SINGULARITIES OF FUNCTIONS ON BOUNDARIES, CORNERS, ETC.

By DIRK SIERSMA

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0. Introduction. Functions on boundaries, corners etc., can be lifted to symmetric functions on n-space. One can apply Beer's and Wassermann's classification of singularities of symmetric functions in order to list all singularities of codimension ≤ 4. The paper follows the suggestion of Arnol'd in [1]. An alternative approach is the study of composed mapgerms $0 \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$. The case of an ordinary boundary (without corners) is already treated by Pitt and Poston [5], see also Poston and Stewart [6], Arnol'd [2].

1. Definitions. Let $\mathbb{H}^q = \{u \in \mathbb{R}^q \mid u_j \geq 0 \ j = 1, \ldots, q\}$, a q-dimensional corner. Let $\mathscr{E}(p, q)$ be the set of mapgerms $\mathbb{R}^p \times \mathbb{H}^q \to \mathbb{R}$ at $(0, 0)$. The group $G = (\mathbb{Z}_2)^q$ acts linearly on $\mathbb{R}^p \times \mathbb{R}^q$ as follows: Let $(\varepsilon_1, \ldots, \varepsilon_q) \in (\mathbb{Z}_2)^q$ then for $x \in \mathbb{R}^p$ and $u = (u_1, \ldots, u_q) \in \mathbb{R}^q$:

$$(\varepsilon_1, \ldots, \varepsilon_q)(x, u_1, \ldots, u_q) = (x, \varepsilon_1 u_1, \ldots, \varepsilon_q u_q)$$

We shall write:

$$\sigma_j(u_1, \ldots, u_q) = (u_1, \ldots, u_{j-1}, -u_j, u_{j+1}, \ldots, u_q)$$

Let ρ: $\mathbb{R}^{p+q} \to \mathbb{R}^p \times \mathbb{H}^q$ be the so called *orbitmapping*, defined by:

$$\rho(x, u_1, \ldots, u_q) = (x, u_1^2, \ldots, u_q^2)$$

also notated as:

$$\rho(x, u) = (x, u^2)$$

Let $\mathscr{E}(p+q) = \mathscr{E}(p+q, 0)$. A germ $\psi \in \mathscr{E}(p+q)$ is called *symmetric* if $\psi\sigma = \psi$ for every $\sigma \in G$. Let $\mathscr{E}_G(p+q)$ be the set of symmetric germs in $\mathscr{E}(p+q)$.

From Schwarz's finitude lemma [7] we have:

2. Lemma. ρ^*: $\mathscr{E}(p, q) \to \mathscr{E}_G(p+q)$ *is bijective where*

$$\rho^* f(x, u_1, \ldots, u_q) = f(x, u_1^2, \ldots, u_q^2)$$

3. Definitions. Let $\mathscr{D}(p, q)$ be the set of germs at $(0, 0)$ of diffeomorphisms $(\mathbb{R}^p \times \mathbb{H}^q, (0, 0)) \to (\mathbb{R}^p \times \mathbb{H}^q, (0, 0))$. Let $S(q) \subset \mathscr{D}(p, q)$ be the subgroup consisting of the permutations of the coordinates in $\mathbb{H}^q$.

Let $\mathscr{D}^*(p, q)$ be the normal subgroup of $\mathscr{D}(p, q)$, consisting of thos elements which map the hyperplanes $\{(x, u) \mid u_k = 0\}$ onto themselves.

4. Lemma. *$\mathscr{D}(p, q) = S(q) \circ \mathscr{D}^*(p, q)$ (no direct product) moreover th decomposition of $\mathscr{D}(p, q)$ is unique.*

Proof. Elements of $\mathscr{D}(p, q)$ map corners into corners of the sam dimension; moreover $\mathscr{D}^*(p, q) \cap S(q)$ consists of the identity only.

5. Lemma. *Let the smooth germ ϕ: $\mathbb{R}^p \times \mathbb{H}^q \to \mathbb{R}^p \times \mathbb{R}^q$ be such that*

$$\phi(x, u) = (\alpha(x, u), \beta(x, u))$$

Then $\phi \in \mathscr{D}^*(p, q)$ if and only if:

(a) $\left(\frac{\partial \alpha}{\partial x}\right)(0, 0)$ *has maximal rank*

(b) *there is $\tilde{\beta}_j \in \mathscr{E}(p, q)$ such that $\beta_j(x, u) = u_j \tilde{\beta}_j(x, u)$*

(c) $\tilde{\beta}_j\ (0, 0) > 0$.

