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ISOLATED LINE SINGULARITIES

DIRK SIERSMA

Introduction. In this paper we study germs of functions f: $(\mathbf{C}^{n+1}, 0) \to \mathbf{C}$ with a smooth 1-dimensional critical set Σ. After a change of coordinates we can suppose

$$\Sigma = L = \{(x, y) \in \mathbf{C} \times \mathbf{C}^n \mid y = 0\}.$$

We call those singularities *line singularities*.

Isolated line singularities are defined by the condition that for every $x \neq 0$ the germ of f at $(x, 0) \in \mathbf{C} \times \mathbf{C}^n$ is equivalent to $y_1^2 + \cdots + y_n^2$ and so is a Morse singularity in the transversal direction. In a certain sense isolated line singularities are the first generalizations of isolated (point) singularities. For isolated line singularities we prove that the Milnor fibre of f is homotopy equivalent to a bouquet of spheres.

For the study of nonisolated singularities in general we refer to the work of Lê Dũng Tráng [**Lê-1**, **2**] and Randell [**Ra**]. The special case of singularities with a 1-dimensional critical locus is especially studied by Iomdin [**Io-1–4**], who gave formulas for the Euler characteristic of the Milnor fibre; see also Lê Dũng Tráng [**Lê-3**]. Kato and Matsumoto [**K-M**] proved that in this case the Milnor fibre is $(n-2)$-connected. Moreover it is proved in [**Lê-Sa**] that the Milnor fibre is simply connected when $n = 2$ and f is irreducible.

The paper is organized as follows.

In §1 we treat line singularities from the point of view of Thom-Mather theory. Let $(y) = (y_1, \ldots, y_n)$ and $\mathfrak{m} = (x, y_1, \ldots, y_n)$ be ideals in $\mathcal{E} = \mathcal{E}_{n+1}$ = the ring of germs of holomorphic functions at $0 \in \mathbf{C}^{n+1}$. The action of germs of diffeomorphisms of $\mathbf{C}^{n+1}$ preserving L define an orbit structure in $\mathcal{E}$. For a singular germ $f \in (y)^2$ we define the codimension of f as $\dim_{\mathbf{C}}(y)^2/\tau(f)$ where $\tau(f)$ is the tangent space to the orbit of f in $(y)^2$.

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Just as in the case of isolated singularities we define k-determinacy and obtain algebraic conditions for being finitely determined. There is also a list of singularities of low codimension. The beginning of this list is as follows:

codimension	0	(A_∞)	$y_1^2 + y_2^2 + \cdots + y_n^2$,
codimension	1	(D_∞)	$xy_1^2 + y_2^2 + \cdots + y_n^2$,
codimension	2	(J_∞)	$x^2y_1^2 + y_1^3 + y_2^2 + \cdots + y_n^2$.

In §2 we give some other characterizations of finite codimension. Among others we show that equivalent are:

(a) $\operatorname{cod}(f) < \infty$,

(b) f has an isolated line singularity.

In §3 we study the topology of the Milnor fibre. We mimic a construction of Lê (cf. [**Br**]) and construct a nice approximation of f having only a finite number of A_1-points and a finite number of D_∞-points. With use of hyperplane sections $x = c$ we show

THEOREM. *Let f be an isolated line singularity (not of type A_∞), then the Milnor fibre of f is homotopy equivalent to a bouquet of μ spheres S^n,*

$$\mu = \sigma + 2\tau - 1$$

where σ is the number of A_1-points and τ is the number of D_∞-points in a generic approximation of f.

§4 contains remarks and questions.

I thank Lê for his remark, which simplified the proof of (2.5).

1. Line singularities.

(1.1) We consider the ring $\mathcal{E}_{n+1}$ of germs at 0 of holomorphic functions $f\colon \mathbf{C}^{n+1} \to \mathbf{C}$. We write $(x, y) = (x, y_1, \ldots, y_n)$ for coordinates in $\mathbf{C}^{n+1}$ and $L = \{(x, y) \mid y = 0\}$. We set $\mathcal{E} = \mathcal{E}_{n+1}$ and define ideals:

$$\mathfrak{m} = \mathcal{E}(x, y_1, \ldots, y_n) = \{f \in \mathcal{E} \mid f(0) = 0\},$$
$$(y) = \mathcal{E}(y_1, \ldots, y_n) = \{f \in \mathcal{E} \mid f(x, 0) = 0 \text{ for all } x\}.$$

The objects of our study are elements of $(y)^2$.

Let $\mathcal{D} = \mathcal{D}_{n+1}$ be the group of germs at 0 of local diffeomorphisms of the source space and let $\mathcal{D}_L$ be the subgroup of $\mathcal{D}$, consisting of $\phi \in \mathcal{D}$ with $\phi(L) = L$. There is a right action of $\mathcal{D}_L$ on $\mathcal{E}$. The orbit of f in $\mathcal{E}$ under $\mathcal{D}_L$ is denoted by $\operatorname{Orb}(f)$.

