Vanishing Cycles and Special Fibres

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Abstract

We show that the homotopy type of certain special fibres in a perturbation of a holomorphic function is a wedge of spheres of middle dimension. We also define a basis of the homology of the special fibre.

1 Introduction

In the case of an isolated singularity $\mathbf{C}^{n+1} \to \mathbf{C}$ the vanishing cycles play an important role in the description of the general non-singular fibre. This Milnor fibre F is homotopy equivalent to a wedge of spheres:

$$F \overset{h}{\simeq} S^n \vee \cdots \vee S^n$$

where the wedge is taken over μ spheres in the middle dimension. Here μ is Milnor's number, which can be identified with the number of Morse-points (or A_1-points) in a generic perturbation. The vanishing cycles correspond to the spheres in the wedge. We show in this note, that the same conclusion is true for certain singular fibres which occur in the case of non-isolated singularities. The proof is similar to the case of the Milnor fibre of an isolated singularity ([AGV-II], [Lo]).

We also define a basis of the homology of the special fibre. Each basis element corresponds to an A_1-point. At the end we discuss monodromies of the special fibre, which occur in 1-dimensional families.

This note was written after discussions with David Mond and Duco van Straten about Mond's Theorem, that the homotopy type of the generic fibre of a disentanglement is a wedge of spheres.

2 Homotopy type of the special fibres

2.1 We consider non-zero holomorphic function germs $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ and allow arbitrary singularities (isolated or non-isolated). We recall the definition of the Milnor fibration. For $\epsilon > 0$ small enough there exist an ϵ-ball B_ϵ in $\mathbf{C}^{n+1}$ and an η-disc D_η in $\mathbf{C}$ such that the restriction:

$$f : f^{-1}(D_\eta) \cap B_\epsilon \longrightarrow D_\eta$$

is a locally trivial fibre bundle over $D_\eta \setminus \{0\}$. The fibres are called Milnor fibres. The boundary ∂B_ϵ is called the Milnor sphere.

2.2 We next consider a deformation F of f, i.e. a holomorphic mapgerm:

$$F : (\mathbf{C}^{n+1} \times \mathbf{C}^r, 0) \to (\mathbf{C} \times \mathbf{C}^r, 0)$$

of the form

$$F(x,a) = (f_a(x), a)$$

such that $f_0(x) = f(x)$.
The mapgerm f_a is called a *perturbation* of f.

We require that the deformation be *topologically trivial over the Milnor sphere* ∂B_ϵ. This means: For η and ρ small enough

$$F : \partial B_\epsilon \times D_\rho \cap F^{-1}(D_\eta \times D_\rho) \to D_\eta \times D_\rho$$

must be a stratified submersion with strata $\{0\} \times D_\rho$ and $(D_\eta \setminus \{0\}) \times D_\rho$ on $D_\eta \times D_\rho$ and some stratification on $\partial B_\epsilon \times D_\rho \cap F^{-1}(D_\eta \times D_\rho)$.
This condition implies:

- $f_a^{-1}(t)$ is (stratified) transversal to ∂B_ϵ for all $|t| < \eta$ and for all $|a| < \rho$.
- $f_a^{-1}(D_\eta) \cap \partial B_\epsilon$ is homeomorphic to $f^{-1}(D_\eta) \cap \partial B_\epsilon$ and therefore contractible.
- the Milnor fibres of f and f_a are diffeomorphic.

2.3 Theorem *Let F be a deformation of f, which is topologically trivial over the Milnor sphere as defined in 2.2. Let $a \in D_\rho$ and suppose that all fibres of f_a are smooth or have isolated singularities except for one special fibre $X_t = f_a^{-1}(t) \cap \partial B_\epsilon$. Then X_t is homotopy equivalent to a wedge of spheres:*

$$X_t \overset{h}{\simeq} S^n \vee \cdots \vee S^n$$

The number of spheres is equal to the sum of the Milnor numbers in the fibres different from X_t.

Proof In the following we use the notation:

$$g : X \to D$$

for the perturbation:

$$f_a : f_a^{-1}(D_\eta) \cap \partial B_\epsilon \to D_\eta.$$

We denote: $X_Y = g^{-1}(Y)$.