Proof. Let $\phi \in \mathscr{D}^*(p, q)$; $\beta_j(x, u) = 0$ whenever $u_j = 0$. So β_j is divisibl by u_j and $\beta_j(x, u) = u_j \tilde{\beta}_j(x, u)$ for certain $\tilde{\beta}_j \in \mathscr{E}(p, q)$. Since $\beta_j(x, u) \geqslant ($ and $\beta_j(x, 0) = 0$ we have $(\partial\beta_j/\partial u_j)(x, 0) \geqslant 0$. Moreover $(\partial\beta_j/\partial u_j)(x, 0) =$ $\tilde{\beta}_j(x, 0)$, especially $\tilde{\beta}_j(0, 0) \geqslant 0$. The Jacobian matrix of ϕ at $(0, 0)$ is:

$$\begin{pmatrix} \dfrac{\partial \alpha}{\partial x}(0,0) & \dfrac{\partial \alpha}{\partial u}(0,0) \\ 0 & \dfrac{\partial \beta}{\partial u}(0,0) \end{pmatrix} \quad \text{where} \quad \left(\frac{\partial \beta}{\partial u}\right)(0,0) = \begin{pmatrix} \tilde{\beta}_1(0,0) & & \bigcirc \\ & \ddots & \\ \bigcirc & & \tilde{\beta}_q(0,0) \end{pmatrix}$$

So $\tilde{\beta}_j(0, 0) > 0$.

On the other hand, let ϕ satisfy (a), (b) and (c), then the Jacobian-matrix at $(0, 0)$ has maximal rank. From (b) and (c) we find locally $\phi(\mathbb{R}^p \times \mathbb{H}^q) = \mathbb{R}^p \times \mathbb{H}^q$.

6. Definition. A germ ψ: $(\mathbb{R}^p \times \mathbb{R}^q, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ is callec *equivariant* if $\psi\sigma = \sigma\psi$ for all $\sigma \in G$.

It is clear that ψ maps orbits into orbits. So there is a well-defined map germ: ${}^\rho\psi$: $(\mathbb{R}^p \times \mathbb{H}^q, (0, 0)) \to (\mathbb{R}^p \times \mathbb{H}^q, (0, 0))$ such that ${}^\rho\psi \cdot \rho = \rho \cdot \psi$.

7. Lemma. *Let ψ be a germ of equivariant diffeomorphism, then ${}^\rho\psi$ is a germ of diffeomorphism.*

Proof. Let $\psi(y, v) = (\gamma(y, v), \delta(y, v)) \in \mathbb{R}^p \times \mathbb{R}^q$. From the equivariance it follows that

$$\gamma(y, \sigma_j v) = \gamma(y, v) \qquad j = 1, \ldots, q$$
$$\delta(y, \sigma_j v) = \sigma_j\, \delta(y, v) \qquad j = 1, \ldots, q$$

So γ is symmetric for every y and so there is a $\tilde{\gamma} \in \mathscr{E}(p, q)$ with $\gamma(y, v) = \tilde{\gamma}(y, v^2)$.

Moreover $\delta_j(y, v) = 0$ if $v_j = 0$, so there is a $\hat{\delta}_j \in \mathscr{E}(p, q)$ such that $\delta_j(y, v) = v_j \hat{\delta}_j(y, v)$. We also see that $\hat{\delta}_j$ is symmetric for every y, so there are $\bar{\delta}_j \in \mathscr{E}(p, q)$ with $\hat{\delta}_j(y, v) = \bar{\delta}_j(y, v^2)$. So:

$$\delta_j(y, v) = v_j \hat{\delta}_j(y, v) = v_j \bar{\delta}_j(y, v^2).$$

The Jacobian-matrix of ψ at $(0, 0)$ is:

$$\begin{pmatrix} \frac{\partial \gamma}{\partial y}(0,0) & \frac{\partial \gamma}{\partial v}(0,0) \\ 0 & \frac{\partial \delta}{\partial v}(0,0) \end{pmatrix} \quad \text{where} \quad \frac{\partial \delta}{\partial v}(0,0) = \begin{pmatrix} \bar{\delta}_1(0,0) & & \bigcirc \\ & \ddots & \\ \bigcirc & & \bar{\delta}_q(0,0) \end{pmatrix}$$

So $\bar{\delta}_j(0, 0) \neq 0$.