(1.2) We next define the *tangentspace* $\tau(f)$ to $\operatorname{Orb}(f)$ at f. Let ϕ_t be a curve in $\mathcal{D}_L$ with $\phi_0 = \mathrm{Id}$.

The chain rule gives:

$$\left.\frac{df\phi_t(p)}{dt}\right|_{t=0} = \left.\frac{\partial f}{\partial x}\frac{d\phi_t^0}{dt}(p)\right|_{t=0} + \sum_{j=1}^{n} \left.\frac{\partial f}{\partial y_j}\frac{d\phi_t^j}{dt}(p)\right|_{t=0}$$
$$= \xi(x, y)\frac{\partial f}{\partial x}(x, y) + \sum_{j=1}^{n} \eta_j(x, y)\frac{\partial f}{\partial y_j}(x, y)$$

with

$$\xi(0,0) = \frac{d\phi_t^0}{dt}(0,0)\Big|_{t=0} = 0 \quad \text{and} \quad \eta_j(x,0) = \frac{d\phi_t^j}{dt}(x,0)\Big|_{t=0} = 0.$$

So $\xi \in \mathfrak{m}$ and $\eta_j \in (y)$. For this reason we define

$$\tau(f) = \mathfrak{m}\frac{\partial f}{\partial x} + (y)\left(\frac{\partial f}{\partial y}\right) \quad \text{where } \left(\frac{\partial f}{\partial y}\right) = \left(\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n}\right).$$

(1.3) DEFINITION. For $f \in (y)^2$ we define the *codimension*

$$c(f) = \operatorname{codim}(f) = \dim \frac{(y)^2}{\tau(f)}.$$

(1.4) DEFINITION. $f \in (y)^2$ is called *k-determined* if

$$f + \mathfrak{m}^{k-1}(y)^2 \subset \operatorname{Orb}(f)$$

(so every $g \in (y)^2$ with the same k-jet as f is right-equivalent with f).

(1.5) PROPOSITION. *Let $f \in (y)^2$.*
(a) *If f is k-determined then $(y)^2\mathfrak{m}^{k-1} \subset \tau(f) + (y)^2\mathfrak{m}^k$.*
(b) *If $(y)^2\mathfrak{m}^{k-1} \subset \mathfrak{m}\tau(f) + (y)^2\mathfrak{m}^k$ then f is k-determined.*

PROOF. (a) We work modulo $(y)^2\mathfrak{m}^k$ in a finite dimensional subspace $j^{k+1}((y)^2)$ of $J^{k+1}(n,1)$. Since $j^{k+1}(f + \mathfrak{m}^{k-1}(y)^2)$ is an affine subspace of $J^{k+1}(n,1)$ its tangent space at $j^{k+1}f$ is $j^{k+1}(\mathfrak{m}^{k-1}(y)^2)$. The tangent space to $j^{k+1}(\operatorname{Orb}(f))$ at $j^{k+1}f$ is $j^{k+1}(\tau(f))$. Since f is k-determined, we have

$$(y)^2\mathfrak{m}^{k-1} \subset \tau(f) + (y)^2\mathfrak{m}^k.$$

(b) Let $f \in (y)^2$ and suppose for $g \in (y)^2$ we have $j^kf = j^kg$, so $g - f \in (y)^2\mathfrak{m}^{k-1}$

$$F(x,y,t) = f(x,y) + t(g(x,y) - f(x,y)).$$

We consider F as an element of $\mathcal{E}_{n+2}$, the ring of germs at $(0, t_0)$. We denote its maximal ideal by $\mathfrak{m}_{n+2}$.

We have inclusions: $\mathcal{E} = \mathcal{E}_{n+1} \subset \mathcal{E}_{n+2}$ and $\mathfrak{m} = \mathfrak{m}_{n+1} \subset \mathfrak{m}_{n+2}$. In the rest of the proof the notations $\mathfrak{m}$, (y), $(y)^2$, etc. are actually used for $\mathcal{E}_{n+3}\mathfrak{m}$, $\mathcal{E}_{n+2}(y)$, $\mathcal{E}_{n+2}(y)^2$, etc. Let

$$\tau^*(F) = \left\{\xi\frac{\partial F}{\partial x} + \sum \eta_j \frac{\partial F}{\partial y_j} \,\middle|\, \xi \in \mathfrak{m} \text{ and } \eta_j \in (y)\right\} \subset \mathcal{E}_{n+2}.$$