Let $x_1, \ldots, x_\sigma$ be the critical points outside X_t and $c_1, \ldots, c_\tau$ be the critical values, different from t. Take small disjoint discs $D_0, D_1, \ldots, D_\tau$ around $t, c_1, \ldots, c_\tau$ and join them with a point s on ∂D_0 with the help of a system of non-intersecting paths Γ (in the usual way, cf. figure 1). Call the endpoints $s_1, \ldots, s_\tau$.

We mention the homotopy equivalence:

$$X_t \overset{h}{\simeq} X_{D_0}.$$

This equivalence is well known in the local case (i.e. in a small neighborhood of a singular point),see proposition 2.A.3.(b) of [GM]. Since our map is proper one can patch together these local equivalences to a global homotopy equivalence. One can also apply directly lemma 2.A.2. of [GM].

Next we use homotopy lifting properties and have first:

$$X_{D_0} \overset{h}{\simeq} X_{D_0 \cup \Gamma}$$

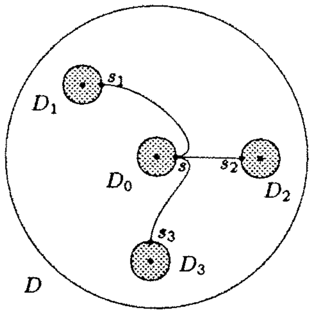

Figure 1: Perturbation g.

and second:

$$X_D \overset{h}{\simeq} X_{D_0 \cup \Gamma} \cup X_{D_1} \cup \cdots \cup X_{D_r}$$

Similar homotopy equivalences occur in [Lo] and [Si-1].

All X_{c_i} contain only isolated singularities. Let μ_i be the sum of the Milnor numbers in the fibre X_{c_i}. Each X_{D_i} can be obtained (up to homotopy equivalence) from X_{s_i} by attaching μ_i cells of dimension $n+1$ in order to kill the vanishing cycles.

After retraction of Γ to the point s, it follows that

$$X_D \overset{h}{\simeq} X_{D_0} \cup \{\sum \mu_i \text{ cells of dimension } n+1\ \}.$$

Since X_D is diffeomorphic to $f^{-1}(D_\eta) \cap \partial B_\epsilon$, which is contractible (as total space of the Milnor fibration), we have that:

$$\pi_k(X_t) = \pi_k(X_{D_0}) = 0 \text{ for all } k < n.$$

Since X_t has the homotopy type of a CW-complex of dimension n, see [GM](p.152), it follows that:

$$X_t \overset{h}{\simeq} S^n \vee \cdots \vee S^n,$$

the number of spheres being equal to $\nu = \sum \mu_i$. □

2.4 Remark In the case of an isolated singularity $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ the above theorem shows that for any perturbation $g : X \to D$ of f, *all* fibres X_s (including the singular fibres) have the homotopy type of a wedge of n-spheres. The same conclusion can be deduced from corollary (5.10) on page 75 of Looijenga's book [Lo], where the following is proved:

Let $f : X \to D$ be a proper good representative of a germ $f : (\mathbf{C}^{n+k}, x) \to (\mathbf{C}^k, 0)$ defining an isolated complete intersection singularity of dimension n. Then every fibre X_s has the homotopy type of a wedge of n-spheres.

2.5 Remark If X is not smooth but if $X \setminus X_t$ is smooth the same theorem as 2.3 applies since X_D is still contractible.

3 Homology of special fibres

3.1 We are especially interested in describing a basis for the homology of the special fibre in such a way that this basis is related to the A_1-points of the perturbation.

We have according to [Si-2] the following direct sum decomposition of the vanishing homology:

$$H_*(E, X_s) \cong H_*(\tilde{X}_t, X_s) \oplus \oplus_{i=1}^{\sigma} H_*(E_i, F_i)$$

where

$$\begin{array}{rcl} X_t & = & \text{special fibre} \\ \tilde{X}_t & = & X_{D_0}\text{ , a neighbourhood of } X_t \\ X_s & = & \text{Milnor fibre} \\ E & = & \text{Milnor ball} \\ E_i & = & \text{Milnor ball of the isolated singularity at } x_i \\ F_i & = & \text{Milnor fibre of the isolated singularity at } x_i \end{array}$$

This direct sum decomposition can depend on the choice of the system of paths Γ.