Let ${}^p\psi(x, u) = (\alpha(x, u), \beta(x, u)) \in \mathbb{R}^p \times \mathbb{H}^q$, then (after writing u for v^2):

$$\alpha(x, u) = \tilde{\gamma}(x, u)$$

$$\beta_j(x, u) = \delta_j^2(y, v) = v_j^2 \, \bar{\delta}_j^2(x, v^2) = u_j \, \bar{\delta}_j^2(x, u) \qquad j = 1, \ldots, q.$$

This implies that ${}^p\psi$ is smooth. Since:

(a) $\left(\frac{\partial \alpha}{\partial x}\right)(0, 0) = \left(\frac{\partial \gamma}{\partial y}\right)(0, 0)$ has maximal rank

(b) $\beta_j(x, u) = u_j \bar{\beta}_j(x, u)$ with $\bar{\beta}_j = \bar{\delta}_j^2$

(c) $\bar{\beta}_j(0, 0) = \bar{\delta}_j^2(0, 0) > 0$

Lemma 5 implies that ${}^p\psi \in \mathscr{D}^*(p, q)$.

8. Definitions. Let $\mathscr{D}(p+q)$ be the group of germs of diffeomorphisms of $(\mathbb{R}^{p+q}, 0)$ and $\mathscr{D}^*_G(p+q)$ its subgroup of equivariant diffeomorphisms. Let $\mathscr{D}_G(p+q) = \{\psi \in \mathscr{D}(p+q) \mid \psi^{-1} G \psi \subset G\}$ and $S(q)$ the subgroup of $\mathscr{D}(p+q)$, which consists of permutations of the coordinates in $\mathbb{R}^q$. The definition of ^{p} extends to $\mathscr{D}_G(p+q)$.

9. Lemma. *$S(q)$ and $\mathscr{D}^*_G(p+q)$ are subgroups of $\mathscr{D}_G(p+q)$ and $\mathscr{D}_G(p+q) = S(q) \circ \mathscr{D}^*_G(p+q)$ (no direct product) moreover the decomposition of $\mathscr{D}_G(p+q)$ is unique.*

10. Lemma. (*a*) *The map ^{p}: $\mathscr{D}^*_G(p+q) \to \mathscr{D}(p, q)$ has image $\mathscr{D}^*(p, q)$, its kernel is $G = (\mathbb{Z}_2)^q$.* (b) *The map ^{p}: $\mathscr{D}_G(p+q) \to \mathscr{D}(p, q)$ is surjective, its kernel is $G = (\mathbb{Z}_2)^q$.*

Proof. (a) Let $\phi = (\alpha, \beta) \in \mathscr{D}^*(p, q)$
Consider:

$$\left.\begin{aligned} \gamma(y, v) &= \alpha(y, v^2) \\ \delta_j^2(y, v) &= \beta_j(y, v^2) \end{aligned}\right\} *$$

So

$$\delta_j^{\pm}(y, v) = \pm v_j \sqrt{\bar{\beta}_j(y, v^2)}$$

are well-defined and smooth near (0, 0) since $\bar{\beta}_j(0, 0) > 0$. The Jacobiar matrix is non-degenerate so $\psi = (\gamma, \delta)$ defines a diffeomorphism whic projects to $\phi = (\alpha, \beta)$. Clearly ψ is equivariant. If $(\alpha, \beta) =$ identity, the the solutions of $(*)$ are just the elements of G. (b) follows from th decompositions of $\mathscr{D}(p, q)$ and $\mathscr{D}_G(p+q)$ (Lemma 4 and 9).

11. Conclusion. Orbits in $\mathscr{E}(p, q)$ under $\mathscr{D}(p, q)$ correspond to orbit in $\mathscr{E}_G(p+q)$ under $\mathscr{D}_G(p+q)$ and vice versa.