Remark that $\tau(f) \subset \tau^*(F) + (y)^2\mathfrak{m}^{k-1}$. So $(y)^2\mathfrak{m}^{k-1} \subset \mathfrak{m}\tau(f) + (y)^2\mathfrak{m}^k \subset \mathfrak{m}\tau^*(F) + (y)^2\mathfrak{m}^k \subset \mathfrak{m}\tau^*(F) + \mathfrak{m}_{n+2}(y)^2\mathfrak{m}^{k-1}$. By Nakayama's lemma:

$$(y)^2\mathfrak{m}^{k-1} \subset \mathfrak{m}\tau^*(F) \subset \tau^*(F).$$

So there exist time dependent vector fields $(\xi, \eta_1, \dots, \eta_n)$ defined in a neighborhood U of $(0, 0, t_0)$ such that

$$\frac{\partial F}{\partial x}(x, y, t)\xi(x, y, t) + \sum_{j=1}^{n} \frac{\partial F}{\partial y_j}(x, y, t)\eta_j(x, y, t) + g(x, y) - f(x, y) = 0$$

for all $(x, y, t) \in U$.

Moreover $\xi \in \mathfrak{m}$ and $\eta_j \in (y)$. So $\xi(0, 0, t) = 0$ for all $(0, 0, t) \in U$ and $\vec{\eta}(x, 0, t) = 0$ for all $(x, 0, t) \in U$. The differential equation

$$\begin{cases} \dfrac{\partial h^x}{\partial t}(x, y, t) = \xi(h(x, y, t), t), \\ \dfrac{\partial h^y}{\partial t}(x, y, t) = \vec{\eta}(h(x, y, t), t), \\ h(x, y, t_0) = (x, y), \end{cases}$$

has a unique solution, generating a family of local diffeomorphisms h_t (for all t near t_0) satisfying

$$\begin{cases} F_{t_0} = F_t h_t, \\ h_t \in \mathfrak{D}_L. \end{cases}$$

By "continuous induction" over the interval $[0, 1]$ we find that $g = F_1$ and $f = F_0$ are right equivalent.

(1.6) Corollary. *Let* $f \in (y)^2$. *Then* $\operatorname{codim}(f) < \infty \Leftrightarrow f$ *is* k*-determined for some* $k \in \mathbf{N}$.

Proof. f is k-determined for some $k \in \mathbf{N} \Leftrightarrow$

$$\exists k\ (y)^2 \mathfrak{m}^{k-1} \subseteq \tau(f) \Leftrightarrow \operatorname{cod}(f) < \infty.$$

Classification of line singularities

The same computational methods as in the case of ordinary singularities can be applied. We find the following beginning of the list:

Type	Residual singularity	codim(f)	determined jet
A_∞	0	0	2
D_∞	xy^2	1	3
$J_{k,\infty}$ $(k \geqslant 2)$	$y^3 + x^k y^2$	$k \geqslant 2$	$k + 2$
$T_{\infty,k,2}$ $(k \geqslant 4)$	$x^2y^2 + y^k$	$k - 1 \geqslant 3$	k
$Z_{k,\infty}$ $(k \geqslant 1)$	$xy^3 + x^{k+2}y^2$	$k + 3 \geqslant 4$	$k + 4$
$W_{1,\infty}$	$y^4 + y^2x^3$	5	5
$T_{\infty,q,r}$ $(q \geqslant 3, r \geqslant 3)$	$xyz + y^q + z^r$	$q + r - 3 \geqslant 3$	$\max(q, r)$
$Q_{k,\infty}$ $(k \geqslant 2)$	$xz^2 + y^3 + x^k y^2$	$k + 2 \geqslant 4$	$k + 2$
$S_{1,\infty}$	$y^2z + xz^2 + x^2y^2$	5	4

The list contains all simple singularities and all line-singularities of codimension $\leqslant 6$. All nonsimple line singularities are adjacent to one of the following three families of one modular singularities.

$y^4 + Ay^3x^2 + y^2x^4$	7	6	$(A^2 \neq 4)$
$x^3y^2 + Axy^4 + \varepsilon y^5$	7	5	$(\varepsilon^2 = \varepsilon)$
$y^2z + xz^2 + Axy^3 + x^3y^2$	7	5	

2. Isolated line singularities.

(2.1) We shall give some other characterizations for finite codimension. Let $f \in (y)^2$. Write $f(x, y) = \sum_{i,j=1}^{n} g_{ij}(x, y) y_i y_j$ with $g_{ij} = g_{ji}$ e.g. take

$$g_{ij}(x, y) = \int_0^1 \int_0^1 \frac{\partial^2 f(x, sty)}{\partial y_i \partial y_j} \, ds \, dt.$$

We define the *Hessian of* f (relative x) by

$$h_f(x, y) = \det\big(g_{ij}(x, y)\big).$$

The 2-jet of f in $(c, 0)$ is equal to $\sum g_{ij}(c, 0) y_i y_j$. So we see $h_f(c, 0) \neq 0 \Leftrightarrow f$ has type A_∞ at $(c, 0)$.