3.2 Since $H_*(E_i, F_i)$ is concentrated in dimension $n+1$ it follows from 3.1 that:

$$\begin{array}{rcll} H_{n+1}(E, X_s) & = & H_{n+1}(\tilde{X}_t, X_s) \oplus \mathbf{Z}^{\nu} & \text{where } \nu = \sum_{i=1}^{\sigma} \mu_i \\ H_k(E, X_s) & = & H_k(\tilde{X}_t, X_s) & \text{if } k \neq n+1 \end{array}$$

All these isomorphisms are induced by inclusions. Therefore the exact sequence of the triple $(E, \tilde{X}_t, X_s)$ splits into short pieces:

$$0 \longrightarrow H_n(\tilde{X}_t, X_s) \longrightarrow H_n(E, X_s) \longrightarrow 0$$

and

$$0 \longrightarrow H_{n+1}(\tilde{X}_t, X_s) \xrightarrow{\alpha} H_{n+1}(E, X_s) \xrightarrow{\beta} H_{n+1}(E, \tilde{X}_t) \longrightarrow 0$$

Remark that independently from theorem 2.3 this gives again $\tilde{H}_k(X_t) = 0$ for $k \neq n$.

3.3 This above sequence extends to the diagram:

$$\begin{array}{ccccccc} & & H_n(X_s) & \xrightarrow{\beta} & H_n(X_t) & \longrightarrow & 0 \\ & \nearrow & \Delta \uparrow \cong & & \Delta \uparrow \cong & & \\ 0 \longrightarrow & H_{n+1}(\tilde{X}_t, X_s) & \xrightarrow{\alpha} H_{n+1}(E, X_s) & \xrightarrow{\beta} & H_{n+1}(E, \tilde{X}_t) & \longrightarrow & 0 \\ & & j \uparrow & & & & \\ & & \oplus_{i=1}^{\sigma} H_{n+1}(E_i, F_i) = \mathbf{Z}^{\nu} & & & & \end{array}$$

Because of the direct sum splitting 3.1 it follows that the composition

$$\oplus_{i=1}^{\sigma} H_{n+1}(E_i, F_i) \xrightarrow{j} H_{n+1}(E, X_s) \xrightarrow{\beta} H_{n+1}(E, X_t)$$

is an isomorphism. Moreover also the induced map

$$\oplus_{i=1}^{\sigma} H_n(F_i) \xrightarrow{j} H_n(X_s) \xrightarrow{\beta} H_n(X_t)$$

is an isomorphism. We summarize:

3.4 Proposition

$$H_n(X_t) \cong \oplus_{i=1}^{\sigma} H_n(F_i).$$

3.5 From now on we assume that all critical points outside X_t are of type A_1.
Definition

$$L = j(\oplus_{i=1}^{\sigma} H_n(F_i)) \subset H_n(X_s)$$

is called the *A_1-lattice in $H_n(X_s)$*. It has a natural basis (up to a sign), which depends via the isomorphism 3.4 on the choice of the paths, which constitute Γ. The A_1-lattice L is via β isomorphic to $H_n(X_t)$.

If necessary we write L_Γ and j_Γ to distinguish between different systems of paths.

3.6 We next adapt the definition of vanishing cycle to our situation. Let again $g : X \to D$ be a perturbation with the properties of 2.3. Let γ be a continuous path in $D \setminus \{t, c_1, \ldots, c_\tau\}$ from s to some s_i (cf. figure 2).

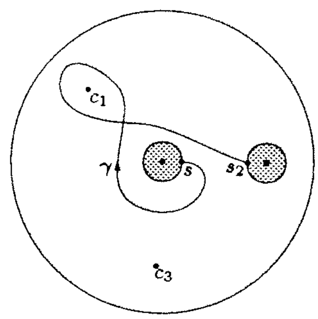

Figure 2: The path γ.

There are induced maps

$$\mathbf{Z} = H_n(F_i) \xrightarrow{j_\gamma} H_n(X_s) \xrightarrow{\beta} H_n(X_t)$$

Definition Let δ be a generator of $H_n(F_i)$.

$$\begin{aligned} \delta_\gamma &:= j_\gamma(\delta) \in H_n(X_s) \\ \Delta_\gamma &:= \beta(\delta_\gamma) \in H_n(X_t) \end{aligned}$$

$\Delta_\gamma \in H_n(X_t)$ is called the *cycle vanishing along γ*.

