comparison of list II of the diagrams of intersectionmatrices and list III of the μ-adjacency raises the question if the following remains true:

g is (μ)-adjacent to f if and only if the diagram of g is contained in the diagram of f.

In list III there is no counterexample to this conjecture.

(10.15) Remark (added in proof):

A'CAMPO informed me that he has developed a new geometric way of computing the intersectionmatrix for singularities with corank 0 an 1. With this method he can also show that A_8 is adjacent to X_{10}.

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SAMENVATTING

We bestuderen de Rechts-equivalentie van kiemen van reële en complexe functies.

In deel I geven we de volledige classificatie voor codimensie kleiner dan of gelijk aan negen. Speciale aandacht wordt besteed aan de equivalentie in k-parameter families, het verschil tussen Rechts-equivalentie en Rechts-links-equivalentie en aan de algebraïsche conditie voor k-bepaaldheid.

In deel II beschouwen we in het complex-analytische geval benaderingen van een functiekiem. We bestuderen de relatie tussen de intersectievormen en de monodromiegroepen van een kiem en zijn benaderingen. Als toepassing behandelen we stellingen over de nabijheidsrelatie van simpele en simpele elliptische singulariteiten van Arnold en Saito. We besluiten met een gedeeltelijke beschrijving van de topologie van de ruimte van de Rechts-equivalentieklassen.

SUMMARY

We study the Right-equivalence of germs of real and complex functions.

In part I the complete classification for codimension smaller than or equal to nine is given. Special attention is given to equivalence in k-parameterfamilies, the difference between Right-equivalence and Right-left-equivalence, and to the algebraic condition for a germ to be k-determined.

In part II we consider in the complex-analytic case approximations of a functiongerm. We study the relation between intersectionforms and monodromygroups of a germ and its approximations. As an application we cover theorems on the adjacencyrelation of simple and simple elliptic singularities by Arnold and Saito. We conclude with a partial description of the topology of the orbitspace.

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STELLINGEN

1. Laten f en g elementen van $\mathcal{E}_n$ zijn met de eigenschap, dat de algebra's $\frac{m_n}{\Delta(f)}$ en $\frac{m_n}{\Delta(g)}$ isomorf zijn. De vraag van Takens of dan f en $\pm$ g rechts-equivalent zijn, moet in het algemeen ontkennend beantwoord worden.

   Takens, F.: Singularities of functions and vectorfields.
   Nieuw Archief voor Wiskunde (3), XX, (1972), 107-130.

2. Stel V een algebraïsch oppervlak in $\mathbb{C}^3$ met een geisoleerde singulariteit in de oorsprong. Laat f de intersectiematrix zijn van de locale naburige vezel en zij f' de intersectiematrix van een goede resolutie van V.
   De bewering van Durfee, dat f en f' stabiel equivalent zijn, is onjuist.

   Durfee, A.H.: Diffeomorphism classification of isolated hypersurface singularities.
   Thesis, Cornell University (1971).

3. Zij G een eindige ondergroep van GL(n). Noteer door $\mathcal{E}(G)_n$, resp. $L(G)_n$ de elementen van $\mathcal{E}_n$, resp. L_n, die invariant zijn onder alle elementen van G.
   $f \in \mathcal{E}(G)_n$ heet k-G-bepalend als voor elke $g \in \mathcal{E}(G)_n$ geldt:
   Als $f_k = g_k$ dan is er een ϕ in $L(G)_n$ met $f\phi = g$.
   Er geldt dan:
   1) Als $m_n^{k+1} \cap \mathcal{E}(G)_n \subset m_n(m_n\Delta(f) \cap \mathcal{E}(G)_n)$ dan is f k-G-bepalend.
   2) Als f k-G-bepalend is, dan is $m_n^{k+1} \cap \mathcal{E}(G)_n \subset m_n(\Delta(f) \cap \mathcal{E}(G)_n)$.

4. Het tegenvoorbeeld (5.2) uit dit proefschrift toont tevens voor elke $p > 0$ de onjuistheid aan van de bewering:
f is k-bepalend dan en slechts dan als $m^{k+p} \subset m^{1+p}\Delta(f) + m^{k+p+1}$.

5. De Boardmansymbolen van f en van zijn universele ontvouwing F zijn gelijk.

Mather, J.: On Thom-Boardman singularities.
Proc. of Dyn. Systems Conference in
Salvador, Brazil.

6. De opgave 4a van het herexamen Wiskunde I van het V.W.O. in 1972 (Gymnasium en Atheneum) luidde als volgt:
"Een functie f is voor $-6 \leq x \leq 3$ gedefinieerd door
$f(x) = 2x + 3\sqrt[3]{(x-2)^2}$.
Onderzoek of de functie differentieerbaar is voor $x = 2$."
De commissie bedoeld in art. 27 lid 5 van het Besluit eindexamens V.W.O.-H.A.V.O.-M.A.V.O. maakt een essentiële gedachtenfout als zij in de bindende normen voor de beoordeling van het schriftelijk werk aangeeft, dat er 2 punten moeten worden afgetrokken indien $\lim_{x\downarrow 2} f'(x)$ en $\lim_{x\uparrow 2} f'(x)$ niet apart onderzocht zijn.

7. Het door Hadeler gegeven bewijs van de stelling, dat elke continue functie op [a,b] daar ook integreerbaar is, is onvolledig.