(2.2) DEFINITION. $f \in (y)^2$ has a *line singularity* if its singular locus is

$$\Sigma(f) = L = \{(x, y) \in \mathbf{C} \times \mathbf{C}^n \mid y = 0\}.$$

The *line singularity* is called *isolated* if for $c \neq 0$ the germ of f at $(c, 0)$ has only A_∞-singularities.

(2.3) EXAMPLES. (a) A_∞: $f(x, y) = y^2$,

(b) D_∞: $f(x, y) = xy^2$,

(c) $f(x, y) = x^2y^2$ is not a line singularity,

(d) $f(x, y) = y^3$ is not an isolated line singularity.

(2.4) We consider the following ideals in $\mathcal{E}$:

$$\tau(f) = \mathfrak{m}\frac{\partial f}{\partial x} + (y)\left(\frac{\partial f}{\partial y}\right) \subset (y)^2 \quad \text{(tangent space)},$$

$$J(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n}\right) \subset (y), \quad \text{(Jacobian ideal)},$$

$$h(f) = \left(h_f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n}\right) \quad \text{(Hessian ideal)}.$$

(2.5) THEOREM. *Let* $f \in (y)^2$. *Equivalent are*:

(A) $c(f) := \dim_{\mathbf{C}} (y)^2/\tau(f) < \infty$,

(B) $j(f) := \dim_{\mathbf{C}} (y)/J(f) < \infty$,

(C) $\lambda(f) := \dim_{\mathbf{C}} \mathcal{E}/h(f) < \infty$ *and* $\Sigma(f) = L$.

(D) *f has an isolated line singularity at* $\underline{0}$.

PROOF. We take a representative f of the germ. Define sheafs of $\mathcal{E}_{n+1}$-modules as follows:

$$\mathcal{F}^1(U) = \frac{(y)}{J(f)} \quad \text{and} \quad \mathcal{F}^2(U) = \frac{(y)^2}{\tau(f)}$$

where (y), $(y)^2$, $J(f)$ and $\tau(f)$ are considered as modules over the holomorphic functions on U. It is clear that $\mathcal{F}^1$ and $\mathcal{F}^2$ are coherent. We intend to use the fact that $\mathcal{F}^i$ is concentrated in a point $\Leftrightarrow \dim \Gamma(\mathcal{F}^i) < \infty$.

(i) (D) $\Rightarrow$ (A) and (B). For $y \neq 0$ f is regular at (x, y) and we have $\dim \mathcal{F}^1_{(x,y)} = \dim \mathcal{F}^2_{(x,y)} = 0$ since $(y) \cong \mathcal{E}_{n+1}$, $J(f) \cong \mathcal{E}_{n+1}$, $(y)^2 \cong \mathcal{E}_{n+1}$ and $J(f) \cong \mathcal{E}_{n+1}$ at

(x, y). If $y = 0$ and $x \neq 0$ then f is of type A_∞ at $(x, 0)$ and we have $\dim \mathfrak{F}^1_{(x,0)} = \dim \mathfrak{F}^2_{(x,0)} = 0$, since $(y) \cong J(f)$ and $(y)^2 \cong \tau(f)$ at $(x, 0)$. So both $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are concentrated at 0 and this implies $c(f) < \infty$ and $j(f) < \infty$.

(ii) (B) $\Rightarrow$ (D). Since $j(f) < \infty$ we have $\dim \mathfrak{F}^1_{(x,y)} = 0$ for $(x, y) \neq (0, 0)$. Since $\dim \mathcal{O}/J(f) = 0$ implies f is regular, and $\dim(y)/J(f) = 0$ implies f is of type A_∞, we have (D).

(iii) (A) $\Rightarrow$ (D) is similar.

(iv) (C) $\Leftrightarrow$ (D) is trivial.

3. The topology of the Milnor fibre.

(3.1) We recall that we consider an *isolated line singularity*, that is an analytic germ f with singular locus the line $L = \{(x, y) \in \mathbf{C} \times \mathbf{C}^n \mid y = 0\}$ such that for every $x \neq 0$ the germ of f at $(x, 0)$ is of type A_∞, i.e. equivalent to $y_1^2 + \cdots + y_n^2$. Let B_ε be the closed ε-ball in $\mathbf{C}^{n+1}$ and D_η be the closed 1-disc in $\mathbf{C}$. We select $\varepsilon > 0$ and $\eta > 0$ such that the restriction

$$f\colon B_\varepsilon \cap f^{-1}(D_\eta) \to D_\eta$$

satisfies the conditions for the Milnor construction, and so f is a C^∞-locally trivial fibre bundle above $D_\eta - 0$.

(3.2) In the case of an ordinary isolated singularity it is useful to consider a generic approximation g of f with only ordinary Morse points (cf. Brieskorn [**Br**]). At every Morse point one can study its local Milnor fibration, with Milnor fibre homotopy equivalent to one n-sphere S^n ("the vanishing cycle"). The Milnor fibre of the original f then has the homotopy type of the wedge of those μ spheres.