The classification of singularities on boundaries, corners, etc. follow from the classification of singularities of symmetric functions, as given b Beer [3] and Wassermann [11] who considered $\mathscr{D}_G^*(p+q)$-action.

Since $S(q)$ is a finite group, orbits under $\mathscr{D}_G(p+q)$ are finite unions o orbits under $\mathscr{D}_G^*(p+q)$.

As a corollary we have:

12. Splitting Lemma. *Let $f \in \mathscr{E}(p, q)$ with $f(0, 0) = 0$ and $(\partial f/\partial x_1)(0, 0) =$ $\cdots = (\partial f/\partial x_p)(0, 0) = 0$ then there are a diffeomorphism $\phi \in \mathscr{D}(p, q)$ numbers r and s and a germ $g \in \mathscr{E}(r, s)$ such that $f\phi = q + \ell + g$ wit*

$$q(x_{r+1}, \ldots, x_p) = \varepsilon_{r+1}x_{r+1}^2 + \cdots + \varepsilon_p x_p^2 \qquad \varepsilon_j = \pm 1.$$
$$\ell(u_{s+1}, \ldots, u_q) = \delta_{s+1}u_{s+1} + \cdots + \delta_q u_q \qquad \delta_j = \pm 1.$$

$g = g(x_1, \ldots, x_r, u_1, \ldots, u_s)$; *the so called residual singularity, with*

$$\frac{\partial g}{\partial x_i}(0, 0) = 0 \qquad i = 1, \ldots, r.$$

$$\frac{\partial^2 g}{\partial x_i\, \partial x_j}(0, 0) = 0 \qquad i = 1, \ldots, r; \qquad j = 1, \ldots, r.$$

$$\frac{\partial g}{\partial u_k}(0, 0) = 0 \qquad k = 1, \ldots, s.$$

$r + s$ is called the corank of the singularity, r the free corank and s th boundary corank. The number of negative linear terms in ℓ is the boundar index, the number of negative quadratic terms in q is the free index Moreover the splitting is essentially unique, in the sense that the corank and indices are unique and the residual singularity is unique up to th action of $\mathscr{D}(r, s)$.

13. Codimension. Let

$$\mathscr{M}(p, q) = \{f \in \mathscr{E}(p, q) \mid f(0, 0) = 0\}$$

and

$$\mathcal{M}_G(p+q)=\{g\in\mathscr{E}_G(p+q)\mid g(0,0)=0\}$$

then

$$\rho^*\mathcal{M}(p,q)=\mathcal{M}_G(p+q).$$

Let

$$\Sigma(p,q)=\left\{f\in\mathcal{M}(p,q)\;\middle|\;\frac{\partial f}{\partial x_1}(0,0)=\cdots=\frac{\partial f}{\partial x_p}(0,0)=0\right\}$$

and

$$\Sigma_G(p+q)=\left\{g\in\mathcal{M}_G(p+q)\;\middle|\;\frac{\partial g}{\partial y_1}(0,0)=\cdots=\frac{\partial g}{\partial y_p}(0,0)=0\right\}.$$

$\Sigma(p,q)$ and $\Sigma_G(p+q)$ are called *singularity subsets.*

For $f\in\Sigma(p,q)$ and $g\in\Sigma_G(p,q)$ we define:

$$J_{p,q}(f)=\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_p},u_1\frac{\partial f}{\partial u_1},\ldots,u_q\frac{\partial f}{\partial u_q}\right)\subset\mathscr{E}(p,q)$$

$$J_G(g)=\left\{\Sigma\zeta_i\frac{\partial g}{\partial y_i}+\Sigma\eta_j\frac{\partial g}{\partial v_j}\;\middle|\;(\zeta_1,\ldots,\zeta_p,\eta_1,\ldots,\eta_q)\text{ equivariant}\right\}$$

$J_{p,q}(f)$ and $J_G(g)$ are called *Jacobi-ideals* and we have: $\rho^*J_{p,q}(f)=J_G(\rho^*f)$. We define next:

$$\operatorname{cod}(f)=\dim_{\mathbf{R}}\frac{\mathcal{M}(p,q)}{J_{p,q}(f)};\quad\text{compare [2].}$$

$$\operatorname{cod}_G(f)=\dim_{\mathbf{R}}\frac{\mathcal{M}_G(p+q)}{J_G(g)};\quad\text{cf. [11].}$$