Hadeler, K.P.: "Mathematik für Biologen".
Heidelberger Taschenbücher Band 129 (1974).

8. Het M.O.-A examen Wiskunde dient steeds te worden aangepast aan de ontwikkelingen in de wiskunde. Met name het vak projectieve en beschrijvende meetkunde moet vervangen worden; bijvoorbeeld door topologie, statistiek en/of computerkunde.

9. Een verdere ontsluiting van het gebergte door wegen, kabelbanen en hotels in de hoogalpine regionen dient voorkomen te worden.

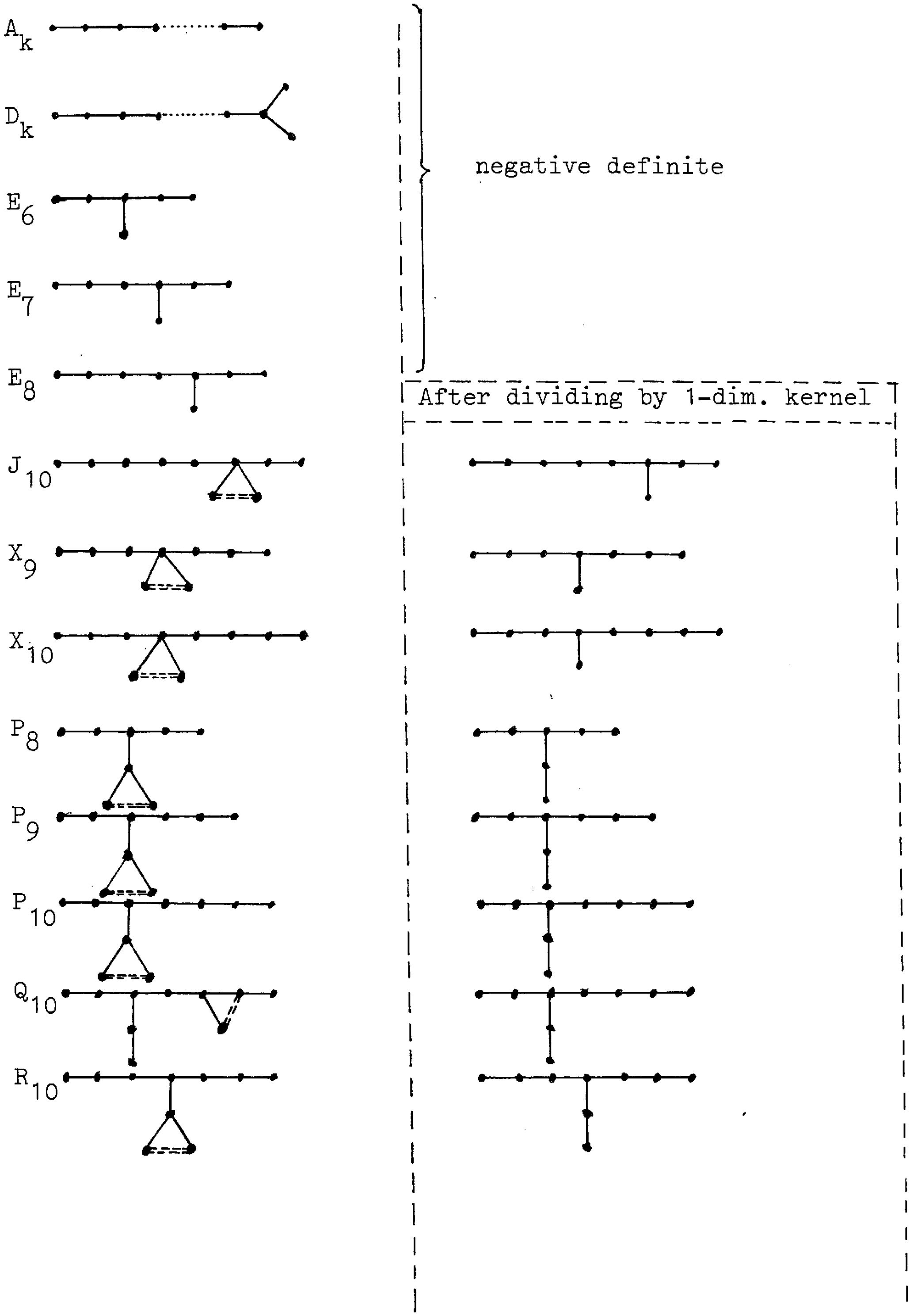
10. Gezien de hoogte van de prijzen van wiskundeboeken in Nederland kan men deze beter uit het buitenland betrekken. Met name de Universiteitsbibliotheek zou van deze mogelijkheid gebruik moeten kunnen maken.

11. Het is merkwaardig, dat in het verplichte wiskunde-programma voor scheikunde-studenten aan de Universiteit van Amsterdam geen lineaire algebra voorkomt.

12. De periode van 3 jaar, waarin een eervol ontslagen hoogleraar als promotor kan optreden, dient verlengd te kunnen worden.

Stellingen behorende bij het proefschrift "Classification and deformation of singularities" van D. Siersma, Amsterdam, juli 1974.

II <u>List of diagrams of intersectionmatrices</u>.



III. Tabel of μ-adjacency
and information about the topology in U_8.