We like to mimic the construction in the case of an isolated line singularity. First we prove the existence of a nice approximation.

(3.3) LEMMA. *Let f have an isolated line singularity. There exist a deformation g of f such that g has*:

(i) *only D_∞ and A_∞ singularities on L*,

(ii) *only A_1 (Morse) singularities outside L.*

[*Recall D_∞ singularity is locally given by $xy_1^2 + y_2^2 + \cdots + y_n^2$.*]

PROOF. Define $F\colon \mathbf{C} \times \mathbf{C}^n \times S \times T \to \mathbf{C}$ by

$$F(x, y, (a_{ij}), (b_{ij})) = f(x, y) + \sum_{i,j} (a_{ij} + b_{ij}x) y_i y_j.$$

A computation shows that the 2-jet extension

$$j^2F\colon \mathbf{C} \times \mathbf{C}^n \times S \times T \to J^2(n+1, 1)$$

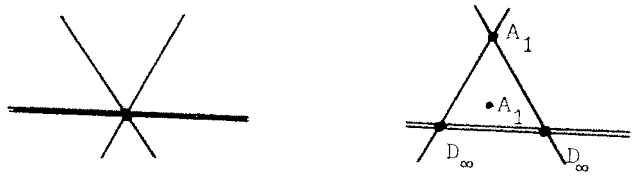
is transversal to the A_1-stratum outside L and transversal to the D_∞ stratum on L. The assertion follows now as an application of Sard's theorem.

EXAMPLE.

$$f(x, y) = y^2(x^2 - y^2),$$

$$g(x, y) = y^2(x^2 - (y - t)^2).$$

$f^{-1}(0)$ $\qquad\qquad$ $g^{-1}(0)$



(3.4) PROPOSITION. *There exists an approximation g of f (as in Lemma (3.3)) with the additional property: The Milnor fibrations of g and f above the boundary ∂D_η are equivalent.*

PROOF. Let $\varepsilon > 0$ such that $(f^{-1}(0)\setminus L) \pitchfork S_{\varepsilon'}$ for all $0 < \varepsilon' \leq \varepsilon$. Let $f_\lambda(x, y)$ be a 1-parameter deformation of f, which satisfies (3.3) for $\lambda \neq 0$. We claim now, that there exist $\delta > 0$ and $\eta > 0$ such that $f_\lambda^{-1}(t) \pitchfork S_\varepsilon$ for all $0 < \lambda \leq \delta$ and $0 < |t| \leq \eta$. This follows from:

(1) $f^{-1}(0) \cap S_\varepsilon$ is compact,

(2) $f | S_\varepsilon$ is a submersion in points of $f^{-1}(0)\setminus L$,

(3) on $L \cap S_\varepsilon$ we have only A_∞-points, so near points of $L \cap S_\varepsilon$ we can change coordinates $(y_1, \ldots, y_n)$, smoothly depending on (x, λ) such that

$$f_\lambda(x, y) \sim y_1^2 + \cdots + y_n^2$$

(parameter version of the Morse lemma).

For $t \neq 0$ the tangent space in (x, y) to $f_\lambda^{-1}(t)$ contains L, since it is given by $\xi_1 y_1 + \cdots + \xi_n y_n = 0$. So we find locally

$$f_\lambda^{-1}(t) \pitchfork S_\varepsilon \quad \text{for } t \neq 0.$$

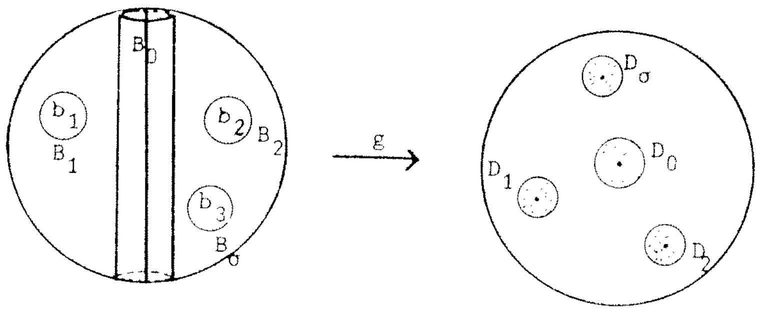
Let $F(x, y, \lambda) = f_\lambda(x, y), \lambda)$. The proposition follows now from the fact that

$$F\colon F^{-1}(\partial D_\eta \times [0, \delta]) \cap (B_\varepsilon \times [0, \delta]) \to \partial D_\eta \times [0, \delta] \to [0, \delta]$$

and the restriction to $S_\varepsilon \times [0, \delta]$ are submersions.

(3.5) REMARK. The equivalence of f and an approximation g is generally nonvalid for nonisolated singularities. Here we have the equivalence because of special properties of isolated line singularities and of the approximation g.