As a consequence we have: $\operatorname{cod}(f)=\operatorname{cod}_G(\rho^*f)$. From [11] it follows that:

$$\operatorname{cod}(f)<\infty\Leftrightarrow f\textit{ is finitely determined.}$$

Consider next the projection j^k: $\mathscr{E}(p,q)\to J^k(p+q)$ on k-jet-space. Let $f\in\mathscr{E}(p,q)$ and Orb (f) its orbit in $\mathscr{E}(p,q)$. Let $\text{Orb}^k(f)=j^k\,\text{Orb}(f)$ and $\Sigma^k(p,q)=j^k\Sigma(p,q)$, then $\text{Orb}^k(f)$ is a smooth semi-algebraic submanifold of $J^k(p+q)$. Let $\operatorname{cod}(f)<\infty$ and let f be k-determinate. Then the codimension of $\text{Orb}^k(f)$ in $\Sigma^k(p,q)$ is equal to $\operatorname{cod}(f)$. (The proof is analogous to the case without boundary [4], p. 126).

The condition coc $(f)=c$ defines a subset Σ_c of $\mathscr{E}(p,q)$. Its projection Σ_c^k on $J^k(p+q)$ is a semi-algebraic set, whose connected components are called *μ-classes* or *bundles.* Let $f\in\Sigma_c$ and $B^k(f)$ the bundle containing f.

Let f be determined by its k-jet. the codimension of B^k in Σ^k is called the *bundle codimension of f* and denoted by $b(f)$.

14. Unfoldings. Let $f \in \mathscr{E}(p, q)$ with $f(0,0)=0$. A *k-dimensional unfolding* of f is a germ $F \in \mathscr{E}(p+k, q)$ such that $F \mid \mathbb{R}^p \times \mathbb{H}^q = f$. An unfolding F of f is called *universal* if given any other unfolding $H \in (p+l, q)$ of f, there exists a pair (Φ, α) where Φ and α, satisfy the following conditions:

(i) If for $(x, u) \in \mathbb{R}^p \times \mathbb{H}^q$ and $\lambda \in \mathbb{R}^l$ we write: $\Phi(x, u, \lambda) = (\phi(x, u, \lambda), \psi(x, u, \lambda)) \in (\mathbb{R}^p \times \mathbb{H}^q) \times \mathbb{R}^k$ then:
 (a) $\psi(x, u, \lambda)$ is independent of (x, u), i.e. $\psi(x, u, \lambda) = \psi(\lambda)$
 (b) $\phi \mid \mathbb{R}^p \times \mathbb{H}^q \times 0 = \text{identity}$
 (c) $\psi(0) = 0$.

(ii) for $(x, u) \in \mathbb{R}^p \times \mathbb{H}^q$, $\lambda \in \mathbb{R}^l$ we have:

$$H(x, u, \lambda) = F(\Phi(x, u, \lambda)) + \alpha(\lambda)$$

15. Remark. Let F be an unfolding of f, then ρ^*F is a G-unfolding of ρ^*f in the sense of Wassermann **[11]**. It is not difficult to prove that *F is universal $\Leftrightarrow$ ρ^*F is G-universal.*

So the existence of universal unfoldings in our case can be checked by considering the symmetric germ $f(x, u^2)$ and applying the symmetric theory. For example one has:
If cod $(f) < \infty$ then the minimal unfolding dimension of a universal unfolding of f is cod (f). If $\phi_1, \ldots, \phi_k \in \mathscr{M}(p+q)$ are representatives of a basis of $\mathscr{M}(p, q)/J_{p,q}(f)$ then the k-dimensional unfolding

$$H(x, u, \lambda) = f(x, u) + \lambda_1 \phi_1(x, u) + \cdots + \lambda_k \phi_k(x, u)$$

is universal.