The fundamental group $\pi_1(D \setminus \{t, c_1, \ldots, c_r\}, s)$ acts on the homotopy classes from paths from s to s_i. The next lemma shows that loops around t do not affect a vanishing cycle.

3.7 Lemma *The definition of the vanishing cycle Δ_γ depends only on the homotopy class of γ in $D \setminus \{c_1, \ldots, c_r\}$.*

Proof A loop u_0 which goes from s around ∂D_0 induces a monodromy homomorphism:

$$h_0 : H_n(X_s) \longrightarrow H_n(X_s).$$

We claim that this monodromy is the identity modulo Ker $(\beta : H_n(X_s) \to H_n(X_t))$. This is intuitively clear for geometrical reasons, since one can choose a geometric monodromy, which is the identity on a sufficient big part of X_s. To be more precise:
Let E_0 be a tubular neighbourhood of the critical locus of X_t. We denote:

$$\begin{array}{lcl} Y_s & = & X_s \cap E_0 \\ A_s & = & \overline{X_s \setminus Y_s} \\ \partial_1 E_0 & = & \overline{\partial E_0 \setminus \partial X} \text{ , the inner boundary of } E_0 \\ \partial_1 Y_s & = & X_s \cap \partial_1 E_0 \text{ , the inner boundary of } Y_s \end{array}$$

Since the restriction of g to $\overline{f^{-1}(D_0) \setminus E_0}$ has maximal rank over D_0 there exists a geometric monodromy

$$h_0 : (X_s, A_s) \longrightarrow (X_s, A_s)$$

which is the identity on A_s. We can also consider the restriction

$$h_0' : (Y_s, \partial_1 Y_s) \longrightarrow (Y_s, \partial_1 Y_s),$$

which is the identity on $\partial_1 Y_s$.
We can use now the theory of the variation mapping. We refer to Lamotke [La] or [Si-3] and especially to [Lo] p.35. The properties of the variation mapping, which we denote by VAR^1 imply that $h_0 - 1$ is the following composition

$$h_0 - 1 : H_n(X_s) \longrightarrow H_n(X_s, A_s) \xrightarrow{\text{exc}} H_n(Y_s, \partial_1 Y_s) \xrightarrow{\mathrm{VAR}^1} H_n(Y_s) \xrightarrow{j_*} H_n(X_s),$$

We claim that

$$\text{Im } j_* \subset \text{Ker } \beta.$$

Consider the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{n+1}(E_0, Y_s) & \longrightarrow & H_n(Y_s) & \longrightarrow & H_n(E_0) \\ & & \downarrow \alpha & & \downarrow j & & \\ 0 & \longrightarrow & H_{n+1}(E, X_s) & \xrightarrow{\cong} & H_n(X_s) & \longrightarrow & 0 \end{array}$$

Since $H_{n+1}(E_0, Y_s) = H_{n+1}(\tilde{X}_t, X_s)$ the vertical map is indeed α. So one has:

$$\text{Ker } \beta = \text{Im } \alpha \supset \text{Im } j$$

This completes the proof. □

3.8 Definition Let Γ be a system of non-intersecting paths. According to proposition 3.4 we get in $H_n(X_t)$ a basis of vanishing cycles $\Delta_1, \ldots, \Delta_\sigma$. This basis is called *distinguished* if we take into account a cyclic numbering condition as well. Compare ([AGV-II], p.14).

A distinguished basis only depends on the relative position of Γ with respect to $c_1, \ldots, c_\tau$ and not with respect to t. But the A_1-lattice $L_\Gamma \subset H_n(X_s)$ shall in general also depend on the relative position of t.

3.9 Theorem *A system of paths Γ gives a well-defined (distinguished) basis of vanishing cycles in $H_n(X_t)$ depending only on the isotopy class in $D \setminus \{c_1, \ldots, c_\tau\}$.*

4 Concluding remarks and questions

4.1 Remarks about disentanglements

Mond [Mo] considered in his work finitely determined map germs $F : \mathbf{C}^2 \to \mathbf{C}^3$. The image $F(\mathbf{C}^2)$ is a hypersurface $f = 0$ and has a 1-dimensional singular locus Σ with transversal singularity type A_1 on $\Sigma \setminus \{0\}$.