(3.6) We now take a generic approximation g of f as in the above lemma and suppose moreover that the approximation is so close that the Milnor fibrations of g and f above the boundary ∂D_η are the same. We can also suppose that all critical values of g are different (this is mostly for notational convenience). The critical value 0 corresponds to the nonisolated singularities on the line L.



Let $b_1, \ldots, b_\sigma$ be the Morse points of g with critical values $g(b_1), \ldots, g(b_\sigma)$. Define $B_1, \ldots, B_\sigma$ disjoint $(2n+2)$ balls around $b_1, \ldots, b_\sigma$ and inside $B = B_\varepsilon$. Let $D_1, \ldots, D_\sigma$ be disjoint 2-discs around $g(b_1), \ldots, g(b_\sigma)$ and inside $D = D_\eta$, chosen in such a way that we get local fibrations

$$g\colon B_i \cap g^{-1}(D_i) \to D_i \qquad (i = 1, \ldots, \sigma)$$

satisfying the usual transversality condition

$$\partial B_i \pitchfork g^{-1}(t) \qquad \text{if } t \in D_i.$$

We also define a small cylinder B_0 around L and a 2-disc D_0 around 0 such that

$$\partial B_0 \pitchfork g^{-1}(t) \qquad \text{if } t \in D_0.$$

Of course we can take all $B_0, \ldots, B_\sigma$ and $D_0, \ldots, D_\sigma$ to be disjoint. We first study the fibres of

$$g\colon B_0 \cap g^{-1}(D_0) \to D_0.$$

We take hyperplane sections $x = c$. A fibre $g^{-1}(t) \cap B_0$ is now fibered by the projection π on L. This projection can have singularities. It is convenient to consider the map

$$\Phi_g\colon g^{-1}(D_0) \cap B_0 \to \mathbf{C} \times \mathbf{C}$$

defined by $\Phi_g(x, y) = (g(x, y), x)$. The singular locus of Φ_g consists of the line L and the so called *polar curve* Γ. The projection

$$\pi\colon g^{-1}(t) \cap B_0 \to L$$

is smooth outside points of Γ_g $(t \neq 0)$.

(3.7) LEMMA. *The polar curve Γ_g can cut L only in the D_∞ points of g.*

PROOF. We have to show, that if g is of type A_∞, then L is locally the discriminant locus of Φ_g.

So let $g(x, y) = \sum g_{ij}(x, y) y_i y_j$ with $\det(g_{ij}(0,0)) \neq 0$

$$\begin{cases} \dfrac{\partial g}{\partial y_1} = \sum \dfrac{\partial g_{ij}}{\partial y_1} y_i y_j + \sum g_{1j} y_j \\ \vdots \\ \dfrac{\partial g}{\partial y_n} = \sum \dfrac{\partial g_{ij}}{\partial y_n} y_i y_j + \sum g_{nj} y_j. \end{cases}$$

$\det(g_{ij}(0,0)) \neq 0$ now implies that we can modulo $(y)^2$ solve for $y_1, \ldots, y_n$. So $(y) \subset (\partial g/\partial y) + (y)^2$.

Nakayama's lemma now gives $(y) = (\partial g/\partial y)$. So the variety defined by $\partial g/\partial y_1 = \cdots = \partial g/\partial y_n = 0$ is just L and it is clear that this is the critical set of Φ_g.

Next we study the D_∞-points.

(3.8) PROPOSITION. *Let g be of type D_∞ and let $\Phi_g(x, y) = (g(x, y), x)$.*

(a) *The diffeomorphism type of the pair of Milnor fibres of Φ_g and g is independent of the choice of g within the type D_∞.*

(b) *The pair of Milnor fibres of g and Φ_g is homotopy equivalent to the pair of standard spheres (S^n, S^{n-1}).*

PROOF. (a) Let g_0 and g_1 be of type D_∞. Select representatives such that the transversality conditions for the Milnor fibrations of g_0, g_1, Φ_{g_0} and Φ_{g_1} are satisfied for certain $(s_0, t_0) \in L \times \mathbf{C}$. Consider the complex family

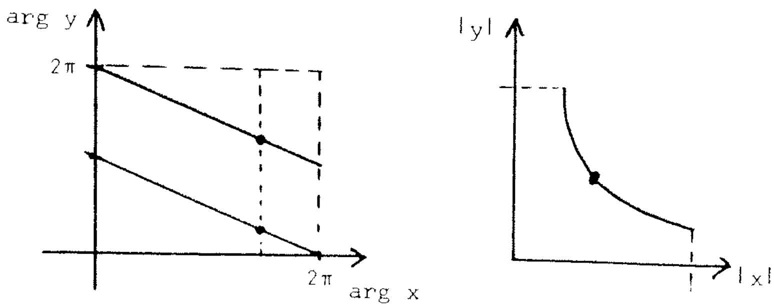
$$\overline{\Psi}(x, y, \tau) = (x, \tau g_1(x, y) + (1-\tau)g_0(x, y)) = (x, \Psi'(x, y, \tau)).$$

The variety $\Psi^{-1}(s_0, t_0)$ intersects

$$\left\{(x, y, \tau) | 0 = \frac{\partial \Psi'}{\partial y_1} = \cdots = \frac{\partial \Psi'}{\partial y_n}\right\}$$

only in a finite number of points. Choose a path $\lambda(t)$ from 0 to 1 in the τ-plane missing τ-coordinates of those points. The real homotopy $g_{\lambda(t)}$ between g_0 and g_1 induces a diffeomorphism between the pairs of Milnor fibres.