16. Classification. In the next section the classification of bundles up to codimension 4 is given. I agree with Wassermann, that one must choose some limit and that germs of codimension ≤ 4 are of particularly interest for applications in catastrophe **[10]**. I don't take ordinary codimensions but bundle codimension, since "moduli" appear already in codimension 3, that is, there are infinitely many orbits of codimension 3 in a certain bundle of codimension 2. The difference $m(f) = \text{cod}(f) - b(f)$ is called the *modality of f*. We use the following abbrevations

r = free corank
s = boundary corank
c = codimension
m = modality
b = bundle codimension

For every bundle we give a normal form. They contain variable coefficients of two types: Either they take on a discrete set of values; in this case they are usually denoted by Greek letters: $\alpha, \beta, \gamma, \delta, \varepsilon$, or $\pm$ (all $\pm$ signs are independent). Or they may vary continuously over some range (indicated by the conditions).

Variable coefficient of this second type will be called *modules*; they are always denoted by capital Roman letters A, B, C, ... In our list the number of modules is equal to the modality of the bundle and the map from modules to orbits is finite to one. The universal unfolding parameters $\phi_1, \ldots, \phi_k$ are of two kinds:

(1) internal ones: the monomials behind the modules $A, B, \ldots$

(2) external ones: listed separately.

The last column refers to the running numbers in Arnol'd's, Beer's and Wassermann lists.

17. List of residual singularities on boundaries, corners, etc. of bundle-codimension $\leqslant 4$.

r	s	germ g	c	m	b	external unfolding-parameters	conditions	
0	0	0	0	0	0			A_1
1	0	x^3	1	0	1	$x,$		A_2
		$\pm x^4$	2	0	2	x, x^2		A_3
		x^5	3	0	3	x, x^2, x^3		A_4
		$\pm x^6$	4	0	4	x, x^2, x^3, x^4		A_5
2	0	$x_1^2x_2 \pm x_2^3$	3	0	3	x_1, x_2, x_2^2		D_4
		$x_1^2x_2 \pm x_2^4$	4	0	4	x_1, x_2, x_2^2, x_2^3		D_5
0	1	$\pm u^2$	1	0	1	u		i_2
		$\pm u^3$	2	0	2	u, u^2		i_3
		$\pm u^4$	3	0	3	u, u^2u^3, u^3		i_4
		$\pm u^5$	4	0	4	u, u^2, u^3, u^4		i_5
1	1	$xu \pm x^3$	2	0	2	x, x^2		ii_3
		$xu \pm x^4$	3	0	3	x, x^2		ii_4
		$xu \pm x^5$	4	0	4	x, x^2, x^3, x^4		ii_5
		$x^3 \pm u^2$	3	0	3	x, u, xu	$\begin{cases} \varepsilon^3 = \varepsilon \\ 4\varepsilon + 27A^2 \neq 0 \end{cases}$	iii_2
		$x^3 + \varepsilon xu^2 + Au^3$	5	1	4	x, u, u^2, xu		
		$\pm x^4 + Ax^2u \pm u^2$	5	1	4	x, u, x^2, xu	$A^2 \neq \pm \cdot \pm .4$	
2	1	$x_1^3 + Ax_1x_2^2 + \varepsilon x_2^3 + x_2u$	5	1	4	x_1, x_2, x_2^2, x_1x_2	$\begin{cases} 27\varepsilon^2 + 4A^3 \neq 0 \\ \varepsilon^3 = \varepsilon \end{cases}$	

r	s	germ g	c	m	b	external unfolding-parameters	conditions	
0	2	$\pm u_1^2 + Au_1u_2 \pm u_2^2$	3	1	2	u_1, u_2	$A^2 \neq \pm \cdot \pm .4$	12
		$\pm(u_1^2 \pm 2u_1u_2 + u_2^2) + Au_2^3$	4	1	3	u_1, u_2, u_2^2	$A \neq 0$	13_3
		$\pm(u_1^2 \pm 2u_1u_2 + u_2^2) + Au_2^4$	5	1	4	u_1, u_2, u_2^2, u_2^3	$A \neq 0$	13_4
		$\pm u_1^2 \pm u_1u_2 + Au_2^3$	4	1	3	u_1, u_2, u_2^2	$A \neq 0$	$14_{2,3}$
		$\pm u_1^3 \pm u_1u_2 + Au_2^3$	5	1	4	u_1, u_2, u_1^2, u_2^2	$A \neq 0$	$14_{3,3}$
		$\pm u_1^2 \pm u_1u_2 + Au_2^4$	5	1	4	u_1, u_2, u_2^2, u_2^3	$A \neq 0$	$14_{2,4}$
		$\pm u_1^2 + Au_1u_2^2 \pm u_2^3$	5	1	4	u_1, u_2, u_1u_2, u_2^2	$A \neq 0$	
1	2	$x^3 \pm xu_1 \pm xu_2 + Au_2^2$	4	1	3	x, u_1, u_2	$A \neq 0$	36_2
		$x^3 \pm xu_1 \pm xu_2 + Au_2^3$	5	1	4	x, u_1, u_2, u_2^2	$A \neq 0$	36_3
		$Ax^4 \pm xu_1 \pm xu_2 \pm u_2^2$	5	1	4	x, x^2, u_1, u_2	$A \neq 0$	
		$x^3 \pm xu_1 + Ax^2u_2 \pm u_2^2$	5	1	4	x, u_1, u_2, xu_2		