A stable perturbation G of F induces a *disentanglement* $G(\mathbf{C}^2)$ of $F(\mathbf{C}^2)$. A disentanglement has only singularities of type A_∞, D_∞ or $T_{\infty,\infty,\infty}$ (ordinary double curve, ordinary pinch point, ordinary triple point). Let $g = 0$ be the equation of $G(\mathbf{C}^2)$. According to [Mo] the function $g : \mathbf{C}^3 \to \mathbf{C}$ has outside $g^{-1}(0)$ only isolated singularities.

The notion of disentanglement was introduced in a more general context by De Jong en Van Straten [Jo-St]. Our theorem 2.3 implies that the surface $G(\mathbf{C}^2) = g^{-1}(0)$ is homotopy equivalent to a wedge of spheres:

$$g^{-1}(0) \overset{h}{\simeq} S^2 \vee \cdots \vee S^2$$

This was proved by Mond [Mo].

More general one can consider the versal unfolding:

$$\widetilde{G} : \mathbf{C}^2 \times \mathbf{C}^d \longrightarrow \mathbf{C}^3 \times \mathbf{C}^d$$

Let $\widetilde{G}(x, a) = (G_a(x), a)$ and let the image $G_a(\mathbf{C}^2)$ be the hypersurface with equation $g_a = 0$. According to Mond the map $g_a : \mathbf{C}^3 \to \mathbf{C}$ has for all $a \in \mathbf{C}^d$ only one fibre with non-isolated singularities. According to theorem 2.3 all the surfaces $G_a(\mathbf{C}^2) = g_a^{-1}(0)$ have the homotopy type of a wedge of spheres.

One can also consider the non-singular fibres of g_a. The general theory tells us, that they must be connected, but not necessarily simply connected. But in Mond's case one knows that $f(\mathbf{C}^2)$ is irreducible and so the Lê-Saito theorem implies that the fundamental group is trivial. So also the general fibre is a wedge of spheres. The number of these spheres is equal to:

$$2\#D_\infty - 1 + 2\#T_{\infty,\infty,\infty} - \chi(\widetilde{\Sigma}) + \#A_1,$$

where $\widetilde{\Sigma}$ is the normalisation of Σ. This formula can be shown in the same way as the formulas in [Si-2].

The only fibres we haven't discussed are those with isolated singularities. One can obtain such a fibre X_c from nearby smooth fibres by attaching 3-cells in order to kill the vanishing homology. Since the general smooth fibre is simply connected this implies that X_c is also simply connected.

Conclusion *In the case of disentanglements all fibres of g_a are wedges of 2-spheres.*

4.2 Remarks about monodromies of the special fibre

Let $G_a : \mathbf{C}^{n+1} \to \mathbf{C}$ be a 1-dimensional family of functions such that the map

$$\tilde{G} : \mathbf{C} \times \mathbf{C}^{n+1} \to \mathbf{C} \times \mathbf{C},$$

defined by $\tilde{G}(a,x) = (G_a(x), a)$ satisfies the conditions of theorem 2.3. We can suppose that for $a \neq 0$ small enough the singularity types of the isolated critical points $x_1, \ldots, x_\sigma$ are constant.

Consider in $\mathbf{C} \setminus \{0\}$ a small loop ω around $a = 0$. If we follow the loop, the critical values of the corresponding functions g_a will behave like braids (cf. figure 3).

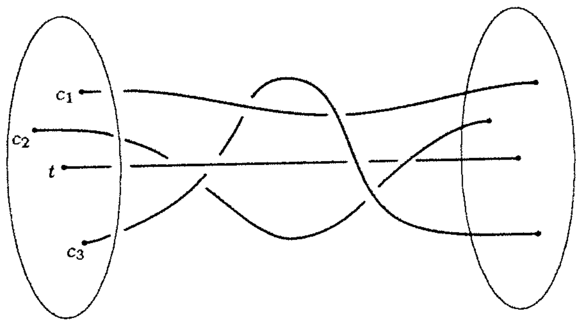

Figure 3: The moving critical values.