(b) It is sufficient to study $g = xy_1^2 + y_2^2 + \cdots + y_n^2$. Take first $n = 1$, $g(x, y) = xy^2 = \delta$, $|x| \cdot |y|^2 = \delta$, $\arg x + 2 \arg y = 0 \bmod 2\pi$.



In (torus) $\times$ $\mathbf{R}^2$ it is clear that the Milnor fibre of g is homotopy equivalent to S^1, the hyperplane section $x = s_0$ is homotopy equivalent to S^0. In the general case we have to take double suspensions of (S^1, S^0).

(3.9) LEMMA. *Let f be not of type A_∞. Let $s_0 \in \partial(B_0 \cap L)$. The fibre*

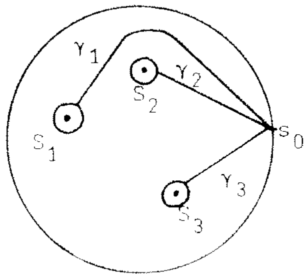
$$X_t' = g^{-1}(t) \cap B_0$$

is homotopy equivalent to 2τ n-balls glued together along their common boundary $S^{n-1} \overset{h}{\simeq} Y_t' = \Phi_g^{-1}(t, s_0) \cap B_0$. So X_t is homotopy equivalent to a bouquet of $(2\tau - 1)$ n-spheres, where τ is the number of D_∞-points in a generic approximation g.

PROOF. Set $S = B_0 \cap L$. Let $S_1, \ldots, S_\tau$ be small disjoint discs inside S around the D_∞ points $c_1, \ldots, c_\tau$. Choose B_0 so small that above $S \setminus \bigcup_{i=1}^{\tau} S_i$ the projection

$$\pi: g^{-1}(t) \cap B_0 \to L \qquad (t \neq 0)$$

is locally trivial.



Choose a system of paths $\gamma_1, \ldots, \gamma_\tau$ from s_0 to $S_1, \ldots, S_\tau$ (in the usual way; see the diagram). Set $Z = S_1 \cup \cdots \cup S_\tau \cup \gamma_1 \cup \cdots \cup \gamma_\tau$, $W = \gamma_1 \cup \cdots \cup \gamma_\tau$. Z is a deformation retract of S; $\{s_0\}$ is a deformation retract of W. Since π is locally trivial over the complement of $S_1 \cup \cdots \cup S_\tau$ it follows from the homotopy lifting property that $\pi^{-1}(Z)$ is a deformation retract of X_t' and Y_t' is a deformation retract of $\pi^{-1}(W)$. Moreover Y_t' is the Milnor fibre of the hyperplane section of an A_∞-singularity and so homotopy equivalent to an S^{n-1} (since the hyperplane section has an A_1-singularity). It follows that $(\pi^{-1}(Z), \pi^{-1}(W))$ is relatively homotopy equivalent to

$$\left(\pi^{-1}(W) \cup e_1^+ \cup e_1^- \cup \cdots \cup e_\tau^+ \cup e_\tau^-, \pi^{-1}(W)\right),$$

where for each D_∞ point two n-cells e^+ and e^- are attached to the vanishing cycle S^{n-1} in the standard way. So

$$(X_t', Y_t') \overset{h}{\simeq} \left(S^{n-1} \cup_{\gamma_1} e_1^+ \cup e_1^- \cup \cdots \cup_{\gamma_\tau} e_\gamma^+ \cup_{\gamma_\tau} e_\tau^-, S^{n-1}\right).$$

So

$$X_t' \overset{h}{\simeq} S^n \vee \cdots \vee S^n \quad (2\tau - 1 \text{ copies}).$$

(3.10) THEOREM. *Let f be an isolated line singularity (not of type A_∞); then the Milnor fibre of f is homotopy equivalent to a bouquet of μ spheres S^n, $\mu = \sigma + 2\tau - 1$, where σ is the number A_1-points and τ is the number D_∞-points in a generic approximation of f.*

PROOF. Take D, D_0, $D_1, \ldots, D_\sigma$ and B, B_0, $B_1, \ldots, B_\sigma$ as before. Let $t \in \partial D_0$. Choose a system of paths $\psi_1, \ldots, \psi_\sigma$ from t to $D_1, \ldots, D_\sigma$. For $T \subset D$ set $X_T = g^{-1}(T) \cap B$. As in the preceding lemma there is a homotopy equivalence

$$(X_D, X_t) \overset{h}{\simeq} \left(X_{D_0} \cup_{\psi_1} e_1^{n+1} \cup \cdots \cup_{\psi_\sigma} e_\sigma^{n+1}, X_t\right).$$