r	s		c	m	b	external unfolding parameters	
0	3	$\alpha u_1^2 + \beta u_2^2 + \gamma u_3^2 + Au_1u_2 + Bu_2u_3 + Cu_1u_3 + Du_1u_2u_3$	7	4	3	u_1, u_2, u_3	I
		$\alpha u_1u_2 + \beta u_2^2 + \gamma u_3^2 + Au_1^3 + Bu_2u_3 + Cu_1u_3 + Du_1u_3^2$	8	4	4	u_1, u_2, u_3, u_1^2	II_a
		$\alpha(u_1 + \beta u_2)^2 + \gamma u_3^2 + Au_1u_2^2 + Bu_2u_3 + Cu_1u_3 + Du_1u_2^3$	8	4	4	u_1, u_2, u_3, u_1u_2	II_b
		$\alpha(u_1 + \beta u_2 + \gamma u_3)^2 + \delta(Au_2 + Bu_3)^2 + Cu_3^3 + Du_3^4$	8	4	4	u_1, u_2, u_3, u_3^2	II_c
1	3	$x^3 + \alpha xu_1 + \beta xu_2 + \gamma xu_3 + Au_2^2 + Bu_2u_3 + Cu_3^2 + Dxu_2u_3$	8	4	4	x, u_1, u_2, u_3	III
0	4	$\alpha u_1^2 + \beta u_2^2 + \gamma u_3^2 + \delta u_4^2 + Au_1u_2 + Bu_1u_3 + Cu_1u_4 + Du_2u_3 + Eu_2u_4 + Fu_3u_4 + Gu_1u_2u_3 + Hu_1u_2u_4 + Ku_1u_3u_4 + Lu_2u_3u_4 + Mu_1u_2u_3u_4$	15	11	4	u_1, u_2, u_3, u_4	IV

Conditions: general: $\alpha, \beta, \gamma, \delta$ are ± 1.

(I) The matrix $\begin{pmatrix} 2\alpha & A & C \\ A & 2\beta & B \\ C & B & 2\gamma \end{pmatrix}$

and all square submatrices centered on the diagonal must be nonsingular.

(II_a) $AC(B^2 - 4\beta\gamma)(\beta C^2 - \alpha BC + \gamma) \neq 0$

(II_b) $A(B^2 - 4\alpha\gamma)(C^2 - 4\alpha\gamma)(\beta B - C) \neq 0$

(II_c) $ABC(\gamma A - \beta B)(\alpha + \delta A^2)(\alpha + \delta B^2) \neq 0$

(III) $AC[B^2 - 4AC][A - \beta\gamma B + C] \neq 0$

(IV) The matrix $\begin{pmatrix} 2\alpha & A & B & C \\ A & 2\beta & D & E \\ B & D & 2\gamma & F \\ C & E & F & 2\delta \end{pmatrix}$

and all square submatrices centered on the diagonal must be nonsingular

For the proof one has to extend Wassermann's and Beer's lists to germs of higher codimensions. We omit the proof, the techniques can be found in [**11**].

Case $s = 0$ are Thom's elementary catastrophes [**10**].

Case $p = 1$ is covered by Pitt and Poston [**5**].

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Mathematich Instituut,
Rijksuniversiteit Utrecht,
The Netherlands.