We can follow this by an isotopy of the disc D. At the end we get a permutation of the critical values. This permutation has to respect the "type" of the singularities in the singular fibre.

We now consider the case of one special fibre X_t with non-isolated singularities and/or several isolated singularities. Moreover we assume that all other singular fibres have only one singularity, which must be of type A_1. The above loop ω defines a monodromy map $X_t \to X_t$, which induces:

$$h_\omega : H_n(X_t) \longrightarrow H_n(X_t)$$

During this *special fibre monodromy* not only X_t is moving, but also the A_1-points move! Let a system of paths Γ be given and suppose that during the isotopy of the disc the system Γ move to a system Γ' (cf. figure 4).

Using the isomorphisms 3.5 of $H_n(X_t)$ with the A_1-lattices L_Γ and $L_{\Gamma'}$,we see that the map $h_\omega : H_n(X_t) \to H_n(X_t)$ is just given by the base change from L_Γ to $L_{\Gamma'}$.

It could be interesting to study the special fibre monodromies in more detail.

4.3 Remarks about Picard-Lefschetz transformations

Let γ_i be a path joining s with s_i, defining vanishing cycles $\delta_i \in H_n(X_s)$ and $\Delta_i \in H_n(X_t)$. The simple loop u_i around the critical value c_i which corresponds to γ_i defines a Picard-Lefschetz transformation:

$$h_{u_i} : H_n(X_s) \longrightarrow H_n(X_s)$$

for which the Picard-Lefschetz formula holds:

$$h_{u_i}(x) = x + (x \cdot \delta_i)\delta_i,$$

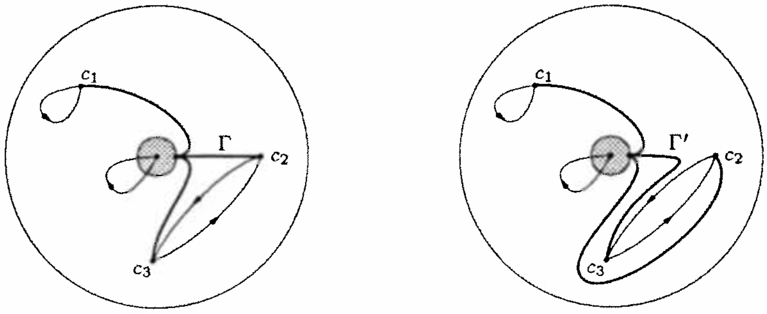

Figure 4: Γ and Γ'.

where $(\cdot)$ denotes the intersection product. This Picard-Lefschetz transformation is independent of the system of paths Γ in which γ_i can be embedded.

For the special fibre X_t the situation is different. Given a system of paths Γ, containing γ_i it follows e.g. from the Picard-Lefschetz formula, that we have the following restriction:

$$h_{u_i} : L_\Gamma \longrightarrow L_\Gamma.$$

Given an other system Γ', containing γ_i, we get the restriction:

$$h'_{u_i} : L_{\Gamma'} \longrightarrow L_{\Gamma'}.$$

Warning: The induced mappings h_{u_i} and $h'_{u_i} : H_n(X_t) \longrightarrow H_n(X_t)$ can in general be different.

This phenomenon already occurs in simple examples, such as isolated plane curve singularities with several A_1-points in the special fibre.

We conclude that there do not exist Picard-Lefschetz transformations of the above type on $H_n(X_t)$.

4.4 Remarks about special fibre monodromy groups

We consider next a d-parameter deformation:

$$\mathbf{C}^{n+1} \times \mathbf{C}^d \longrightarrow \mathbf{C} \times \mathbf{C}^d,$$

which satifies the conditions of theorem 2.3. We have two special examples in mind:

- The universal unfolding of isolated singularities.
- The universal disentanglement unfolding in the Mond examples.

One should stratify the parameter space $\mathbf{C}^d$ such that a stratum corresponds to a partition of the singular set into types. The fundamental groups of the strata act now on the homology of the corresponding special fibres, like in 4.2. In this way we get several special fibre monodromy groups related to our family.

Already in the case of isolated singularities it could be interesting to study these groups.

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