Moreover,

$$(X_{D_0}, X_t) \overset{h}{\simeq} (X_{D_0} \cap B_0 \cup X_t, X_t).$$

Let $\phi_1, \ldots, \phi_{2\tau-1}$: $S^n \to X_t' = X_t \cap B_0$ represent the $2\tau - 1$ generators of $\pi_n(X_t \cap B_0)$. Use $\phi_1, \ldots, \phi_{2\tau-1}$ to attach $(n+1)$-cells $f_1^{n+1}, \ldots, f_{2\tau-1}^{n+1}$ to $X_t \cap B_0$.

The inclusion mapping

$$X_t \cap B_0 \hookrightarrow X_{D_0} \cap B_0$$

extends to a homotopy equivalence

$$X_t \cup_{\phi_1} f_1^{n+1} \cup \cdots \cup_{\phi_{2\tau-1}} f_{2\tau-1}^{n+1} \to X_{D_0} \cap B_0,$$

since both spaces are contractible. So we get a homotopy equivalence

$$\left(X_{D_0}, X_t\right) \overset{h}{\simeq} \left(X_t \cup_{\phi_1} f_1^{n+1} \cup \cdots \cup_{\phi_{2\tau-1}} f_{2\tau-1}^{n+1}, X_t\right).$$

X_D is obtained from X_t by attaching $\sigma + 2\tau - 1$ $(n+1)$-cells. So X_t is $(n-1)$-connected, since X_D is contractible. Since X_t has the homotopy type of an n-dimensional finite CW-complex it follows that X_t has the homotopy type of a bouquet of $\mu = \sigma + 2\tau - 1$ n-spheres.

4. Remarks and questions.

(4.1) In the case of isolated (point) singularities, there is the algebraic description of the Milnor number

$$\mu = \dim \mathcal{E}_{n+1} / \left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

For isolated line singularities we have

$$\tau = {}^{\#}(D_\infty \text{ points}) = \dim \mathcal{E}_1 / \left(h_f(x, 0)\right).$$

A question to prove is

$$c(f) = {}^{\#}(A_1 \text{ points}) + 1 = \sigma + 1,$$

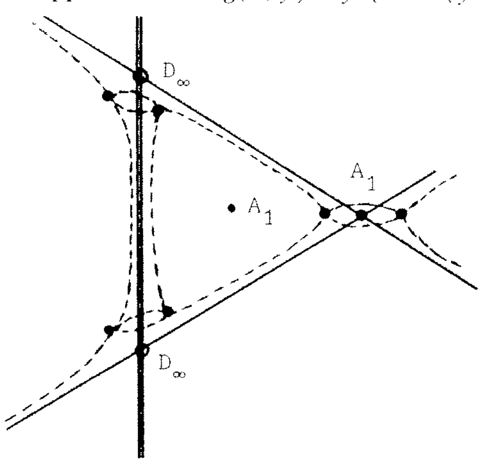
$$j(f) = {}^{\#}(A_1 \text{ points}) + {}^{\#}(D_\infty \text{ points}) = \sigma + \tau,$$

which is true in all known examples.

(4.2) Find the intersection forms for isolated line singularities. For $n = 2$ one can use the method of A'Campo and Guzein-Zade. Here follows an example:

$$f(x, y) = x^2y^2 + y^4.$$

Nice approximation: $g(x, y) = y^2(x^2 - (y - t)^2)$. Level curves:



$^{\#}D_\infty = 2$

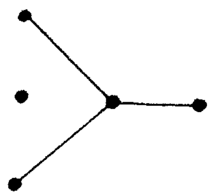
$^{\#}A_1 = 2$

$\mu = 5$

$c(f) = 3 = {}^{\#}A_1 + 1$

$j(f) = 4 = {}^{\#}A_1 + {}^{\#}D_\infty$

In the diagram you can easily find the 5 cycles.
There is some freedom in choice.
A diagram for the intersection matrix is:



(c) Find the relation between isolated line singularities and certain series of isolated singularities, especially relate the topology of their Milnor fibres. From Iomdin [**Io-4**] it already follows that for k sufficiently large $\chi(F) = \chi(F'_k) - k$ where χ is the Euler characteristic, F is the Milnor fibre of $f(x, y)$ and F'_k is the Milnor fibre of $f(x, y) + x^k$.

(d) Study line singularities which have other transversal singularities than A_1 (outside 0).

(e) Study in general singularities with 1-dimensional critical locus, which have transversally A_1-singularities (outside 0).

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Rijksuniversiteit Utrecht, The Netherlands